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THE CHARGE-ASYMMETRICAL PARASTATISTICS

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1. Introduction

A possibility of the generalization of the usual Fermi- and Bose-quantization of (free) fields has been considered by the author[1] and by Greenberg and Mohapatra[2] on the basis of trilinear commutation relations which are more general than the Green ones[3]

$$[a_m^+ a_l + c a_l a_m^+, a_k]_- = -\alpha\, \delta_{km} a_l, \qquad (1)$$

where α and c are some real numbers. The reason for this consideration was a general formulation of the principle of indistinguishability of identical particles in the form of the symmetry of the density matrix[1,4]. It turns out that the same relations can be derived from the requirement of the (strong) locality of (free) fields[2].

It was proved on the basis of the positive definiteness of the norm of state vectors in the Fock space that $c=\pm 1$[1] unless $c=0$ or $c \to \infty$ ($\alpha \to \infty$, α/c - finite). The cases $c=\pm 1$ correspond to the Green quantization[3], and, as it was proved by Greenberg and Messiah[5], any of its representation is equivalent to the so-called Green ansatz which is the direct sum of ordinary (Fermi- or Bose-) operators with anomalous mutual commutation relations[3]. In this sense the Green quantization turns out to be equivalent to the usual quantization in the presence of some internal degrees of freedom (see[4,6,7]).

For the completeness of the classification of possible schemes of quantization we need to consider two cases $c \to 0$ and $c \to \infty$ which have been rejected from the outset in[1]. Moreover, in[1] there is an inaccurate assertion about the insufficiency of relations (1) for the eliminated case $c=0$ for calculations of the state vector norms (or any general matrix elements). In fact, in this case the relations (1) do not allow those calculations if the vacuum state vector is situated on the right-hand side (r.h.s.). However, as it will be shown, those calculation can be a success if the vacuum vector is settled down on the left-hand side (l.h.s.). Moreover, this case can be reduced to another above-mentioned excluded case $c \to \infty$ by some redefinition of creation and annihilation operators. In this later case the vacuum state vector occupies a standard place on the r.h.s. Thus, we can consider these two equivalent quantization schemes on the same footing with the Green quantization.

Similar to the Green quantization the new one corresponds to para-Fermi or para-Bose statistics for which the number of particles in a symmetric or in antisymmetric state cannot exceed a given integer r which is called the order (sometimes rank) of parastatistics. However, the new quantization does not include any additional requirements characteristic of the Green quantization which make some symmetrized combinations of many particle states to vanish (see[4,6,7]). On the other hand, this new quantization turns out C-noninvariant from the beginning in contrast with the Green one.

In the limit case $r \to \infty$ (which is equivalent to the limit $|\varepsilon| \to \infty$ and $\alpha/\varepsilon \to 0$) the new quantization corresponds to the infinite statistics in which all representations of the symmetric group are allowed. It turns out that this limit case coincides exactly with a quantization recently proposed by Greenberg[8] directly for the description of infinite statistics. It is interesting that in the case of infinite statistics the theory ceases to be local. Note that in our limiting approach the treatment of antiparticles differs from the Greenberg one.

The paper is organized as follows. In Section 2 we formulate the new field quantization corresponding to cases excluded from previous investigations[1]. Therein we are convinced of this theory being local but not C-invariant.

In Section 3 our aim is to build the Fock representation for this quantization and to compare it with the analogous representation for the Green quantization.

In Section 4 we consider the limit case corresponding to infinite statistics interpreted as classical statistics and comper our approach with the Greenberg one in the description of particles and antiparticles in this limit.

In conclusion we discuss our results.

2. The new paraquantization

Under $\varepsilon=0$ Eq.(1) becomes

$$a^+_m a_l a_k - a_k a^+_m a_l = -\alpha\, \delta_{km} a_l \tag{2}$$

with the Hermitian-conjugate

$$a^+_k a^+_l a_m - a^+_l a_m a^+_k = -\alpha\, \delta_{km} a^+_l . \tag{3}$$

Relations (2) and (3) do not allow the calculation of any matrix elements if, as usual, a and a^+ are annihilation and creation operators, respectively, and the vacuum state vector is situated on the r.h.s.[1]:

$$a_k|0\rangle = 0 \quad \textit{for all } k. \tag{4}$$

But if we redefine a and a^+ as creation and annihilation operators, respectively, on condition that

$$a_k^+|0\rangle = 0 \quad \text{or} \quad \langle 0|a_k = 0 \quad (\textit{"left vacuum"}) \tag{5}$$

then we can calculate the action of a^+ onto polynomial $\mathcal{P}(a\ldots a)|0\rangle$ by means of Eq.(2). Evidently, we can come back to the usual definition of the vacuum state (4) if we rename operators: $a \leftrightarrow a^+$. But after this renotation we have instead of relations (2) and (3)

$$a_m a_l^+ a_k^+ - a_k^+ a_m a_l^+ = -\alpha\ \delta_{km} a_l^+ \tag{6}$$

$$a_k a_l a_m^+ - a_l a_m^+ a_k = -\alpha\ \delta_{km} a_l. \tag{7}$$

These relations are just those following from Eq.(1) in the limit $\varepsilon \to \infty$, $\alpha \to \infty$ on condition that $\beta = -\alpha/\varepsilon$ is a finite (positive or negative) number. We should only put β in Eqs.(6) and (7) instead of α. Thus, these two ways of quantization are fully equivalent. To retain the usual notation, we shall follow the second way with trilinear commutation relations (6) and (7) and vacuum condition (4).

For definiteness, as an exaple of integer spin fields we shall consider the (charge) scalar field

$$\phi(x) = (2\pi)^{-3/2} \int d^3k(2\omega(\mathbf{k}))^{-1/2}[a(\mathbf{k})e^{-ikx} + b^+(\mathbf{k})e^{ikx}] \tag{8}$$

and as an example of half-integer spin fields, the Dirac field

$$\psi(x) = (2\pi)^{-3/2} \int d^3k(m/E(\mathbf{k}))^{1/2} \sum_{\sigma = \pm 1/2} [a(\sigma,\mathbf{k})u(\sigma,\mathbf{k})e^{-ikx} + b^+(\sigma,\mathbf{k})v(\sigma,\mathbf{k})e^{ikx}], \tag{9}$$

where $\mathbf{k}$ is a momentum and σ is a spin state. In accordence with the previous discussion we propose the following commutation relations for the spinor field:

$$[\psi(x)\bar{\psi}(y), \psi(z)]_- = -i\alpha\ S(z-y)\psi(x), \tag{10a}$$

and Hermitian-conjugate relations:

$$[\psi(x)\bar{\psi}(y), \bar{\psi}(z)]_- = +i\alpha\ S(x-z)\bar{\psi}(y), \tag{10b}$$

where $\bar{\psi} = \psi^+\gamma^0$, and $-iS(x)$ is the well-known singular function for the Dirac field. For the scalar field it is necessary to exchange in Eqs.(10) $\psi \to \phi$, $\bar{\psi} \to \phi^+$, and $-iS \to i\Delta$, where $\Delta(x)$ is a singular function for the scalar field. Of course, usual fermionic (or bosonic) fields satisfy Eqs.(10) identically with $\alpha=1$ ($\alpha=-1$).

After substitution of the field decomposition (8) or (9) into Eqs.(10) we get the following relations for operators of the creation and annihilation of particles and antiparticles

$$[a_r a_s^+, a_t]_- = + \alpha\ \delta_{st} a_r, \qquad [a_r a_s^+, a_t^+]_- = - \alpha\ \delta_{rt} a_s^+, \tag{11a}$$

$$[a_r a_s^+, b_t]_- = 0, \qquad [a_r a_s^+, b_t^+]_- = 0, \tag{11b}$$

$$[a_r b_s, a_t]_- = 0, \qquad [b_r^+ a_s^+, a_t^+]_- = 0, \tag{11c}$$

$$[a_r b_s, b_t^+]_- = \pm\ \alpha\ \delta_{st} a_r, \qquad [b_r^+ a_s^+, b_t]_- = \mp\ \alpha\ \delta_{rt} a_s^+, \tag{11d}$$

$$[a_r b_s, a_t^+]_- = - \alpha\ \delta_{rt} b_s, \qquad [b_r^+ a_s^+, a_t]_- = + \alpha\ \delta_{st} b_r^+, \tag{11e}$$

$$[a_r b_s, b_t]_- = 0, \qquad [b_r^+ a_s^+, b_t^+]_- = 0, \tag{11f}$$

$$[b_r^+ b_s, a_t]_- = 0, \qquad [b_r^+ b_s, a_t^+]_- = 0, \tag{11g}$$

$$[b_r^+ b_s, b_t]_- = \mp\ \alpha\ \delta_{rt} b_s, \qquad [b_r^+ b_s, b_t^+]_- = \pm\ \alpha\ \delta_{st} b_r^+, \tag{11h}$$

where the upper and lower sign corresponds to the spinor and scalar field, respectively. The symbol δ_{st} means $\delta_{\sigma_s \sigma_t} \delta^{(3)}(\mathbf{k}_s - \mathbf{k}_t)$ for the spinor field and $\delta^{(3)}(\mathbf{k}_s - \mathbf{k}_t)$ for the scalar field.

The comparison of Eqs.(11a) with Eqs.(6) and (7) and Eqs.(11h) with Eqs.(2) and (3) shows that quantization rules for particles corresponds to the limit case $\varepsilon \Rightarrow \infty$ whereas rules for antiparticles correspond to the case ε=0. Thus particles and antiparticles obey quantization rules of different kinds. It is important that both of them cannot satisfy two kinds of relations simultaneously. Otherwise, a contradiction arises. If, for example, particle operators satisfy both Eqs.(11a) and (11h), then the summation and subtraction of these equations leads to the Green para-Bose and para-Fermi quantization, respectively, for the same particle operators at the same time.

Nevertheless, as we shall see below, the existence of mutual commutation relations (11b-g) already allow us to perform calculations of any matrix elements for systems containing particles and antiparticls.

The Hamiltonian and charge operator are to be written as

$$\mathcal{H}_{spinor} = - \alpha^{-1} \int d^3x (-i\vec{\gamma}\cdot\nabla + m)_{\mu\nu} \psi_\nu(x) \bar{\psi}_\mu(x) + const, \tag{12}$$

$$Q_{spinor} = e\ \alpha^{-1} \int d^3x\ \psi_\nu(x) \gamma^0_{\mu\nu} \bar{\psi}_\mu(x) + const, \tag{13}$$

and

$$\mathcal{H}_{scalar} = - \alpha^{-1} \int d^3x\ [\partial_t \phi(x) \partial_t \phi^+(x) + \nabla\phi(x)\cdot\nabla\phi^+(x) + m^2 \phi(x)\phi^+(x)] + const, \tag{14}$$

$$Q_{scalar}=ie\ \alpha^{-1}\int d^3x\ [\phi(x)\partial_t\phi^*(x) - \partial_t\phi(x)\phi^*(x)] + const. \quad (15)$$

Under substitution of Eqs.(8) and (9) into these expressions we have (merely due to orthogonality of solutions)

$$\mathcal{H} = -\alpha^{-1}\sum_\sigma \int d^3k\ E(\mathbf{k})[a(\sigma,\mathbf{k})a^*(\sigma,\mathbf{k})\mp b^*(\sigma,\mathbf{k})b(\sigma,\mathbf{k})] + const, \quad (16)$$

$$Q = -e\ \alpha^{-1}\sum_\sigma \int d^3k[a(\sigma,\mathbf{k})a^*(\sigma,\mathbf{k})\pm b^*(\sigma,\mathbf{k})b(\sigma,\mathbf{k})] + const, \quad (17)$$

where the upper sign corresponds to the spinor field ($\sigma=\pm1/2$) and the lower sign corresponds to the scalar field ($\sigma=0$).

Owing to Eqs.(11) the following relations

$$\begin{aligned}[][\mathcal{H},a(\sigma,\mathbf{k})]_- &= -E(\mathbf{k})a(\sigma,\mathbf{k}),\\ [\mathcal{H},b(\sigma,\mathbf{k})]_- &= -E(\mathbf{k})b(\sigma,\mathbf{k}),\end{aligned} \quad (18)$$

and

$$\begin{aligned}[][Q,a(\sigma,\mathbf{k})]_- &= -e\ a(\sigma,\mathbf{k}),\\ [Q,b(\sigma,\mathbf{k})]_- &= +e\ b(\sigma,\mathbf{k}),\end{aligned} \quad (19)$$

and Hermitian cojugate relations for a^* and b^* are valid. So both $a(\sigma,\mathbf{k})$ and $b(\sigma,\mathbf{k})$ are annihilation operators whereas $a^*(\sigma,\mathbf{k})$ and $b^*(\sigma,\mathbf{k})$ are creation ones.

The theory is local if currents take a form $\psi\bar{\psi}$. Really, due to Eq.(10) we have

$$[\psi(x)\bar{\psi}(y),\psi(z)\bar{\psi}(u)]_- = [\psi(x)\psi(y),\bar{\psi}(z)]_-\bar{\psi}(u)+\psi(z)[\psi(x)\bar{\psi}(y),\bar{\psi}(u)]_- =$$

$$= -i\alpha\ S(z-y)\psi(x)\bar{\psi}(u) + i\alpha\ S(x-u)\psi(z)\bar{\psi}(y). \quad (20)$$

The r.h.s. vanishes at spacelike separations of $x - u$ and $z - y$. At the same time this is not valid for currents of the form $\bar{\psi}\psi$. So, we can apply only currents of the former kind.

Commutation relations (10) are not invariant under the charge conjugate transformation which we write out for the Dirac field

$$\begin{aligned}\psi(x) &\Rightarrow \eta_c\psi_c(x) = \eta_c C\bar{\psi}^T(x),\\ \bar{\psi}(x) &\Rightarrow \eta_c^*\bar{\psi}\ (x) = \eta_c^*[C^{-1}\psi(x)]^T,\end{aligned} \quad (21)$$

where C is the charge-cojugation matrix and T means the trasposition matrix or spinor, $|\eta_c|^2=1$. Equations (10) change to *other* relations

$$\begin{aligned}[][\bar{\psi}_c(x)\psi_c(y),\bar{\psi}_c(z)]_- &= -i\alpha\ S(y-z)\bar{\psi}_c(x),\\ [\bar{\psi}_c(x)\psi_c(y),\psi_c(z)]_- &= +i\alpha\ S(z-x)\psi_c(y).\end{aligned} \quad (22)$$

Thus, this theory is invalid for the description of any true neutral (scalar or Majorana) fields, when $\psi_c = \psi$ and $b^* = a^*$, unless they are usual fermionic or bosonic fields. Only the Green paraquantization remains valid for these true neutral parafields.

Remark that under the transformation (21) the relations (11) for operators of creation and annihilation of particles and antiparticles remain invariable, which could be proved by substitution of a decomposition of type (9) for trasformed fields (21) into Eq.(22).

On the other hand, the theory (Lagrangian and commutation relations) is invariant under the space reflection: $\psi(x) \to \eta_p \gamma^0 \psi(i_s x)$, $\eta_p^2=\pm 1$, but is not invariant under the *antiunitary* time reflection: $\mathcal{T}(i_t)\psi(x)\mathcal{T}^{-1}(i_t)=\eta_t C^{-1}\gamma_5\psi(i_t x)$. Due to locality this theory is invariant under the *CPT*-transformation and thus under the combined *CT*-transformation:with the time reflection the (para)particles and (para)antiparticles must exchange with one another.

3. The Fock representation

We can construct the Fock representation of relations (11) in the same manner as done by Greenberg and Messiah[5] for the Green paraquantization. We assume the existence of a *unique* vacuum vector such that

$$a_r|0\rangle = b_r|0\rangle = 0 \qquad \textit{for all } r. \tag{23}$$

At first, we consider states containing only particles. By applying the first Eq.(11a) to the vacuum vector due to its uniqueness we arrive at

$$a_r a_s^+|0\rangle = c_{rs}|0\rangle, \tag{24}$$

where c_{rs} are some numbers. Then, we have an identity

$$[a_l a_m^+, a_r a_s^+]_- \equiv [a_l a_m^+, a_r]_- a_s^+ + a_r[a_l a_m^+, a_s^+]_- =$$
$$= \alpha\, \delta_{mr} a_l a_s^+ - \alpha\, \delta_{ls} a_r a_m^+.$$

By applying this identity to the vacuum vector we get (when $\alpha\neq 0$) $\delta_{mr}c_{ls}=\delta_{ls}c_{rm}$, i.e. $c_{ls}=p\delta_{ls}$, where p is a common multiplier. Now the following theorem can be proved

T h e o r e m 1. *The condition positive definiteness of the norm of a symmetric, if $\alpha\geq 0$, or an antisymmetric, if $\alpha\leq 0$, vector for r+1 particles implies that $p\geq r|\alpha|$ $(r=1,2,\ldots)$.*

P r o o f. Let $\alpha\geq 0$. The relation

$$a_s a_{l_1}^+ \ldots a_{l_n}^+|0\rangle = p\, \delta_{sl_1} a_{l_2}^+ \ldots a_{l_n}^+|0\rangle -$$
$$- \sum_{k=2}^{n} \alpha\, \delta_{sl_k} a_{l_2}^+ \ldots a_{l_{k-1}}^+ a_{l_1}^+ a_{l_{k+1}}^+ \ldots a_{l_n}^+|0\rangle \tag{25}$$

can be obtained by the repeated application of the second Eq.(11a). The sum taken over all $n!$ permutations $(\mathcal{P}1,\ldots,\mathcal{P}n)$ of indices $(1,\ldots,n)$ gives

$$a_s \sum_{\mathcal{P}} a^+_{l_{\mathcal{P}_1}}\ldots a^+_{l_{\mathcal{P}_n}}|0\rangle = [p-(n-1)\alpha] \sum_{\mathcal{P}} \delta_{s l_{\mathcal{P}_1}} a^+_{l_{\mathcal{P}_2}}\ldots a^+_{l_{\mathcal{P}_n}}|0\rangle.$$

By using this relation n times we obtain

$$\sum_{\mathcal{P}} \langle 0|a_{s_n}\ldots a_{s_1} a^+_{l_{\mathcal{P}_1}}\ldots a^+_{l_{\mathcal{P}_n}}|0\rangle = [p-(n-1)\alpha][p-(n-2)\alpha]\ldots$$
$$\ldots(p-\alpha)p \sum_{\mathcal{P}} \delta_{s_1 l_{\mathcal{P}_1}}\ldots\delta_{s_n l_{\mathcal{P}_n}}. \qquad (26)$$

For the norm calculation $l_i=s_i$ $(i=1,\ldots,n)$. Taking in succession $n=1,2,\ldots,r+1$, in consequence of the requirement of positiveness of corresponding norms, we get

$$p\geq 0,\ p\geq\alpha,\ p\geq 2\alpha,\ldots,\ p\geq r\alpha, \qquad (27)$$

which proves the theorem for $\alpha=|\alpha|$. The case $\alpha=-|\alpha|$ can be proved for antisymmetric vectors analogously.

Now we have two possibilities:either p is a finite (positive) number or it is infinite. The later case will be considered in the next section. Here we consider the case with a finite p. In this case we can always choose a sufficiently large number of particles r in a symmmetric (antisymmetric) state such that $r>p/|\alpha|$, and theorem 1 is broken unless the norm for $r+1$ particles vanishes and $p=r|\alpha|$. Thus Eq.(24) becomes

$$a_r a^+_s|0\rangle = r|\alpha|\delta_{rs}|0\rangle, \qquad (28)$$

where the number r takes any integer 1,2,... The following theorem is valid

T e o r e m 2. *If Eq.(28) is fulfilled, the norm of any vector with the number of particles* $n\geq r+1$ *under symmetrization for* $\alpha\geq 0$ *and antisymmetrization for* $\alpha\leq 0$ *over any* $r+1$ *of them vanishes.*

The proof of this theorem is presented in Appendix A.

Thus at a finite $p=r|\alpha|$ the number of particles in a symmetric (for $\alpha\geq 0$) or in an antisymmetric (for $\alpha\leq 0$) state cannot exceed a certain given integer r. In this case one speaks about para-Fermi or para-Bose statistics of order (or rank) r, respectively (see, for example, [4,9]). We conclude that our theory is convenient for the description of C-noninvariant parastatistics.

Now we include antiparticles into our consideration. For the same reasons employed for deriving Eq.(24), via applying of the second Eq.(11d) we arrive at

$$b_r b^+_t|0\rangle = d_{rt}|0\rangle, \qquad (29)$$

where d_{rt} are some numbers. But the action of the second Eq.(11h) on the vacuum vector gives at once

$$b_r b_t^+|0\rangle = \pm\, \alpha\, \delta_{rt}|0\rangle, \tag{30}$$

where the up and down sign correspons to the spinor and scalar field, respectively.

The requirement of the positiveness of a norm of an antiparticle vector gives the condition

$$\| \sum_l f_l b_l^+|0\rangle \|^2 = \sum_{l,m} f_l^* f_m \langle 0|b_l b_m^+|0\rangle = \pm \sum_l \alpha\ |f_l|^2 \geq 0,$$

that means $\pm\alpha\geq 0$. Thus, we have for the spinor field $\alpha\geq 0$ and for the scalar field $\alpha\leq 0$. Then in accordance with theorems 1 and 2 spinor parafields must obey para-Fermi statistics whereas scalar parafields must obey para-Bose statistics. Thus, we have a generalization of the Pauli spin-statistics theorem to parastatistics. Now in Eqs.(11d,h) we can write $|\alpha|$ instead of $\pm\alpha$. Then Eq.(29) becomes

$$b_r b_t^+|0\rangle = |\alpha|\delta_{rt}|0\rangle. \tag{31}$$

The comparison of Eqs.(28) and (31) reveals different vacuum conditions for particles and antiparticles unless r=1 (for usual statistics).

Analogously, by applying the first Eq.(11e) to the vacuum vector we arrive at

$$b_s a_t^+|0\rangle = f_{st}|0\rangle,$$

and then the action of the second Eq.(11g) gives f_{st}=0. So we have

$$b_s a_t^+|0\rangle = 0. \tag{32}$$

Remark that an analogous relation for $a_s b_t^+|0\rangle$ cannot be derived.

In general, we cannot calculate the norm of a state vector containing only antiparticles because the relation (11h) does not allow us to remove the annihilation operator to right vacuum vector. However, we can do this for each particle-antiparticle pair by means of Eq.(11d). Thus, we conclude that the present theory admits only states with the number of antiparticles which can exceed the number of particles in a system not more than unity. The number of particles in the system can be arbitrary. We have two types of allowed states with antiparticles:

$$b_{t_1}^+ a_{l_1}^+\ b_{t_2}^+ a_{l_2}^+ \ldots b_{t_m}^+ a_{l_m}^+\ a_{l_{m+1}}^+ \ldots a_{l_{m+n}}^+|0\rangle, \tag{33a}$$

$$b_{t_1}^+ a_{l_1}^+\ b_{t_2}^+ a_{l_2}^+ \ldots b_{t_m}^+ a_{l_m}^+\ a_{l_{m+1}}^+ \ldots a_{l_{m+n}}^+\ b_{t_{m+1}}^+|0\rangle. \tag{33b}$$

Due to the second relations (11c,f) these vectors obey the symmetry under permutations of any pairs $b^+_{t_k} a^+_{l_k}$ and $b^+_{t_s} a^+_{l_s}$ $(k,s=1,\dots,m)$. With the help of the same relations all antiparticle creation operators can be gathered in front of particle operators (except $b^+_{t_{m+1}}$ in (33b)).

As follows from the norm calculation for vector (33), the number of antiparticles in a symmetric state for $\alpha\geq 0$ (the spinor field) or in antisymmetric state for $\alpha\leq 0$ (the scalar field) cannot exceed the number r (but can be $r, r-1, r-2$, etc.). So orders of particles and antiparticles parastatistics coincide.

Thus, we can construct the Fock representation for the parastatistics of finite order with the above-mentioned restriction on the number of antiparticles as compared with the number of particles in a system, and now we can work within this space not worse than in the case of usual statistics. We can also get rid of the factor $|\alpha|$ making the renormalization of operators: $a_l \to a_l |\alpha|^{-1/2}$, $b_l \to b_l |\alpha|^{-1/2}$. Then we have merely $p=r$.

There are no futher restrictions within the present quantization scheme in contrast with the Green one. Therefore there is no additional disappearance of multiparticle states characteristic of the Green paraquantization. For illustration of this situation we consider a system of three particles obeying para-Fermi statistics of order two. A common vector of this system has the form

$$|\Psi\rangle = \sum_{l_1,l_2,l_3} \Psi(l_1 l_2 l_3) a^+_{l_1} a^+_{l_2} a^+_{l_3} |0\rangle, \tag{34}$$

where the sum on each of l_1, l_2, l_3 is taken over all one-particle states. No symmetry properties of function $\Psi(l_1 l_2 l_3)$ are implied beforehand. Due to Eqs.(11a) and (28) (set $|\alpha|=1$) we have the projection of this vector on certain one-particle states r,s,t

$$\langle 0|a_t a_s a_r|\Psi\rangle = 8\Psi(rst) - 4\Psi(rts) - 4\Psi(srt) + 2\Psi(str) - 4\psi(tsr) + 2\Psi(trs). \tag{35}$$

Only symmetric combination vanishes in consequence of this relation. Other five combinations form an antisymmetric representation and two irreducible representations of a mixed symmetry. These later are written in the explicit form

$$\Psi^{(1)}_{m'}(rst) = N_{m'}/(12\sqrt{3})\ \langle 0|2a_t a_s a_r - a_s a_t a_r + 2a_t a_r a_s - a_r a_t a_s - a_r a_s a_t - a_s a_r a_t|\Psi\rangle = N_{m'}/(2\sqrt{3})[2\Psi(rst) - \Psi(rts) + 2\Psi(srt) - \psi(str) - \psi(tsr) - \Psi(trs)], \tag{36a}$$

$$\Psi^{(2)}_{m'}(rst) = N_{m'}/12\ \langle 0|-a_s a_t a_r - a_r a_t a_s + a_r a_s a_t + a_s a_r a_t|\Psi\rangle = N_{m'}/2\ [-\Psi(rts) - \Psi(str) + \Psi(tsr) + \Psi(trs)], \tag{36b}$$

and

$$\Psi^{(1)}_{m''}(rst) = N_{m''}/12 \ \langle 0|-a_s a_t a_r + a_r a_t a_s + a_r a_s a_t - a_s a_r a_t|\Psi\rangle =$$
$$= N_{m''}/2 \ [-\psi(rts)+\Psi(str)+\Psi(tsr)-\Psi(trs)] \quad (37a)$$

$$\Psi^{(2)}_{m''}(rst) = N_{m''}/(12\sqrt{3}) \ \langle 0|2a_t a_s a_r + a_s a_t a_r - 2a_t a_r a_s - a_r a_t a_s +$$
$$+ a_r a_s a_t - a_s a_r a_t|\Psi\rangle = N_{m''}/(2\sqrt{3})[2\Psi(rst)+\Psi(rts)-2\Psi(srt)-$$
$$-\Psi(str)+\Psi(tsr)-\Psi(trs)], \quad (37b)$$

where $N_{m'}=(1+\delta_{rs}-\delta_{rt}/2-\delta_{st}/2)^{-1/2}$ and $N_{m''}=(1+\delta_{rt}/2+\delta_{st}/2)^{-1/2}$.

We emphasize that these combinations form irreducible representations under place permutations of operators but not under permutations of one-particle states (r,s,t). For instance any two states of particles could be even equal. Thus, for example, the $\mathcal{P}_{12}$ - transposition means the exchange of any two operators standing on (from right to left) first and second positions that is $a_t a_s a_r \Rightarrow a_t a_r a_s$, $a_s a_t a_r \Rightarrow a_s a_r a_t$, etc.; accordingly, in functions - two arguments standing on (from left to right) first and second positions: $\Psi(rst) \Rightarrow \Psi(srt)$, $\Psi(rts) \Rightarrow \Psi(trs)$, etc.

For the two equivalent representations (36) and (37) the operator (or argument) place transpositions have the usual matrix form

$$\mathcal{P}_{12}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \mathcal{P}_{13}=\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}, \ \mathcal{P}_{23}=\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}. \quad (38)$$

Any other permutations could be composed of traspositions.

The probability of three particles described by anyone of these two mixed representations $m=m'$ or m'' to be found in states r,s,t is

$$W_m(rst) = |\Psi^{(1)}_m(rst)|^2 + |\Psi^{(2)}_m(rst)|^2. \quad (39)$$

Evidently, this expression is invariant under place permutations of operators (or arguments) such as (38). But this is just our goal for the theory of identical particles. Thus, our second-quantized theory based on the relations (11a) with the subsequent symmetrization of many particle states according to Young-diagrams is really consistent with first quantized parastatistics of identical particles. For the later the many particle states are described by, the so-called, "generelized rays" in the many particle Hilbert space consisting of the set of basic vectors of irreducible representations[9]. In distinction to the Green paraquantization, the new one does not imply any supplementary restrictions on irreducible representations such as vanishing of one of mixed representations of three parafermions of order two[7].

The absence of the interference term in Eq.(39) indicates the possible interpretation of this expression as a result of averaging over some hidden internal degrees of freedom like isospin in the framework of ordinary Fermi statistics for complete functions. However, the consideration of this possibility is beyond the scope of this paper.

4. The infinite statistics

Now we examine consequences of theorem 1 when $p \to \infty$. In this case there are no restrictions on the number of particles (or antiparticles) in the symmetric or antisymmetric state in accordance with inequalities (27). Thus, we can speak about *infinite* statistics in this case (see, for example,[4]). However, it is necessary to accomplish a renormalization of operators

$$a_s \to a_s /\sqrt{p}, \tag{40}$$

otherwise, according to Eq.(26), norms of symmetric or antisymmetric vectors tend to infinity. Under renormalization (40) Eq.(25) takes a simple form

$$a_s a^+_{l_1} \ldots a^+_{l_n} |0\rangle = \delta_{s l_1} a^+_{l_2} \ldots a^+_{l_n} |0\rangle, \tag{41}$$

which means that an annihilation operator "kills" the nearest (from the right) creation operator and the result of its action is merely δ. Since the relation (41) holds for all particle vectors, this implies the existence of the algebraic equality

$$a_s a^+_l = \delta_{sl}. \tag{42}$$

This is just the equation which Greenberg (as he noted, by the suggestion of R.Hegstrom)[8] has directly assumed for his description of infinite statistics.

Note, in the limit $p \to \infty$ Eq.(11a) turns into an identity because its l.h.s. vanishes identically due to Eq.(42) and r.h.s. goes to zero as α/p.

However, in this limit we cannot define the Hamiltonian and charge operator (in their particle parts) in their previous bilinear forms (16) and (17) owing to their becoming infinite as p/α when $p \to \infty$ under the renormalization (40). Assuming Eq.(42) as origin Greenberg[8] supposed another expression for the particle number operator (consequently, the Hamiltonian, the charge operator, and so on) in the form of the infinite sequence

$$n_l = a_l^+ a_l + \sum_k a_k^+ a_l^+ a_l a_k + \sum_{k_1,k_2} a_{k_1}^+ a_{k_2}^+ a_l^+ a_l a_{k_2} a_{k_1} + \ldots +$$
$$+ \sum_{k_1,k_2,\ldots,k_s} a_{k_1}^+ \ldots a_{k_s}^+ a_l^+ a_l a_{k_s} \ldots a_{k_1} + \ldots \qquad (43)$$

It is easy to verify that these operators satisfy the necessary properties

$$[n_l, a_j]_- = - \delta_{lj}\, a_j . \qquad (44)$$

Greenberg[8] has shown also that the partition function of particles obeying infinite statistics corresponds to the Boltzmann statistics without the famous Gibbs 1/N! factor. It is well known that the introduction of this factor is necessary for avoiding the Gibbs paradox which consists in the increase of the entropy when two volumes of the identical molecule gas at the same temperature and density are mixed. One can suggest the following interpretation of infinite statistics. Any parastatistics can be interpreted as the usual (Bose or Fermi) statistics when there is the exact degeneracy of particles with respect to an additional internal coordinate, and the number of internal states of this additional degree of freedom is equal to the order of parastatistics[4]. In the case of infinite statistics this number is infinite. Then we can consider infinite statistics as statistics of *non-identical* particles since they are (mentally) distiguishable in their internal states. Then the entropy of the non-identical molecule gas in a larger volume must increase, and the Gibbs paradox does not appear. In my opinion, it is remarkable that the Boltzmann statistics of classical (non-identical) particles can be described by infinite statistics of (identical) paraparticles, called by Greenberg the "quantum Boltzmann statistics", with the help of his operator relation (42)[8].

Now we turn to examination of the behaviour of antiparticles in the limit $p \to \infty$. The vacuum relation (31) does not contain this parameter at all, and the renormalization of antiparticle operators of the type (40) is not required. For this reason Eqs.(11d,h) are not affected by this limit. However, the r.h.s. of Eqs.(11e) vanishes as α/p:

$$[a_r b_s, a_t^+]_- = [b_r^+ a_s^+, a_t]_- = 0. \qquad (45)$$

Thus, though the product $a_r b_s$ contains a_r, it commutes with a_r^+ in this limit $p \to \infty$!

Therefore all our conclusions about the behaviour of antiparticles in this limit hold valid, which represent the rule limiting the number of antiparticles by the number of particles plus unity and correspondence between the sign of α and the field spin: $\alpha\geq 0$ corresponds to spinor fields; and $\alpha\leq 0$, to scalar ones. However, in both cases there are no restrictions on numbers of particles and antiparticles in the symmetric or antisymmetric state.

As in the previous consideration, particles and antiparticles appear in pair states (33) (with an exception for one isolated antiparticles in the state (33b)). So, we can introduce pair operators

$$A_{rs} = a_r b_s,\ A^+_{sr} = b^+_s a^+_r,\ (A^+_{sr})^+ = A_{rs}, \tag{46}$$

Due to Eqs.(11d),(43), and (11c,d) we have Bose-like relations for these operators

$$[A_{rs}, A^+_{s'r'}]_- = |\alpha|\delta_{rr'}\delta_{ss'},\ [A_{rs}, A_{r's'}]_- = 0. \tag{47}$$

The pair number operators

$$P_{sr,rs} = |\alpha|^{-1} A^+_{sr} A_{rs} \tag{48}$$

satisfy the required relations

$$[P_{sr,rs}, A_{pq}]_- = -\delta_{sq}\delta_{rp}A_{pq},\ [P_{sr,rs}, A^+_{pq}]_- = \delta_{rq}\delta_{sp}A^+_{pq}. \tag{49}$$

Moreover, in consequence of Eq.(45) these operators commute with a^+ and a, and, therefore, with particle number operators (43). Thus, we can consider a heap of non-pairing particles and the addition of particle-antiparticle pairs as independent subsystems.

Certainly, in the limit $p \to \infty$ the theory becomes non-local, and we cannot employ bilinear expressions like (12) or (14) for Hamiltonians and other observables. From the beginning positive- and negative-frequency field solutions are separated. Particle operators connected with positive solutions satisfy The Greenberg relations (42) whereas antiparticle operators connected with negative solutions satisfy the relations (11) with the alteration (45) and substitution $\pm\alpha=|\alpha|$, which we put equal to unity. For fields (8) and (9) with these alterations instead of the initial relations (10) we have

$$[\psi(x)\bar{\psi}(y), \psi(z)]_- = -iS^{(-)}(z-y)\psi(x), \tag{50a}$$

$$[\psi(x)\bar{\psi}(y), \bar{\psi}(z)]_- = +iS^{(-)}(x-z)\bar{\psi}(y) \tag{50b}$$

(for scalar fields one should change $S^{(-)} \to \Delta^{(-)}$). Negative-frequency singular functions standing in r.h.s. do not vanish outside of the light-cone, and so the theory becomes really non-local in this limit.

It is necessary to emphasize the difference between our approach and the Greenberg one/8/ concerning to the description of antiparticles. Greenberg has immediately suggested the same relation (42) for antiparticle operators too, and analogous mutual ralations between particle and antiparticle operators with r.h.s.equal to zero. He has shown also that his theory remains *CPT*-invariant though it is non-local. The non-locality of the present theory of infinite statistics as well as the Greenberg one corresponds to the non-existence of infinite statistics within the axiomatic *local* algebra of observables/10/.

In our approach the theory of infinite statistics is *CPT*-invariant too. For free-field expansions (8) or (9) via Eqs.(23),(31), and (42) we arrive at

$$\langle 0|\phi(x)\phi^{+}(y)|0\rangle = \langle 0|\phi^{+}(-y)\phi(-x)|0\rangle = i\Delta^{(+)}(x-y) \qquad (51)$$

(for simplicity we present the two-point vacuum expectation for the scalar field). Then, by the direct calculation, we can be convinced that just as for a free Bose-field an arbitrary vacuum matrix element of a product of free fields is a sum of products of two-point functions. In consequence of this property and Eq.(51) we can prove the requirement of the week locality for any vacuum matrix element, and thus, *CPT*-invariance of our theory for free parafields, although they are not local in the limit $p \Rightarrow \infty$.

5. Conclusion

We are convinced that there exists, side by side with the Green paraquantization, a new generelized quantization without any additional restrictions which are characteristic of Green paraquantization. As well as the later the new theory turns out to be local. However, in distinction to the Green paraquantization, this one appears to be charge-asymmetric: the number of particles in any state can be arbitrary but the number of antiparticles is limited by the number of particles plus unity.

In the limit $p \Rightarrow \infty$ the new quantization coincides with the quantization recently proposed by Greenberg/8/ for description of infinite statistics, he has called the "quantum Boltzmann statistics". It is plausible that this later can be interpreted as the Boltzmann statistics of classical (non-identical) particles which are distinguishable in their (hidden) internal states. In this limiting case the theory becomes non-local, in accordance with the impossibility of the existence of infinite statistics in the framework of the local algebra of observables/10/.

Since infinite statistics has occurred within the present theory as a limiting case, and antiparticles are not affected by this limit, our description of antiparticles is different from the Greenberg one/8/. In our case an antiparticle can be included only in a pair with a particle (except for the only antiparticle isolated from the outset). These pairs form Bose-like objects which are independent of heaps of nonpairing particles.

Thus we have a complete classification of all permitted statistics of identical particles and corresponding schemes of field quantizations. But now we have *two* different quantization schemes which are convenient for the description of the same parastatistics: the Green paraquantization and the present one. Then a question arises whether can we discriminate between these two possibilities by any additional requirements when we consider a few interacting parafields. Maybe, there is a possibility for the employment of both of the schemes for the description of different interactions of the same system of parafields (paraparticles). I propose to consider these questions in the future.

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Appendix A

For simplicity we consider only the case $\alpha\geq 0$. Let one has an n-paricle vector with the symmetrization over $r+1$ particles $l_1, l_2, \ldots, l_{r+1}$ distributed in any order among other $n-r-1$ particles. Now we prove that the action of an annihilation operator a_s on this vector results in a sum of vectors which are symmetric in $r+1$ particles too.

Let the first (from left) creation operator participates in symmetrization. According to Eq.(25) under $p=r|\alpha|$ we have

$$a_s \sum_{\mathcal{P}\in S_{r+1}} a^+_{\mathcal{P}_1} a^+_2 \ldots a^+_{\mathcal{P}_{l_2}} \ldots a^+_{\mathcal{P}_{l_{r+1}}} \ldots a^+_n |0\rangle =$$

$$= |\alpha| \sum_{\mathcal{P}\in S_{r+1}} \Big(r\, \delta_{s\mathcal{P}_1} a^+_2 \ldots a^+_{\mathcal{P}_{l_2}} \ldots a^+_{\mathcal{P}_{l_k}} \ldots a^+_{\mathcal{P}_{l_{r+1}}} \ldots a^+_n |0\rangle -$$

$$- \sum_{k=1}^{r+1} \delta_{s\mathcal{P}_{l_k}} a_2 \ldots a_{\mathcal{P}_{l_2}} \ldots a_{\mathcal{P}_1} \ldots a_{\mathcal{P}_{l_{r+1}}} \ldots a_n |0\rangle -$$

$$- \delta_{s2} a^+_{\mathcal{P}_1} a^+_3 \ldots a^+_{\mathcal{P}_{l_2}} \ldots a^+_{\mathcal{P}_{l_k}} \ldots a^+_{\mathcal{P}_{l_{r+1}}} \ldots a^+_n |0\rangle - \ldots \Big)$$

(instead of states $l_1, \ldots, l_n$ we write merely their numbers $1, \ldots, n$). Terms with the index s coincident with the one of symmetrized indices are collected in the second sum, and in all these terms the operator

$a^+_{\mathcal{P}_1}$ replaces $a^+_{\mathcal{P}_{I_k}}$. The number of these terms is r and due to symmetrization over indices $1, i_2, \ldots, i_{r+1}$ these terms are cancelled out with the first term. Certainly, the remaining terms are symmetric in initial $r+1$ indices.

Now let the first creation operator does not participate in the initial symmetrization. Then we have

$$a_s \sum_{\mathcal{P}\in S_{r+1}} a_1^+ a_2^+ \ldots a^+_{\mathcal{P}_{I_1}} \ldots a^+_{\mathcal{P}_{I_k}} \ldots a^+_{\mathcal{P}_{I_{r+1}}} \ldots a_n^+ |0\rangle =$$

$$= -|\alpha| \sum_{\mathcal{P}\in S_{r+1}} \Big(r\delta_{s1} a_2^+ \ldots a^+_{\mathcal{P}_{I_1}} \ldots a^+_{\mathcal{P}_{I_k}} \ldots a^+_{\mathcal{P}_{I_{r+1}}} \ldots a_n^+ |0\rangle -$$

$$- \delta_{s2} a_1^+ a_3^+ \ldots a^+_{\mathcal{P}_{I_1}} \ldots a^+_{\mathcal{P}_{I_k}} \ldots a^+_{\mathcal{P}_{I_{r+1}}} \ldots a_n^+ |0\rangle - \ldots -$$

$$- \sum_{k=1}^{r+1} \delta_{s\mathcal{P}_{I_k}} a_2^+ \ldots a^+_{\mathcal{P}_{I_1}} \ldots a_1^+ \ldots a^+_{\mathcal{P}_{I_{r+1}}} \ldots a_n^+ |0\rangle \Big).$$

Terms with the index s coincident with the one of symmetrized indices are collected in the last sum, and in all those terms the operator a_1^+ replaces $a^+_{\mathcal{P}_{I_k}}$. Then this sum can be written as

$$\sum_{k=1}^{r+1} \delta_{sI_k} \sum_{\mathcal{P}\in S(I_1,\ldots,I_{k-1},1,I_{k+1},\ldots,I_{r+1})} a_2^+ \ldots a^+_{\mathcal{P}_{I_1}} \ldots a^+_{\mathcal{P}_1} \ldots a^+_{\mathcal{P}_{I_{r+1}}} \ldots a_n^+ |0\rangle.$$

Thus we have the sum of terms which are again symmetric in $r+1$ indices $i_1, \ldots, i_{k-1}, 1, i_{k+1}, \ldots, i_{r+1}$.

Repeating the action of annihilation operators on the initial vector many times we arrive at the sum of vectors which are symmetric in all their $r+1$ states. Under the action of one more annihilation operator on these vectors they vanish according to theorem 1 (at $n=r+1$ and $p=r|\alpha|$).

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