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QUANTIZATION OF CONSTRAINED SYSTEMS AND PATH INTEGRAL IN CURVILINEAR SUPERCOORDINATES

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I. Introduction

It is well-known that elimination of unphysical variables in gauge theories and quantization do not commute (Christ and Lee 1980, Prokhorov 1982, Ashtekar and Horowitz 1982, Isham 1986). In other words, a quantum theory described by the Dirac scheme (Dirac 1965) can differ from that one when unphysical degrees of freedom are eliminated before quantization. However, the standard way of path integral (PI) construction (Faddeev 1970, Faddeev and Slavnov 1980) corresponds just to the last method since in this way unphysical momenta and coordinates are eliminated from the classical action with the help of constraints and supplementary conditions, respectively, and the phase space of physical degrees of freedom is a'priori considered an even-dimensional Euclidean space.

The difference of these quantization methods comes out from a curvilinear character of physical variables (Prokhorov 1982) (it is known that the application of operations of quantization and introduction of curvilinear coordinates in a different order to a classical theory gives different quantum theories), on the one hand, and from their phase space reduction appearing because of a gauge symmetry (Prokhorov 1982, Prokhorov and Shabanov 1989), on the other hand. A modification of PI when a physical phase space is reduced was shown in (Prokhorov and Shabanov 1989, Shabanov 1989). Other examples, of the "quantum dynamical" phase space reduction was given by Dunne, Jackiw and Trugenberger (1989).

The present work is devoted to the consideration of a PI form corresponding uniquely to the Dirac quantization scheme. It turns out that there exists a connection between a PI form in curvilinear coordinates (Sect.2,3) and PI for gauge theories containing both boson and fermion degrees of freedom. Existence of fermions in a theory causes in PI derivation on physical superspace, some specific difficulties, since one cannot decrease the number of anticommuting variables describing fermions by gauge transformations (Sect.4). In Section 5 it is shown that taking into consideration a curvilinear character of physical variables and their phase space reduction, we may explicitly define the gauge-invariant kernel of the evolution operator via PI. A mathematical reason of this is also presented in Sect.5. In Conclusion we suggest a general recipe of the PI construction corresponding to the Dirac quantization scheme for arbitrary picking out physical variables. It should be remembered that there exist invariant and non-invariant ways of picking out physical degrees of freedom. The first corresponds to the introduction of gauge-invariant variables. However, a complete set of gauge invariants is not always known, so we are forced to use the second way when physical variables are separated by supplementary conditions from initial variables, i.e., by gauge fixing. In the recipe suggested below, we show how one should take into account a curvilinear character of physical variables and their phase space reduction in a non-invariant way of separating them.

2. PI in curvilinear coordinates on superspace

Consider a quantum-mechanical system containing boson degrees of freedom as well as Grassman ones. We take the Hamiltonian as follows

$$H = \frac{1}{2} p_a^2 + V(x, \psi^+, \psi), \tag{2.1}$$

where $[x_a, p_b] = i\,\delta_{ab}$ $(a, b = 1, 2, \ldots, M)$ and $[\psi^+_\alpha, \psi_\beta]_+ = \delta_{\alpha\beta}$ $(\alpha, \beta = 1, 2, \ldots, N)$. The operator algebra may be realized in a space of functions on superspace $\Phi = \Phi(x, \bar\psi) \equiv \Phi(Q)$, $Q = (x, \bar\psi)$ ($\bar\psi$ is complex conjugated to ψ) if

$$p_a \Phi = -i \frac{\partial}{\partial x_a} \Phi, \quad \psi^+_\alpha \Phi = \bar\psi_\alpha \Phi, \quad \psi_\alpha \Phi = \frac{\partial}{\partial \bar\psi_\alpha} \Phi. \tag{2.2}$$

Here and below all derivations of Grassman variables are left. The scalar product under which we define Hermitian conjugated operators has the form (Berezin 1966)

$$\langle \Phi_1 | \Phi_2 \rangle = \int dx\, d\bar\psi\, d\psi\, e^{-\bar\psi\psi}\, \bar\Phi_1(Q)\, \Phi_2(Q), \tag{2.3}$$

where the integral is taken over the whole $\mathbb{R}^M$. In accordance with (2.3) the unit operator kernel $\langle Q | Q' \rangle$ has the form

$$\sum_E \Phi_E(Q) \bar\Phi_E(Q') \equiv \delta(Q, \bar Q') = \delta(x - x')\, e^{\bar\psi\psi'}, \tag{2.3a}$$

where Φ_E are eigenfunctions of Hamiltonian (2.1).

In the general case the change of variables is defined by a function on superspace $Q = Q(q)$, $q = (y, \bar\xi)$.

However, we shall consider special forms of Q. But it will be enough for the application of gauge theories. Introduce the new variables

$$x_a = x_a(y)\ , \quad \psi_\alpha = \Omega_{\alpha\beta}\xi_\beta\ , \tag{2.4}$$

where $\Omega \in SU(N)$ and $\Omega = \Omega(y)$. Then $dQ = A\,dq$ and $\partial/\partial Q = A^{-1T}\partial/\partial q$. Here $A^i_j = \partial Q^i/\partial q^j\ ,\ i,j = (a,\alpha)$.
After some calculations we get from (2.4)

$$\frac{\partial}{\partial x_a} = B^b_a\left(\partial_b + i\Pi_b\right), \tag{2.5}$$

where $B^b_a = \left(\left(\partial x/\partial y\right)^{-1}\right)^b_a$, $\Pi_b = i\bar\xi\partial_b\Omega^+\Omega\,\partial/\partial\bar\xi$, $\partial_b = \partial/\partial y_b$. Using (2.5), (2.4) and (2.2), we rewrite Hamiltonian (2.1) in the new variables

$$H = \frac{1}{2}P_a g^{ab} P_b + V_q(y) + V. \tag{2.6}$$

Here $P_a = -i\mu^{-1/2}(\partial_a + i\Pi_a)\circ\mu^{1/2}$, $\mu = \mathrm{sdet}\,A = \sqrt{g}$, $g = \det\|g^{ab}\|^{-1}$, $g^{ab} = B^a_c B^b_c$ and V_q is the effective quantum addend to a potential

$$V_q = \frac{1}{2\sqrt{\mu}}\left(\partial_a g^{ab}\right)\partial_b\sqrt{\mu} + \frac{1}{2\sqrt{\mu}}g^{ab}\partial_a\partial_b\sqrt{\mu}. \tag{2.7}$$

The form of a scalar product in the space of functions $\varphi = \varphi(q)$ follows from (2.3) and (2.4)

$$\langle\varphi_1|\varphi_2\rangle = \int\limits_K dy\,d\bar\xi\,d\xi\,e^{-\bar\xi\xi}\mu(y)\bar\varphi_1(q)\varphi_2(q), \tag{2.8}$$

where K is a region of integration over y, $K \subset \mathbb{R}^M$. The mapping $x = x(y)$ is one-to-one if $x \in \mathbb{R}^M$ and $y \in K$. Let the functions $x(y)$ be analytical for all $y \in \mathbb{R}^M$. So, there exist transformations $\hat S$ acting in $\mathbb{R}^M$ such as

$$x_a(y) = x_a(\hat{s}y). \tag{2.9}$$

For example, M=2 and $y_a = (r, \theta)$ are polar coordinates. In this case K is the strip $r > 0$, $\theta \in (0, 2\pi)$, and transformations $\hat{S}$ have the form $\theta \to \theta + 2\pi n$, $n \in \mathbb{Z}$ and $r \to -r$, $\theta \to \theta + \pi$. Apparently, transformations $\hat{S}$ form a discrete group $\mathcal{S}$ and $K = \mathbb{R}^M \backslash \mathcal{S}$, i.e., K is made from $\mathbb{R}^M$ by identification of points in $\mathbb{R}^M$ connected with each other by S-transformations. In the mathematical language, the mapping $x = x(y)$ gives the projection in the principal fibre bundle (Kobayashi and Nomizu 1963), where the base and fibre bundle coincide with $\mathbb{R}^M$ and the discrete group $\mathcal{S}$ acts along a fibre in the fibre bundle $\mathbb{R}^M(y)$.

The group $\mathcal{S}$ induces discrete transformations of Grassman variables $\hat{T}_s$ such as $\Omega(y)\xi = \Omega(\hat{s}y)\hat{T}_s \xi$, so, $\hat{T}_s = \Omega^+(\hat{s}y)\Omega(y)$. We mark the total group of transformations $\hat{s}$ and $\hat{T}_s$ as $\mathcal{S}^*$ and $\hat{s}^* q = (\hat{s}y, \bar{\xi}\hat{T}_s^+)$.

Hamiltonian (2.6) is written in an explicit Hermitian form since the operators P_a are Hermitian under scalar product (2.8). If $\varphi_{E'}$ are eigenfunctions of (2.6), we may write (since the Hilbert spaces (2.1) with (2.3) and (2.6) with (2.8) are isomorphic)

$$\varphi_{E'}(q) = \sum_E C_{E'E}\, \Phi_E(Q). \tag{2.10}$$

By definition $Q(\hat{s}^* q) = Q(q)$, and we conclude

$$\varphi_{E'}(\hat{s}^* q) = \varphi_{E'}(q). \tag{2.11}$$

Using the property of parity (2.11), we may analytically continue φ_E to the unphysical region $y \in \mathbb{R}^M$. Thus, in accordance with (2.8) we have

$$\sum_E \varphi_E(q)\overline{\varphi}_E(q') = \sum_{S^*} \left[\mu(y)\mu(\hat{s}y')\right]^{-1/2} \delta(q, \hat{s}^* \bar{q}') \tag{2.12}$$

where $y \in \mathbb{R}^M$, $y' \in K$. For physical values of y, y', i.e., $y, y' \in K$, one should only keep the first term in sum (2.12) with $\hat{s}^* = 1$. Formula (2.12) defines an analytical continuation of the unit operator kernel $\langle q | q' \rangle$ to the unphysical region.

Note also that kernel (2.12) can be obtained directly from (2.3a). Since a change of variables in a quantum theory is equivalent to a choice of a new basis in the Hilbert space of states, the left-hand sides of (2.3a) and (2.12) must coincide, hence, the right-hand sides coincide too. Indeed, let us change Q and Q' in (2.3a) by expressions (2.4) and assume $y \in \mathbb{R}^M$, $y' \in K$, then

$$\delta(Q, \bar{Q}') = \sum_S \left[\mu(y)\mu(\hat{s}y)\right]^{-1/2} \delta(y - \hat{s}y') \exp \bar{\xi} \hat{T}_s \xi . \tag{2.12a}$$

The equality follows from the rule of changing an argument of M-dimensional δ-function and the definition of $\hat{T}_s = \Omega^+(\hat{s}y')\,\Omega(y')$ (at $\delta(y - \hat{s}y')$ we may change $\Omega^+(y)$ by $\Omega^+(\hat{s}y')$ in $\exp \bar{\psi}\psi'$).

Let us turn now directly to the PI derivation. The kernel of the infinitesimal evolution operator is

$$U_\varepsilon(q, \bar{q}') = \left[1 - i\varepsilon H(y, \bar{\xi}, \partial_{\bar{\xi}})\right] \langle q | q' \rangle \tag{2.13}$$

where H is given by (2.6) and $\varepsilon \to 0$. We rewrite kernel (2.12) as follows

$$\int \frac{dy'' d\bar{\xi}'' d\xi'' e^{-\bar{\xi}''\xi''}}{(\mu\mu'')^{1/2}} \delta(q,\bar{q}'')\, Q(q'',\bar{q}'), \tag{2.14}$$

where $\mu = \mu(y)$, $\mu'' = \mu(y'')$ and

$$Q(q'',\bar{q}') = \sum_{S^*} \delta(q'', \hat{s}^*\bar{q}'). \tag{2.15}$$

Then, we use the representation of δ-function $\delta(y) = (2\pi)^{-M}\int dp \exp ipy$ in (2.14) and substitute (2.14) into (2.13). For the calculation of action of H on $\delta(q,\bar{q}')$ one should take into consideration a noncommutability $\bar{\xi}$ and ∂_{ξ} and also use the equality

$$\partial_a \circ g^{ab}(y)\partial_b \delta(y-y'') = \left(g^{ab}(y'')\partial_a\partial_b - \partial_a g^{ab}(y'')\partial_b\right)\delta(y-y''), \tag{2.16}$$

where $\partial_a = \partial/\partial y_a$. Thus we find, with an accuracy of $O(\varepsilon^2)$,

$$U_\varepsilon(q,\bar{q}') =$$

$$= \int \frac{dy'' d\bar{\xi}'' d\xi''}{(\mu\mu'')^{1/2}} e^{-\bar{\xi}''\xi''} U_\varepsilon^{eff}(q,\bar{q}'')\, Q(q'',\bar{q}'), \tag{2.17}$$

$$U_\varepsilon^{eff}(q,\bar{q}'') =$$

$$= \int \frac{dp}{(2\pi)^M} e^{\bar{\xi}\xi''} \exp i\varepsilon\left[\frac{p}{\varepsilon}(y-y'') - H^{eff}(p,q,\bar{q}'')\right] \tag{2.18}$$

and the effective Hamiltonian has the form

$$H^{eff}(p,q,\bar{q}'') = H_0(p,q,\bar{q}'') + \tilde{V}(q,\bar{q}'') + \tilde{V}_q(p,q,\bar{q}''), \tag{2.19}$$

$$H_0 = \frac{1}{2}(p_a + \pi_a)\, g^{ab}(y'')\,(p_b + \pi_b). \tag{2.20}$$

where Grassman variables $\xi'' = \bar{\bar{\xi}}''$ stand instead of $\partial_{\bar{\xi}}$ in π_a ; $\tilde{V}$ follows from V if we carry all operators $\partial_{\bar{\xi}}$ to the right and, then, change them by Grassman variables ξ'' ; and finally the expression

$$\tilde{V}_q = V_q(y'') + \frac{i}{2}\partial_a g^{ab}(y'')(p_b + \pi_b) -$$

$$- \frac{1}{2} g^{ab}(y'')\,\bar{\xi}\,\partial_a \Omega^+(y'')\,\partial_b \Omega(y'')\,\xi'' \tag{2.21}$$

takes into account the noncommutability of operators in the kinetic energy operator. If we restore the dependence on $\hbar$, then $V_q \sim \hbar^2$ and other terms in (2.21) $\sim \hbar$. This shows their connection with the operator ordering (see the review by Prokhorov 1982 in Phys.Elem.Part.Atom. Nucl. and references there). When ε tends to zero, we can replace $y - y''$ by $\dot{y}''\varepsilon$ with an accuracy of $O(\varepsilon^2)$.

To obtain the evolution operator kernel for a finite time, we must find the formula for iterations of infinitesimal kernels (2.17). By definition (2.8) we write

$$U_{2\varepsilon}(q,\bar{q}') = \int_K dy''\, d\bar{\xi}''\, d\xi''\, \mu''\, e^{-\bar{\xi}''\xi''}\, U_\varepsilon(q,\bar{q}'')\, U_\varepsilon(q'',\bar{q}') \tag{2.22}$$

Transformations (2.22) are cumbersome enough. However, we may easily control them if we take into account that their main sense is to carry to the right the operator $\hat{Q}$

being between two $U_{\varepsilon}^{\text{eff}}$ in (2.22) (see (2.17)). In this way, we would like to represent the final formula as (2.17), where $\varepsilon \to 2\varepsilon$ and

$$U_{2\varepsilon}^{\text{eff}}(q,\bar{q}') = \int dy_1 d\bar{\xi}_1 d\xi_1 e^{-\bar{\xi}_1\xi_1} U_{\varepsilon}^{\text{eff}}(q,\bar{q}_1) U_{\varepsilon}^{\text{eff}}(q_1,\bar{q}') \tag{2.23}$$

If into (2.22) we place expression (2.17) instead of the first U_{ε}, the integration is carried out over the right argument of the kernel Q and over the left argument of the second U_{ε} entering into (2.22). Let us calculate, at first, the action of $\hat{Q}$ from the left on the function Φ. We have in accordance with (2.15)

$$\hat{Q}\Phi(q) = \int_K dy' d\bar{\xi}' d\xi' e^{-\bar{\xi}'\xi'} Q(q,\bar{q}')\Phi(q') = \tag{2.24}$$

$$= \int_K dy' \sum_S \delta(y - \hat{s}y') \Phi(y', \bar{\xi}\hat{T}_s).$$

To take the integral over y', we rearrange $\sum_S$ and $\int_K$ and change integration variables $z = \hat{s}y'$. In the general case, $\hat{s}$ is not a linear transformation, i.e., $\hat{s}y = s(y)$ is a certain function. So,

$$\hat{Q}\Phi(q) = \sum_S \theta_K(\hat{s}^{-1}y) \frac{1}{J_s(y)} \Phi(\hat{s}^{-1}y, \bar{\xi}\hat{T}_s). \tag{2.25}$$

Here $J_s = \mathcal{D}(\hat{s}y)/\mathcal{D}(y)$ is Jacobian,

$$\theta_K(y) = \begin{cases} 1, & y \in K \\ 0, & y \bar{\in} K. \end{cases} \tag{2.26}$$

Apparently, the measure $dx = dx(y)$ is invariant under the group S hence from the equality $dx(\hat{s}y) = dx(y)$

it follows that

$$\mu(\hat{s}y) = \left(J_s(y)\right)^{-1}\mu(y). \tag{2.27}$$

Using the property (2.27) and $\hat{T}_s = \hat{T}^{+}_{s^{-1}}$ with (2.25) we can take the integrals over y'', $\bar{\xi}''$ and ξ'' in (2.22)

$$U_{2\varepsilon}(q,\bar{q}') = \int \frac{dy_1\, d\bar{\xi}_1\, d\xi_1}{(\mu\mu_1)^{1/2}}\, e^{-\bar{\xi}_1\xi_1}\, U^{eff}_{\varepsilon}(q,\bar{q}_1)\cdot$$

$$\cdot \sum_{S} \mu_1 \theta_K(\hat{s}y_1)\, U_{\varepsilon}(\hat{s}^* q_1, \bar{q}'). \tag{2.28}$$

By construction,

$$U_{\varepsilon}(\hat{s}^* q, \bar{q}') = U_{\varepsilon}(q,\bar{q}'). \tag{2.29}$$

Indeed, since the initial Hamiltonian (2.1) (or, which is the same (2.6)) is invariant under S^* $\left(Q(\hat{s}^* q) = Q(q)!\right)$, by definition (2.13) we conclude that (2.29) follows from the equality $\langle \hat{s}^* q | q' \rangle = \langle q | q' \rangle$ which must take place in accordance with definition (2.12) and the parity (2.11). Of course, we may directly prove the symmetry property of the unit operator kernel under S^* making calculations like (2.24), (2.25). The result of action of kernel (2.12) from the left on the function Φ coincides with the right-hand side of (2.25) if the factor $\left(J_s(y)\right)^{-1}$ is omitted. Thus, the function $\tilde{\Phi}(q) = \sum_{S^*} \theta_K(\hat{s}y)\,\Phi(\hat{s}^* q)$ is invariant under S^*. If Φ belongs to the Hilbert space of the theory, i.e., it is a linear combination of $\varphi_E(q)$, then $\Phi(\hat{s}^* q) = \Phi(q)$. So, the equality $\tilde{\Phi} = \Phi$ follows from

$$\sum_{S} \Theta_K(\hat{s}y) = 1. \tag{2.30}$$

Now we can see from (2.29) and (2.30) that the summation over S^* in (2.28) disappears. After substituting (2.17) into (2.28) we find the required expression for $U_{2\varepsilon}$ coinciding with (2.17) if $\varepsilon \to 2\varepsilon$, and $U_{2\varepsilon}^{eff}$ is defined by (2.28).

Now, clearly, all iterations of U_ε reduce to iterations of U_ε^{eff}. On the other hand, iterations of the kernel U_ε^{eff} give the standard finite-dimensional approximation of PI (Feynman and Hibbs 1965) for the theory with Hamiltonian (2.19). Thus we get for a finite time interval

$$U_t(q,\bar{q}') = \int \frac{dy''}{(\mu\mu'')^{1/2}} d\bar{\xi}'' d\xi'' e^{-\bar{\xi}''\xi''} U_t^{eff}(q,\bar{q}'') Q(q'',\bar{q}') . \tag{2.31}$$

where the kernel U_t^{eff} has the standard PI form

$$U_t^{eff}(q,\bar{q}'') = \int \prod_{t=0}^{t} \left(\frac{dp\,dy}{(2\pi)^M} d\bar{\xi} d\xi \right) e^{\gamma} e^{iS_{eff}} . \tag{2.32}$$

Here $\gamma = 1/2 \left(\bar{\xi}(t)\xi(t) + \bar{\xi}(0)\xi(0) \right)$ takes into account the standard initial conditions in PI containing Grassman variables (Faddeev and Slavnov 1980) $\bar{\xi}(t) = \bar{\xi}$ and $\xi(0) = \xi''$. Moreover, $y(t)=y$ and $y(0)=y''$ are initial conditions for boson variables, and

$$S_{eff} = \int_0^t d\tau \left[p\dot{y} + \frac{1}{2i}\left(\bar{\xi}\dot{\xi} - \dot{\bar{\xi}}\xi \right) - H^{eff} \right] . \tag{2.33}$$

Note that the measure μ (Jacobian) is not contained in the PI measure, but it stays as a factor both at initial and finite points of the transition amplitude. If we omit the dependence of the theory on Grassman degrees of freedom, the boson PI in curvilinear coordinates appears for which the recipe of construction was suggested in (Prokhorov 1984 and see also his review in 1982).

The main difficulty appearing in the PI derivation in curvilinear coordinates is that a physical region of values for new variables is reduced $\mathbb{R}^M \to K \subset \mathbb{R}^M$. Moreover eigenvalues of some canonical momenta become discrete (for example, the angular momentum see Sect.3), i.e., integration over them is replaced by summation.

We have got over these difficulties by using the analytical continuation of the unit operator kernel (2.12) in the PI derivation. We have found that the integration in PI can be carried out over the total phase space $\mathbb{R}^M \otimes \mathbb{R}^M$, however, after calculation of a transition amplitude we must symmetrize it with respect to the group S^* in accordance with (2.31).

3. Example: two-dimensional SUSY-oscillator

In this short section we give a simple illustration of general formulae of sect.2. Consider a two-dimensional SUSY-oscillator. Its Hamiltonian is

$$H = -\frac{1}{2}\Delta + \frac{1}{2}x_a^2 + \psi_a^+ \psi_a - 1 , \quad a = 1,2 . \qquad (3.1)$$

Let us study states of this oscillator with a fixed total angular momentum, i.e., with a total angular momentum of bosons and fermions. For this we introduce the generalized polar coordinates

$$x_1 = r\cos\theta , \quad x_2 = r\sin\theta , \quad \psi_a = e^{i\theta} \xi_a . \qquad (3.2)$$

In this case $\mu = r$, $g^{ab} = \mathrm{diag}(1, r^{-2})$, $P_1 = p_r = -i r^{-1/2} \partial_r \circ r^{1/2}$ is the Hermitian momentum operator for a radial degree of freedom, $P_\theta = p_\theta = -i\partial_\theta + \bar{\xi}_a^{+}\xi_a$ is the angular momentum of a boson (a total angular momentum is $-i\partial_\theta$), and $V_q = -1/8\, r^{-2}$. The Hamiltonian in coordinates (3.2) is defined by (2.6).

To get PI, one should find a group S^*. The structure of the group S for polar coordinates was discussed in Sect. 2 where it was found that $K = R^2 \setminus S$ is the strip $r > 0$, $\theta \in (0, 2\pi)$, so, $\hat{T}_s$ should only be calculated. Apparently, for $\hat{s}\theta = \theta + 2\pi n$, $\hat{s}r = r$ we have $\hat{T}_s = 1$ and for $\hat{s}\theta = \theta + \pi$, $\hat{s}r = -r$ the equality $\hat{T}_s \xi_a = -\xi_a$ takes place. Thus, the operator $\hat{Q}$ has the form

$$Q(q, \bar{q}') = Q_1(\theta, \theta')\,\delta(r - r')\exp \bar{\xi}_a \xi'_a + $$
$$+ Q_1(\theta, \theta' + \pi)\,\delta(r + r')\exp(-\bar{\xi}_a \xi'_a) \qquad (3.3)$$

where

$$Q_1(\theta, \theta') = \sum_{n=-\infty}^{\infty} \delta(\theta - \theta' + 2\pi n). \qquad (3.4)$$

The PI is obtained from (2.31).

4. A simple model with a gauge symmetry

Here we show the connection between the description of dynamical systems in curvilinear coordinates and gauge-invariant description of gauge models. Consider a mechanical model with SO(3)-gauge group and with the Lagrangian

$$L = \frac{1}{2}(D_t x)^2 + i\psi^+ D_t \psi - V(x, \psi^+, \psi), \qquad (4.1)$$

where $x \in R^3$, ψ is a three-dimensional complex Grassman vector, $D_t = \partial_t + y$, y is a real 3x3 antisymmetric matrix. Lagrangian (4.1) is invariant under gauge transformations

$$x \to \Omega x , \quad \psi \to \Omega \psi , \quad y \to \Omega y \Omega^T + \Omega \partial_t \Omega^T . \tag{4.2}$$

Here $\Omega = \Omega(t) \in SO(3)$, and we assume also that V is gauge-invariant,

Let us turn to the Hamiltonian formalism. Canonical momenta are $\pi = \partial L/\partial \dot{y} = 0$, $p = \partial L/\partial \dot{x} = D_t x$ and

$$\pi_{\psi^+} = \frac{\partial L}{\partial \psi^+} = 0 , \quad \pi_\psi = \frac{\partial L}{\partial \psi} = -i\psi^+ . \tag{4.3}$$

Obviously, $\pi = 0$ and (4.3) gives primary constraints. Note, (4.3) are the second-class constraints (Dirac 1965) which appear always since usual Lagrangians for fermions are linear in velocities. To eliminate the second-class constraints, we replace the Poisson brackets (a definition of the Poisson brackets for Grassman variables was given by Martin (1959)) by the Dirac brackets (Dirac 1965). We take ψ^+ and ψ as new canonical conjugate variables (Martin 1959), and their Dirac brackets are

$$\{\psi_a^+ , \psi_b\}_D = \{\psi_b , \psi_a^+\}_D = -i\delta_{ab} , \tag{4.4}$$

a,b=1,2,3. The momenta π_ψ and π_{ψ^+} are eliminated from the theory by using constraints (4.3). The Hamiltonian of the system has the form

$$H = \frac{1}{2} p_a^2 + V(x, \psi^+, \psi) - p y x - i \psi^+ y \psi . \tag{4.5}$$

Put $y_{ab} = y_c \varepsilon_{abc}$, ε_{abc} is a completely antisymmetric tensor, $\varepsilon_{123} = 1$ and also $\pi_{ab} = \pi_c \varepsilon_{cab}$. So, the secondary constraints (Dirac 1965) are

$$\{\pi_a, H\} = \varepsilon_{abc}\left(p_b x_c + i\psi_b^+ \psi_c\right) = G_a = 0. \tag{4.6}$$

Constraints (4.6) are the first-class constraints $\{G_a, G_b\}_D = -\varepsilon_{abc} G_c$, $\{G_a, H\}_D = \varepsilon_{abc}\, y_b G_c$ (Dirac 1965).

After quantisation, when all canonical variables are replaced by operators and $\{\,,\,\}_D \to -i[\,,\,]$ ($[\,,\,]$ is a commutator for bosons and an anticommutator for fermions), constraints (4.6) pick out the physical subspace of states

$$G_a |\Phi_{ph}\rangle = 0\,, \quad |\Phi_{ph}\rangle \in \mathcal{H}_{ph}\,. \tag{4.7}$$

(The constraints $\pi_a |\Phi_{ph}\rangle = 0$ are easily solved, and we shall not turn our attention to them below). To construct $\mathcal{H}_{ph}$, we use the representation of second quantisation $a_b = 1/\sqrt{2}\,(x_b + i p_b)$. By definition, the vacuum is $a_b|0\rangle = \psi_b|0\rangle = 0$. We find $\varepsilon_{abc}\, p_b x_c = -i\,\varepsilon_{abc}\, a_b^+ a_c$ and $|0\rangle \in \mathcal{H}_{ph}$. The operators of the constraints generate $SO(3)$-rotations of the vectors a^+ and ψ^+, hence, $|\Phi_{ph}\rangle$ may be obtained by action on $|0\rangle$ of all combinations of a^+ and ψ^+ which are invariant under $SO(3)$-transformations. The basis for these polynomials of a^+ and ψ^+ is defined by invariant tensors of the group $SO(3)$ δ_{ab} and ε_{abc} (Barut and Raczka 1977), i.e., we may obtain any acting on $|0\rangle$ by the operators

$$b_1^+ = a_a^+ a_a^+\,, \quad b_2^+ = \varepsilon_{abc}\, \psi_a^+ \psi_b^+ a_c^+\,, \tag{4.8a}$$

$$f_1^+ = a_b^+ \psi_b^+\,, \quad f_2^+ = \varepsilon_{abc}\, \psi_a^+ \psi_b^+ \psi_c^+\,. \tag{4.8b}$$

The operators (4.8a) are "boson", i.e., they commute and the operators (4.8b) correspond to physical excitations of a fermion sector, i.e., they anticommute.

Now we return to the PI derivation. Christ and Lee (1980) and Prokhorov (1982) have shown for the model (4.1), but without fermions, that the elimination of unphysical variables and subsequent quantization lead to the results contradicting the Dirac scheme. The main point is as follows. Put, for example, $V=\frac{1}{2}x^2$ (fermions are absent), then the basis in $\mathcal{H}_{ph}$ is $b_1^{+n}|0\rangle$, n=0,1,... (Prokhorov and Shabanov 1989), i.e., the oscillator spectrum is $E_n=2n+3/2$. Now we eliminate unphysical variables before quantization. Since the constraints $G_a=l_a=\varepsilon_{abc}p_b x_c=0$ are projections of the angular momentum of a boson, we conclude that angles of the spherical coordinate system $x\rightarrow(r,\theta,\varphi)$ are unphysical variables (their canonical momenta are $p_\varphi=-l_3=0$, $p_\theta=\sin\varphi\, l_1-\cos\varphi\, l_2=0$). So, the classical physical Hamiltonian depending on physical variables r and $p_r=p_a x_a/r$ is $1/2\,(p_r^2+r^2)$. It coincides with the Hamiltonian of a one-dimensional oscillator, the quantization of which gives the spectrum $E_n=n+1/2$.

On the other hand, as it has been noted in Sect.1, a standard recipe of the PI construction corresponds to a quantum theory obtained from an initial classical one just by eliminating unphysical degrees of freedom before quantization. From this point of view it is interesting to find a PI form which corresponds to the Dirac quantization scheme. With this purpose we, at first, quantize the theory, then we eliminate unphysical variables and construct the quantum Hamiltonian in $\mathcal{H}_{ph}$. At last, using it we derive a PI for the evolution operator kernel in $\mathcal{H}_{ph}$.

Note, to eliminate unphysical variables in a quantum theory, one should introduce curvilinear coordinates. Indeed, if we define new variables so that some of them get shifts (unphysical variables) and others do not change under gauge transformations (physical variables) constraints become diagonal, i.e., they are linear combinations of momentum operators conjugated to unphysical variables. It then follows that constraints are generators of gauge transformations, and momentum operators are generators of translations. However, if gauge transformations are isotopic rotations, we cannot diagonalize constraints without introducing curvilinear coordinates.

On this basis, we define new variables

$$
\begin{aligned} x &= U\rho \\ \psi &= U\xi \end{aligned}
\qquad
U = \begin{pmatrix} \sin\theta\cos\varphi & -\sin\varphi & -\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \cos\varphi & \cos\theta\sin\varphi \\ \cos\theta & 0 & \sin\theta \end{pmatrix} \in SO(3). \tag{4.9}
$$

where $\rho = (r, 0, 0)$, $\xi = (\xi_a)$, $a = 1,2,3$. Apparently, θ and φ get shifts under gauge transformations and r, ξ_1 do not change. The quantum Hamiltonian in the new variables has the form (2.6) if $y_a = (r, \theta, \varphi)$, $\pi_r = 0$, $\pi_\theta = L_2$, $\pi_\varphi = \sin\theta L_3 + \cos\theta L_1$, $L_a = i\varepsilon_{abc}\xi_b^+\xi_c$ ($\xi_c^+ = \partial/\partial\xi_c$), $V_q = 0$, $\mu = r^2$ and the metric tensor is $g^{ab} = \mathrm{diag}\,(1, r^{-2}, (r\sin\theta)^{-2})$.

Nondifficult calculations with the use of (2.5) show that the system of equations (4.7) in the coordinate representation is equivalent to

$$\partial_\varphi \Phi_{ph} = \partial_\theta \Phi_{ph} = 0\,, \quad L_1 \Phi_{ph} = 0. \qquad (4.10)$$

The form of the third equation in (4.10) may easily be understood if we note that using gauge transformations we may always reduce φ, θ to zeros in $x = U\rho$, however, the vector ρ has the stationary subgroup SO(2) with the generator ε_{1ab}, being a subgroup of the gauge group SO(3). These remaining gauge transformations do not change ρ, but they change ξ, hence, physical fermion states should be invariant under it, i.e., $L_1 \Phi_{ph} = 0$. So, the Hamiltonian in $\mathcal{H}_{ph}$ is

$$H_{ph} = -\frac{1}{2r}\partial_r^2 \circ r + \frac{1}{2r}\left(L_2^2 + L_3^2\right) + V(\rho, \xi^+, \xi). \quad (4.11)$$

A gauge symmetry in a pure fermion system was studied in (Shabanov 1989). Using gauge transformations we cannot decrease the number of Grassman variables. Nevertheless, in a classical theory the constraint of the type $L_1 = 0$ leads to that the time evolution of one fermion degree of freedom, for example, $\xi_2(t)$ is determined by the time evolution of the other, i.e., $\xi_3(t)$. In a quantum theory the constraint $L_1 \Phi_{ph} = 0$ is equivalent to the requirement of $\mathbb{Z}_2$-invariance: $\xi_{2,3} \to -\xi_{2,3}$ for Φ_{ph} (the latter was interpreted as a phase space reduction for a fermi-system (Shabanov 1989). Thus, we find

$$\Phi_{ph}(r, \hat{T}_1 \xi) = \Phi_{ph}(r, \xi), \qquad (4.12)$$

where $\hat{T}_1 = \hat{S}_1 = \mathrm{diag}(1,-1,-1) \in SO(3)$. Note that the only invariant of SO(2)-subgroup generating L_1 and of $Z_2 = (1, \hat{S}_1)$ is $\bar{\mathfrak{Z}}_2 \bar{\mathfrak{Z}}_3$.

Let us define now the scalar product in $\mathcal{H}_{ph}$. Since (4.9) is the change of variables, we make it like (2.8). However, now θ and φ are unphysical variables, hence, here K is a physical configuration space of r. The symmetry group of the change of variables (4.9) contains the group $S = Z_2$: $\rho \to \pm \rho$ acting on physical variables; so, $K = R \setminus S$ is a semiaxis $r > 0$.

To find the full group S^* acting in a physical superspace, one should determine all nontrivial transformations $\hat{S} \in SO(3)$ such as $\hat{S}\rho = \pm\rho$. They are $\hat{S}_1 = \mathrm{diag}(1,-1,-1)$, $\hat{S}_2 = \mathrm{diag}(-1,1,-1)$, $\hat{S}_3 = \mathrm{diag}(-1,-1,1)$. Since $\hat{S}_a^2 = 1$ (no summation over a), we see that the following transformations of θ, φ correspond to $\hat{S}_a$ ($x = U\rho = U\hat{S}_a\hat{S}_a\rho$)

$\hat{S}_1 : \theta \to -\theta,\ \varphi \to \varphi + \pi,\ r \to r$; $\hat{S}_2 : \theta \to \theta + \pi,\ \varphi \to \varphi,\ r \to -r$;

$\hat{S}_3 : \theta \to -\theta + \pi,\ \varphi \to \varphi + \pi,\ r \to -r$ (the full symmetry group of (4.9) is obtained by adding to $\hat{S}_a$ the transformation $\theta \to \theta + 2\pi n,\ \varphi \to \varphi + 2\pi m,\ m, n \in Z$). Thus, two different transformations of $\mathfrak{Z}$ correspond to every Z_2-transformation $r \to \pm r$, i.e., $S^* \sim Z_2 \otimes Z_2$ in $\mathcal{H}_{ph}$. Moreover, $\hat{T}_a = \hat{S}$ since both the representations of bosons and of fermions coincide.

All physical states must be invariant under S^* because S^* is a subgroup of the gauge group SO(3). We may also prove it in another way. Consider an eigenfunction Ψ_E of the full Hamiltonian (without constraints) $1/2\, p_a^2 + V(x, \psi^*, \psi)$ in the coordinates (4.9). In accordance with (2.10) Ψ_E

should be invariant under the above-described discrete symmetry group for the change of variables (4.9). The dependence of φ_E on θ and φ is determined by the spherical functions $Y_{\ell m}(\theta,\varphi)$ because $V(x,\psi^+,\psi) = V(\rho,\xi^+,\xi)$ (V is gauge-invariant), i.e., $\varphi_E = \sum R^E_{\ell m} Y_{\ell m}$. So, R^E_{00} are invariant under S^* but they form a basis in $\mathcal{H}_{ph}$, so, any Φ is invariant under S^* if $\Phi \in \mathcal{H}_{ph}$. Note, (4.12) is fulfilled automatically for R^E_{00} . Thus, we write the unit operator kernel in $\mathcal{H}_{ph}$ by analogy with (2.12)

$$\langle q|q'\rangle_{ph} = \frac{1}{2rr'}\Big[\delta(r-r')\left(e^{\bar\xi\xi'} + e^{\bar\xi\hat s_1\xi'}\right) - $$

$$ - \delta(r+r')\left(e^{\bar\xi\hat s_2\xi'} + e^{\bar\xi\hat s_3\xi'}\right); \qquad (4.13)$$

the factor 1/2 in (4.13) follows from the equality

$$\int_0^\infty dr\, r^2 \int d\bar\xi d\xi\, e^{-\bar\xi\xi} \langle q'|q\rangle_{ph}\, \Phi(q) \equiv \Phi(q'), \Phi\in\mathcal{H}_{ph}, \qquad (4.13a)$$

$r\in\mathbb{R}$ and $r'>0$ in (4.13) . Of course, (4.13) can be obtained from (2.12) by averaging over θ and φ since θ and φ are unphysical variables.

The PI derivation for U_t^{ph} coincides with (2.13)-(2.31) if we replace H by H_{ph} (see (4.11)) and $\langle q|q'\rangle$ by (4.13). A final expression has the form (2.31) where $\hat Q$ is given by the expression in brackets of (4.13) if the sign of $\delta(r+r')$ is changed, and

$$H^{eff} = \frac{1}{2}p^2 + \tilde V(r,\bar\xi,\xi) - $$

$$ - \frac{1}{2r^2}\Big[\left(\bar\xi\varepsilon_2\xi\right)^2 + \left(\bar\xi\varepsilon_3\xi\right)^2 + \bar\xi\left(\varepsilon_2^2+\varepsilon_3^2\right)\xi\Big]. \qquad (4.14)$$

Here $\tilde V$ is defined as in (2.19), matrix elements of the matrix ε_a are ε_{abc} , and p is a momentum canonically conjugated to r .

The main point we would like to note is that the PI contains the operator $\hat{Q}$ symmetrizing the transition amplitude over the group S^*. It was shown (Prokhorov and Shabanov 1989 and Shabanov 1989) that $\hat{Q}$ appears for gauge systems when a physical-phase-space reduction takes place. On the other hand, by construction the kernel $U_t^{ph}(q,\bar{q}')$ $(q=(r,\bar{\xi}))$ is invariant under S^*. Then, we state that q and $\bar{q}'$ in it can be replaced by Q and $\bar{Q}'$ respectively ($Q=(x,\bar{\psi})$), and the result does not depend on the unphysical variables θ and φ, i.e.,

$$U_t^{ph}(q,\bar{q}') = U_t^{ph}(Q,\bar{Q}'). \tag{4.15}$$

In other words, there exists one-to-one gauge-invariant analytical continuation of the kernel U_t^{ph} to the total configuration space of the system. To prove this, we note that any polynomial of q invariant under S^* depends only on degrees of the following quantities

$$\delta_{ab}\rho_a\rho_b,\ \varepsilon_{abc}\rho_a\bar{\xi}_b\bar{\xi}_c,\ \delta_{ab}\rho_a\bar{\xi}_c,\ \varepsilon_{abc}\bar{\xi}_a\bar{\xi}_b\bar{\xi}_c, \tag{4.16}$$

where $\rho=(r,0,0)$. We may check this directly. Since δ_{ab} and ε_{abc} are invariant tensors of SO(3) (Barut and Raczka 1977), we conclude that quantities (4.16) are equal respectively to

$$x_a^2,\ \varepsilon_{abc}x_a\bar{\psi}_b\bar{\psi}_c,\ x_a\bar{\psi}_a,\ \varepsilon_{abc}\bar{\psi}_a\bar{\psi}_b\bar{\psi}_c \tag{4.17}$$

in accordance with (4.9). Any gauge-invariant polynomial can be formed from (4.17) (compare with (4.8)!). Moreover, an analytical function of q being invariant under the residual discrete gauge group S^* has the unique analytical gauge-invariant continuation to the space of Q because polynomials form a dense set in the space of analytical functions. So, (4.15) is proved. Note, Q contains six degrees of freedom and a gauge arbitrariness has three parameters, nevertheless, the system has four physical degrees of freedom (see (4.8) or (4.17)). This happens because two first constraints in (4.I0) pick out already the full $\mathcal{H}_{ph}$, as it has been shown above.

Thus, the explicit gauge-invariant form of PI for a transition amplitude can be obtained if we take into consideration a curvilinear character of physical variables and their phase space reduction. These both main moments are usually ignored in the standard PI derivation for gauge theories.

5. The case of an arbitrary group and generalized Shevalley theorem

Here we attempt to reveal a general mathematical origin of equality (4.15). It turns out that the statement like the Shevalley theorem (Partasarathy, Ranga Rao and Varadarajan 1967) makes a basis of equality (4.15) in the general case.

Consider the model with the Lagrangian

$$L = \frac{1}{2} \mathrm{Tr} \left(D_t x \right)^2 + i \, \mathrm{Tr} \, \psi^+ D_t \psi - V(x, \psi^+, \psi). \tag{5.1}$$

Here $D_t = \partial_t + [y, \ ]$; variables x, y, ψ^+, ψ are elements of a Lie algebra X of an arbitrary compact gauge group G, i.e., $x = x_i \lambda_i$ (analogously for y), $\psi = \lambda_i \psi_i$ (analogously for ψ^+), x_i, y_i are real, ψ_i, ψ_i^+ are complex Grassman variables, where λ_i is an orthonormal basis in X : $\mathrm{Tr}\,\lambda_i\lambda_j=\delta_{ij}$, $[\lambda_i,\lambda_j]=f_{ijk}\lambda_k$, f_{ijk} are total antisymmetric structural constants and i,j,k=1,2,... N=dim X . Lagrangian (5.1) is invariant under gauge transformations

$$x\to\Omega x\Omega^{-1},\quad \psi\to\Omega\psi\Omega^{-1},\quad \psi^+\to\Omega\psi^+\Omega^{-1},$$
$$y\to\Omega y\Omega^{-1}+\Omega\partial_t\Omega^{-1}, \tag{5.2}$$

where $\Omega=\Omega(t)\in G$, and we assume that V is invariant under (5.2).

Canonical momenta are $\pi=\partial L/\partial\dot{y}=0$, $p=\partial L/\partial\dot{x}=D_t x$. We describe Grassman degrees of freedom as in Sect.4, i.e., we introduce the Dirac brackets (4.4). So, the Hamiltonian is

$$H=\frac{1}{2}\mathrm{Tr}\,p^2+V(x,\psi^+,\psi)+y_i G_i\ , \tag{5.3}$$

where $G_i=-\{\pi_i,H\}=f_{ijk}\left(p_j x_k+i\psi_j^+\psi_k\right)=0$ are the secondary constraints. As one may check, they are the first-class constraints. After a quantisation of the theory G_i pick out the physical subspace $\mathcal{H}_{ph}$

$$G_i\,|\Phi_{ph}\rangle=\pi_i\,|\Phi_{ph}\rangle=0. \tag{5.4}$$

Our purpose is a PI construction for the evolution operator kernel of physical degrees of freedom. In accordance with the recipe suggested in Sect.4 it is necessary to introduce new curvilinear coordinates in which the constraints (5.4) are diagonalized, then, to write the Hamiltonian in $\mathcal{H}_{ph}$ and to find $\langle q|q'\rangle_{ph}$. At last, $U_t^{ph}(q,\bar{q}')$ may be restored by the method of Sect.2.

Determine new variables as follows (Prokhorov and Shabanov 1989)

$$x = e^{z} h e^{-z} , \quad \psi = e^{z} \xi e^{-z} , \tag{5.5}$$

where $h \in H$ is a Cartan subalgebra in X (Helgason 1984) $z \in X \ominus H$. In accordance with (5.2) z are unphysical variables. Note that like (4.9) h has a stationary subgroup in G, the Cartan subgroup, i.e., maximal Abelian subgroup in G (Helgason 1984). We denote $h = h_\alpha \lambda_\alpha$ (α =1,2,...,l=dim H), $z = z_a \lambda_a$ (a=l+1,l+2,...,N). The metric in the new variables has the block-diagonal form (Prokhorov and Shabanov 1989 and Shabanov 1989), g^{ij} = $= (\delta_{\alpha\beta}, [(F^T\omega^T\omega F)^{-1}]_{ab})$, where $\omega_{ab} = h_\alpha f_{\alpha ab}$,

$$F_{ia} = \mathrm{Tr}\,(\lambda_i e^{z} \partial_a e^{-z}), \quad \partial_a = \partial/\partial z_a .$$

The measure is $dx = \det \omega F\, dh\, dz \equiv \mu^2(h) \tilde{\mu}(z)\, dh\, dz$. The measure in a physical configurational space may be calculated explicitly (Helgason 1984)

$$\mu(h) = \prod_{\alpha>0} (h, \alpha) = (\det \omega)^{1/2} , \tag{5.6}$$

where α are positive roots of X, $(h,\alpha) = h_\beta \alpha_\beta$.

To find the Hamiltonian in $\mathcal{H}_{ph}$, we calculate the constraints in new variables. Since z_a are translated under gauge transformations generated by constraints,

N-1 constraints G_j are linear combinations of $i\partial_a$ (compare with (4.10)). The remaining gauge arbitrariness is connected with the Abelian l-dimensional Cartar group which does not change the physical boson variables h_α, but it changes the fermion variables ξ . So, other ℓ constraints must represent the equalities to zero of generators of Abelian transformations of fermion variables (like (4.10)). Thus, equations (5.4) are equivalent to

$$-i\partial_a \Phi_{ph} = 0\,, \quad L_\alpha \Phi_{ph} \equiv f_{\alpha a b}\, \xi_a^+ \xi_b \Phi_{ph} = 0\,. \tag{5.7}$$

where $\xi_b = \partial/\partial\bar{\xi}_b$. Note, $f_{\alpha\beta i} = 0$, hence, $[L_\alpha, L_\beta] = 0$, i.e., L_α are generators of the Cartan subgroup.

In the quantum Hamiltonian (5.3) rewritten in the form (2.6) for coordinates (5.5) we carry ∂_a and L_α to the right and use (5.7) in $\mathcal{H}_{ph}$, then we get the quantum Hamiltonian in $\mathcal{H}_{ph}$. To simplify calculations, note that in new variables diagonalizing constraints, unphysical variables become cyclic (Dirac 1965), i.e., H_{ph} does not depend on them, So, we may only keep an eye on terms containing h and ξ^+, ξ . We have

$$H_{ph} = -\frac{1}{2\mu}\partial_\alpha \circ \mu \partial_\alpha + \frac{1}{2} L_a (\omega^T\omega)^{-1}_{ab} L_b + V(h, \xi^+, \xi)\,. \tag{5.8}$$

Here $\partial_\alpha = \partial/\partial h_\alpha$, $L_a = -i f_{aij}\, \xi_i^+ \xi_j$.

To find S and S^* , we introduce the Cartan-Weyl basis in X (Barut and Raczka 1977)

$$[e_\alpha, e_{-\alpha}] = \alpha\,, \quad [h, e_\alpha] = (\alpha, h)\, e_\alpha\,,$$

$$[e_\alpha, e_\beta] = N_{\alpha\beta}\, e_{\alpha+\beta}\,, \tag{5.9}$$

where $\alpha > 0$ are positive roots in X , e_α are corresponding root vectors, $h, \alpha \in H$, $N_{\alpha\beta}$ are numbers, $N_{\alpha\beta} \neq 0$ if $\alpha + \beta$ is a root in X . We define also an operator of the adjoint representation $\mathrm{ad}\,x\,y = [x, y]$ for all $x, y \in X$. Any element $h \in H$ can be represented as $h = h_\omega \omega$, where ω are simple roots of X , hence, the set $\{\omega, e_\alpha, e_{-\alpha}\}$ gives a basis in X (Cartan-Weyl basis). However, it is more convenient for us to use the orthogonal basis in $X \ominus H$

$$S_\alpha = \frac{1}{\sqrt{2}}(e_\alpha - e_{-\alpha}), \quad C_\alpha = \frac{1}{\sqrt{2}}(e_\alpha + e_{-\alpha}) \tag{5.10}$$

(the orthogonality is understood with respect to the scalar product in X : $(x, y) = \mathrm{Tr}\,\mathrm{ad}\,x\,\mathrm{ad}\,y$, for compact groups one may normalize so that $(x, y) = \mathrm{Tr}\,x\,y$ in a matrix representation (Barut and Raczka 1977)).

It is well-known that there exists a subgroup of G in H called the Weyl group W which is a group of reflections and rearrangements in the root system. The group W is defined by combinations of the operators (Zhelobenko 1970)

$$\hat{S}^s_\omega = \exp\frac{\pi}{(\omega,\omega)^{1/2}}\mathrm{ad}\,S_\omega, \quad \hat{S}^c_\omega = \exp\frac{i\pi}{(\omega,\omega)^{1/2}}\mathrm{ad}\,C_\omega, \tag{5.11}$$

i.e., any $\hat{s} \in W$ is a combination of $\hat{S}^s_\omega$ or a combination of $\hat{S}^c_\omega$ (ω are simple roots). We may check that $\hat{S}^{c,s}_\omega \omega = -\omega$, i.e., (5.11) are reflections of all simple roots, and they give two equivalent representations of W in H . In accordance with the definition of $\mathrm{ad}\,x$ and (5.5) we conclude that actions of W in H induce transformations in $X \ominus H$, but the left-hand sides of (5.5) are invariant. Hence, transformations (5.11) are generators of a searched discrete group S. Indeed, the change of boson variables (5.5) exists if $h \in K^+ = H \setminus W$ (Helgason

1984) where K^+ is the Weyl camera (physical configurational space (Prokhorov and Shabanov 1989)). In other words, S cannot contain generators except (5.11), otherwise $H\backslash S\subset K^+$ that is wrong. Note, $\hat{S}^c_\omega$ and $\hat{S}^s_\omega$ coincide in H but their actions are different for Grassman elements ξ.

We call the discrete group defined by (5.11) in space $H_g = X_g \otimes H$ $(\xi\in X_g, h\in H)$ the generalized Weyl group W^*. Since boson and fermion representations are identical, $S^* = W^*$. Certainly, to get a full symmetry group of the change of variables we must add to W^* transformations of z inducing shifts $e^z = e^{z+a}$ like $2\pi n$-shifts of θ, φ in (4.9). Using considerations like above-suggested ones for the derivation of (4.13) (we denote N_* the number of different elements of S^* such as $\hat{S}^* q = (h, \bar{\xi} T_s^+)$, $N_* = 2$ in (4.13)) one may write

$$\langle q|q'\rangle_{ph} = \frac{N_*^{-1}}{\mu(h)\mu(h')} \sum_{W^*=S^*} (-1)^{P_s} \delta(q, \hat{s}^* \bar{q}') . \quad (5.12)$$

where $q \in H_g$, $q' \in X_g \otimes K^+$ and $\mu(\hat{s}h) = (-1)^{P_s}\mu(h)$, $\hat{s}\in W$, $P_s = 0$ if $\hat{s}$ is rearrangement of roots without reflections, $P_s = 1$ for $\hat{s}$ including non-even numbers of reflections of roots. Equality (5.12) means that all physical states from $\mathcal{H}_{ph}$ are invariant under the residual discrete gauge group W^*. Moreover, the requirement of the W^*-invariance gives automatically solutions of constraints (5.7) in the Grassman sector. To prove the latter statement, note that $\hat{S}^c_\omega \hat{S}^s_\omega = 1$ in H, however, in X these operators must be elements exp $\mathrm{ad}\,\lambda$, $\lambda \in H$ which are equal to 1 in H. On the other hand, one may check by direct calculations in basis (5.9) that operators (5.11) are reflections with rearrangements in the real basis of $X \ominus H$ (ic_α, s_α),

$\alpha>0$ (Zhelobenko 1970). Then $\exp\, ad\,\lambda$ are also combinations of rearrangements and reflections. Using this we can find explicit forms of λ. Indeed, $\exp(ad\lambda)\, i S_\alpha$ is only $\pm i S_\alpha$ or $\pm C_\alpha$ as it follows from (5.9) and $\lambda \in H$. So, λ can take values $i\pi\alpha(\alpha,\alpha)^{-1}$, α runs over all positive roots, i.e., W^* contains the operators $\hat{S}_\alpha = \exp i\pi(\alpha,\alpha)^{-1} ad\,\alpha$. Further, transformations from the Cartan subgroup $\exp ad\,\chi$ $(\chi \in H)$ generated by L_α in (5.7) in basis (5.10) ($\bar{\xi} = \bar{\xi}_\omega \omega + \bar{\xi}^c_\alpha C_\alpha + \bar{\xi}^s_\alpha S_\alpha$) are rotations of two-dimensional Grassman vectors ($\bar{\xi}^c_\alpha, \bar{\xi}^s_\alpha$) through the angle (χ, α) for every $\alpha>0$. Invariants of these rotations are $\bar{\xi}^c_\alpha \bar{\xi}^s_\alpha$ (α is fixed), but $\hat{S}_\alpha C_\alpha = -C_\alpha$, $\hat{S}_\alpha S_\alpha = -S_\alpha$, hence, $\bar{\xi}^c_\alpha \bar{\xi}^s_\alpha$ are also invariant under W^*, i.e., W^*-invariant functions give solutions of (5.7) in the Grassman sector.

Using the technique of Sect.2, we restore the form of $U_t^{ph}(q,\bar{q}')$ for Hamiltonian (5.8) and kernel (5.12). It has the form (2.31) where

$$Q(q,\bar{q}') = \sum_{W^*=S^*} N_*^{-1}\, \delta(q, \hat{s}^* \bar{q}'), \tag{5.13}$$

$$H^{eff} = \frac{1}{2} p_\alpha^2 + \tilde{V}(h,\bar{\xi},\xi) + \frac{1}{2} L_a (\omega^T\omega)^{-1}_{ab} L_b + V_q \tag{5.14}$$

and $L_a = -i f_{aij} \bar{\xi}_i \xi_j$, $V_q = -\bar{\xi}_i \xi_n f_{aik} f_{bnk} (\omega^T\omega)^{-1}_{ab}$. The constructed kernel U_t^{ph} turns out to be invariant under W^* like (4.15) ($\hat{Q}$ symmetrizes it in W^*). If fermions are absent, $W^* = W$. In this case the Shevalley theorem (Zhlobenko 1970) gives: every analytical function in H being invariant under W has the unique analytical gauge-invariant continuation to X. So, $U_t^{ph}(h,h') = U_t^{ph}(x,x')$. Examples of the construction of gauge-invariant wave functions were given in (Shabanov 1989) and gauge-invariant forms of PI in total (i.e., including also unphysical degrees of freedom) configurational and phase spaces were presented in (Shabanov 1989, preprints JINR).

For the present system there is an analogous statement, we call the generalized Shevalley theorem: every analytical function in H_g being invariant under W^* has the unique analytical gauge-invariant continuation to $X \otimes X_g$ ($Q \in X \otimes X_g$ if $x \in X$, $\bar{\psi} \in X_g$). Consider an oscillator in

(5.3) $V(h, \xi^+, \xi) = 1/2 \, \mathrm{Tr}\, h^2 + \mathrm{Tr}\, \xi^+ \xi - N/2$.

Its wave functions are $p_E(q) \exp(-1/2 \, \mathrm{Tr}\, h^2)$, where $p_E(q)$ are polynomials invariant under W^*. Since H_{ph} is Hermitian, $p_E(q)$ form a basis in the space of all W^*-invariant polynomials in H_g. On the other hand, we may solve the quantum problem in the total Hilbert space, i.e., in the space of functions in $X \otimes X_g$. Then, eigenfunctions of the oscillator are $\tilde{P}_E(Q) \exp(-1/2 \, \mathrm{Tr}\, x^2)$, moreover, $\mathcal{H}_{ph}$ is formed by gauge-invariant polynomials from $\tilde{P}_E(Q)$ which give a basis in the space of all gauge-invariant polynomials (the total Hamiltonian is also Hermitian). Because V is gauge-invariant, we may write in coordinates

(5.5) $\tilde{P}_E(Q) = \sum_n P_E^n(q)\, Y_n(z)$, where $Y_n(z)$ are eigenfunctions of the Laplace-Beltrami operator on a gauge group orbit formed by values of z when h is fixed.

Clearly, $p_E(q) = P_E^0(q)$ (Y_0 = const). Then, in $\mathcal{H}_{ph}$ $\tilde{P}(Q(q)) = \tilde{P}_E(q) = P_E^0(q) = p_E(q)$ because of the gauge invariance, i.e., between polynomials P_E^0 and p_E there exists a one-to-one correspondence, hence, it exists between $\tilde{P}_E(Q) \in \mathcal{H}_{ph}$ and $p_E(q)$. Since polynomials form a dense set in the space of analytical functions, we arrive of the statement of the generalized Shevaley theorem. Thus, formula (4.15) takes place in the general case.

Note a simple consequence. Every polynomial in X_q being invariant under W^* is gauge-invariant, i.e., a gauge symmetry in a pure fermion sector of a theory is equivalent to the discrete symmetry with respect to the generalized Weyl group W^*.

6. Conclusion

Thus, we have seen that the main points of PI derivation corresponding uniquely to the Dirac quantization scheme (i.e., to an explicit gauge- invariant description) are the curvilinear character of physical variables and reduction of both physical configuration and phase spaces. The latter, as it has been shown, is connected with the invariance of PI under residual discrete gauge transformations (the operator $\hat{Q}$ in the expression of U_t^{ph}), and this guarantees an explicit gauge-invariance of PI (the generalized Shevalley theorem).

The recipe may be generalized to any theory with the first-class constraints (i.e., to any gauge theory). Let independent constraints be φ_a which generate gauge transformation (Pyatov and Razumov 1989). The struoture of gauge groups orbits in the total configurational space is not always known, therefore physical variables are picked out with the help of supplementary conditions $\chi_a(x) = 0$. To get the correspondence to the Dirac scheme, one has to do as follows. Let the gauge transformation law be $x \to u x$, $\psi \to T_u \psi$, where $u \in G$, G is a gauge group, T_u is a representation of G . Then after quantization we change variables in a quantum Hamiltonian $(x,\psi) \to (\theta, y, \xi)$, where $x = u(\theta) y$, $\psi = T_u(\theta)\xi$, and y satisfies supplementary conditions $\chi_a(y) = 0$. In this case constraints φ_a become linear combinations of derivatives $\partial/\partial\theta_a$ since θ_a shift under gauge transformations, i.e., θ_a are unphysical variables. Further, one should define a quantum Hamiltonian in the physical subspace, i.e., in the space of analytical functions of y , and find a unit operator kernel in the physical subspace of states, i.e., determine the measure (Jacobian) and the group S^* (the group S may be found from conditions $\chi_a(\hat{s}y) = 0$, $\hat{s} \in G$, where $\hat{s}$ are all residual discrete gauge transformations keeping conditions $\chi_a = 0$). At last, U_t^{ph} can be restored in accordance with the above-suggested recipe. The effective-action form and S^* depend on the χ_a form. However, changing χ_a by χ_a' is equivalent to a passage to other curvilinear coordinates in quantum theory unbreaking the diagonality of quantum constraints ($x = uy = u'y'$, $\chi_a'(y') \equiv 0$ and $\varphi_a \sim \partial/\partial\theta_a \sim \partial/\partial\theta_a'$), hence it is a passage to a new basis in $\mathcal{H}_{ph}$. So, the change of χ_a does not influence the form of the function U_t^{ph} which depends only on gauge-invariant quantities (see (4.15)). Change of χ_a is the change in form of an entry of gauge-invariant quantities (compare (4.16) with (4.17), in this case $\chi_a = 0$ are $x_2 = x_3 = 0$).

Needless to say, quantum theories determined by the elimination of unphysical variables with subsequent quantization and in accordance with the Dirac scheme are free from internal contradictions whereas they can be different. Therefore we may consider them as two quantum versions of one classical theory. However, note that in the case of a quantum gauge field theory we should observe an explicit Lorentz invariance in choosing physical variables. The latter is known to require the introduction of unphysical variables to a theory (Dirac 1967). Otherwise, we cannot impose supplementary conditions on operators since contraditions with commuting relations appear (Dirac 1965, 1967). Therefore the Dirac scheme turns out to be more preferable for formulation of a theory in the total Hilbert space as being free from these contradictions. Thus, PI should be defined according to the Dirac quantization scheme.

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