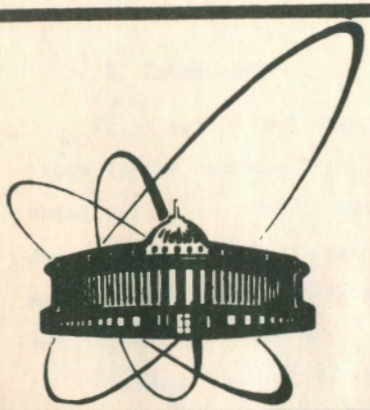

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PHASE SPACE REDUCTION AND THE CHOICE OF PHYSICAL VARIABLES IN GAUGE THEORIES

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I. Introduction

It is well-known that a gauge symmetry in a dynamical system leads to the appearance of first-class constraints [1] for canonical variables. The latter means that a system of that sort does not move in the total phase space (PS), but only on a submanifold in it. So, these systems have "superfluous" or unphysical degrees of freedom.

To describe systems with a gauge group uniquely, one should in a way choose physical variables. This can be done by gauge fixing or passing to gauge-invariant variables. In this way, we pass from the description of dynamics in the total PS to its description in the physical PS. It is usually assumed that in gauge theories the physical PS is an even-dimensional Euclidean space defining an integration region in HPI [2] for systems with constraints. However, it has been shown in [3] for simple models with a gauge group that the physical PS may differ from an ordinary plane and can be a cone. This PS reduction leads to the HPI modification [4,5] . The general cause, for the reduction of the "volume" of the physical PS, is that after the elimination of all unphysical variables in a theory there remainsthe discrete gauge group. This residual discrete gauge group (RDGG) acts in both configuration and phase spaces of physical variables identifying some points of them [4]. To remain into the framework of the Dirac quantization method for systems with constraints, it is necessary to take into consideration the PS reduction for the HPI construction since in this case there exists a one-valued analytic continuation of the operator evolution kernel to the total configuration space (including also unphysical variables) which is explicitly gauge-invariant [5,6]. Note also that the PS reduction takes place for the Higgs field [3] and Yang-Mills fields [7] .

The present paper is devoted to the above problems. In particular, the question about a correct form of HPI for the Yang – – Mills quantum mechanics [8,9] will be solved. The Yang-Mills field theory becomes the Yang-Mills mechanics if fields are required to be homogeneous in space. This model is remarkable since it contains many qualitative peculiarities of dynamics of a non-abelian gauge field theory. That is why it has been used for studying different aspects of the Yang-Mills field dynamics: stochastic behaviour [10], statistical properties [8], classical solutions [11], qualitative consideration of a glueball spectrum [12], Gribov's problem [13], etc. So, the Yang-Mills quantum mechanics is a quite suitable system for a qualitative consideration of an influence of the PS reduction to the quantum description of the Yang-Mills fields in terms of HPI.

The paper is organized as follows. In sect.2, the abelian matrix model with the group SO(2) is considered for the explanation of main difficulties arising at the HPI derivation in the reduced PS. Section 3 is devoted to the Yang-Mills quantum mechanics. And in sect.4, for a simple model, the question is studied about the connection of HPI in gauge invariant variables with the description using an arbitrary gauge condition. In Conclusion we summarize our results. In Appendix, the recipe of the HPI construction in curvelinear coordinates is suggested which is used for the analysis in sections 2–4. This problem was already discussed in literature. In particular, in [14] the HPI form was found in another way in spherical and topologically equivalent coordinates.

2. Abelian matrix model

Consider the system with the Lagrangian

$$L = \frac{1}{2}\mathrm{Tr}\left(\dot{x} + yTx\right)^{T}\left(\dot{x} + yTx\right) - V\left(x^{T}x\right). \qquad (2.1)$$

Here the 2x2 real matrix x and scalar y are dynamical variables of the theory, $T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is a generator of rotations in a two-dimensional plane (a generator of the group $SO(2)$), V is a potential. The Lagrangian (2.1) does not change under gauge transformations

$$y \to y - \dot{\omega}\,, \quad x \to \Omega x\,, \quad \Omega = \exp T\omega \tag{2.2}$$

where $\omega = \omega(t)$ is an arbitrary function of time. The gauge transformations are rotations of the two-dimensional vectors being the matrix x columns.

Let us turn now to the Hamiltonian formalism. The canonical momenta are $p = \partial L/\partial \dot{x} = (\dot{x} + yTx)^T$, $\pi = \partial L/\partial \dot{y} = 0$. So, the Hamiltonian of the system has the following form

$$H = \frac{1}{2} \mathrm{Tr}\, p^T p + V(x^T x) - y\, \mathrm{Tr}\, p^T T x\,. \tag{2.3}$$

Secondary constraints follow from self-consistency conditions of dynamics [1]

$$\dot{\pi} = \{\pi, H\} = \mathrm{Tr}\, p^T T x \equiv G = 0\,. \tag{2.4}$$

Here $\{\,,\,\}$ are the Poisson brackets. It is easy to check that Eq.(2.4) exhausts all secondary constraints. Indeed, $\{G, H\} = 0$.

Let us discuss the question about the physical PS structure in the present model. By the gauge transformation (2.2), the matrix x can always be reformed to a triangular form, for example, $x_{21} = 0$ since the transformations (2.2) are rotations of the columns x_{j1} and x_{j2}. However, gauge arbitrariness is not yet exhausted. Transformations changing the sign of x_{11} remain, and they do not break the equality $x_{21} = 0$ (the rotations through the angle π in the plane (x_{11}, x_{21})). Therefore, the points x_{11} and $-x_{11}$ are gauge-equivalent ($RDGG = \mathbb{Z}_2$).

Note that these $\mathbb{Z}_2$ -transformations can be made at any moment of time (they are gauge transformations!). Since laws of gauge transformations for coordinates x and momenta p coincide, p_{11} changes its sign simultaneously with x_{11} . The latter means that $PS(x_{11}, p_{11})$ is a cone unfoldable into a half-plane $con(\pi)$ [3,7] (if in the plane (x_{11}, p_{11}) we identify the points (x_{11}, p_{11}) and $(-x_{11}, -p_{11})$ which are gauge-equivalent, the phase plane becomes a cone: $con(\pi)$). This exhausts the gauge arbitrariness. Thus, there are only three physical degrees of freedom with $PS = con(\pi) \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ in the present model.

As is shown in [3,7], the distance between energy levels of a quantum oscillator with PS = con (π) is doubled. So, it would be expected that the physical (observed) frequencies will be 2, 1, 1 if $V = \frac{1}{2} Tr\, x^T x$ and what's more, 2 corresponds to the degree of freedom with PS = con (π) . However, it is not the case. It turns out that degrees of freedom with the reduced PS influence the spectrum of the full system.

To make more clear this question, consider a quantum theory. After quantization $p_{ij} \to \hat{p}_{ij}$, $x_{ij} \to \hat{x}_{ij}$, $[\hat{x}_{ij}, \hat{p}_{kn}] = i\delta_{ik}\delta_{jn}$ $(\hbar = 1)$ the constraint operator (2.4) picks out a physical subspace of states $\mathcal{H}_{ph}$

$$\hat{G}\,|\psi_{ph}\rangle = 0 \qquad (2.5)$$

(the constraint $\hat{\pi}\,|\psi_{ph}\rangle = 0$ is solved immediately). Equation (2.5) can easily be solved in the second quantization representaion. In this representation, Eq. (2.5) has the following form

$$\hat{a}^+_{ji}\, T_{jk}\, \hat{a}_{ki}\, |\psi_{ph}\rangle = 0, \qquad (2.6)$$

where $\hat{a}_{kj} = (\hat{x}_{kj} + i\hat{p}_{kj})/\sqrt{2}$. Apparently, the vacuum $|0\rangle \in \mathcal{H}_{ph}$ ($\hat{a}_{kj}\,|0\rangle \equiv 0$), therefore any physical state can be

obtained by applying a function of $\hat{a}^+$ commuting with $\hat{G}$ to $|0\rangle$. Since the gauge transformations are rotations of columns of the matrix $\hat{a}^+_{ij}$, all gauge-invariant quantities depend on scalar products of $\hat{a}^+_{ij}$ columns [15]: $\hat{b}_1^+ = \hat{a}^+_{i1}\hat{a}^+_{i1}$, $\hat{b}_2^+ = \hat{a}^+_{i2}\hat{a}^+_{i2}$, $\hat{b}_3^+ = \hat{a}^+_{i1}\hat{a}^+_{i2}$. However, there exists one more operator commuting with $\hat{G}$. It has the form $\hat{b}_0^+ = \det \hat{a}^+_{ij} = \hat{a}^+_{11}\hat{a}^+_{22} - \hat{a}^+_{12}\hat{a}^+_{21}$ and $\hat{b}_0^{+2} = \hat{b}_1^+\hat{b}_2^+ - \hat{b}_3^{+2}$, i.e., it does not correspond to an independent degree of freedom. Nevertheless the state $\hat{b}_0^+|0\rangle \equiv |b_0\rangle$ satisfies (2.6) and it is orthogonal to $\hat{b}_a^+|0\rangle$ (a=1,2,3). So, full $\mathcal{H}_{ph}$ splits into the orthogonal sum of two subspaces $\mathcal{H}_{ph} = \mathcal{H}^0_{ph} \oplus \mathcal{H}^1_{ph}$ in which bases are formed by following states

$$\hat{b}_1^{+n_1}\,\hat{b}_2^{+n_2}\,\hat{b}_3^{+n_3}\,|0\rangle \in \mathcal{H}^0_{ph}, \qquad (2.7a)$$

$$\hat{b}_1^{+n_1}\,\hat{b}_2^{+n_2}\,\hat{b}_3^{+n_3}\,|b_0\rangle \in \mathcal{H}^1_{ph}, \qquad (2.7b)$$

where $n_a = 0,1,2,\ldots$. Now one can see from (2.7) that the physical spectrum of the system for $V = \frac{1}{2}\mathrm{Tr}\,x^T x$ is

$$E = 2n_1 + 2n_2 + 2n_3 + 2, \qquad (2.7c)$$

i.e., it contains levels of three independent oscillators with frequencies 2. However, the degeneration γ_E of a level E is different from one of the case of three oscillators and it, apparently, is

$$\gamma_E = \gamma(n) + \gamma(n-1), \qquad (2.7d)$$

where $\gamma(n)$ is a number of solutions of Diophantine equation (2.7c) ($n=n_1+n_2+n_3$) at fixed E. Two addendum in (2.7d) correspond to subspaces $\mathcal{H}^0_{ph}$ and $\mathcal{H}^1_{ph}$ ($\gamma(n)$ for $\mathcal{H}^0_{ph}$ and $\gamma(n-1)$ for $\mathcal{H}^1_{ph}$).

The existence of two orthogonal subspaces in $\mathcal{H}_{ph}$ is connected with the difference of the groups O(2) and SO(2). Consider

the parity transformation $P=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Apparently, $P\in O(2)$, but $P\notin SO(2)$. Nevertheless, Lagrangian of the theory (2.1) does not change under transformations from $O(2)$, $x\to u_p x$, $y\to -y+\omega$, $u_p=Pu$, $u=\exp T\omega(t)$ and $\det u_p=-1$. The transformation P changes the sign of the lower line of the matrix x_{ij} (therefore momenta p_{2j} change their sign too). So, $\hat{a}^+_{2j}$ change also the sign under the P-transformation. According to (2.7a,b) the space $\mathcal{H}^0_{ph}$ is P-invariant and states from $\mathcal{H}^1_{ph}$ are P-odd since $P|b_0\rangle=-|b_0\rangle$. Although the transformation P belongs to the gauge group of the Lagrangian, it does not lead to an additional cutting down of $\mathcal{H}_{ph}$. The main point is that operators of constraints in a gauge theory realize always a representation of a Lie algebra for a considered gauge group, hence, taking constraints as generators of gauge transformations, we may only restore a connected component of the identity element of a gauge group. The information about the global group structure is absent in constraints. In this sense, transformations which do not belong to a connected component of a group identity element cannot call gauge (obviously, they form a discrete subgroup). For example, the above-considered transformation $u_p=Pu$ consists of both "ungauge" P and gauge $u\in SO(2)$ transformations for the quantum theory. Of course, we may supplementary require the "P-invariance" for physical states, i.e., we do it "by hand". However, it changes nothing in the dynamical description of the system since transitions between $\mathcal{H}^0_{ph}$ and $\mathcal{H}^1_{ph}$ are absent for a P-invariant potential V (V does not depend on $\det x$), i.e., subspaces $\mathcal{H}^0_{ph}$ and $\mathcal{H}^1_{ph}$ are dynamical unconnected and have the identical structure exept of the vacuum energies of $|0\rangle$ and $|b_0\rangle$. So, quantum dynamics in $\mathcal{H}^0_{ph}$ and $\mathcal{H}^1_{ph}$ coincide. Moreover, if V is not P-invariant, then there exist transitions between $\mathcal{H}^0_{ph}$ and $\mathcal{H}^1_{ph}$, but in this case P is not the gau-

ge transformation ($V(\rho x) \neq V(x)$!). Thus, we may conclude that gauge groups should always be, in this sense, connected components of a group identity element.

Note that doubling of these levels is not connected with the global SO(2)-symmetry with respect to index of columns of x_{ij} . Indeed, the Lagrangian (2.1) can be rewritten as sum of two Lagrangians, and besides, each of them corresponds to one column of x (to one particle on a plane). Then we may introduce different constants of the kinetic terms, i.e., particles x_{i1} and x_{i2} will have different masses. So, the global SO(2) symmetry will be broken, but the gauge SO(2) symmetry will not. However, by this procedure we shall only take off the degeneracy between levels of three oscillators but the distance doubling between the levels will conserve; since the basis of gauge invariants, as before, has the form $x_{ji}x_{jk}$, i.e., is formed out of scalar products of x_{i1} and x_{i2} . The latter entails the level doubling for all physical degrees of freedom. Attention is to be paid to that the system spectrum would contain levels of three oscillators with unit frequencies if one eliminates a nonphysical variable before the quantization, for example, through the gauge condition $x_{21} = 0$.

To reveal the origin of such peculiar kinematic coupling of the degrees of freedom with PS = con $(\overline{\pi})$ and PS = R^2, consider the quantum problem in the coordinate representation, i.e., $\hat{p}_{kj} = -i\,\partial/\partial x_{kj}$. In so doing, we pass to new variables to solve Eq.(2.5)

$$x = \exp(T\theta)\,\rho \equiv u\rho, \qquad (2.8)$$

where ρ is a triangular matrix ($\rho_{21} = 0$). Using rules of derivative transformations one checks easily that $\hat{G} \sim -i\,\partial/\partial\theta$ in coordinates (2.8). Thus, $\mathcal{H}_{ph}$ consists of functions indepen-

dent of ϑ , i.e., $\psi_{ph}(x) = \psi(\rho)$. To calculate the Hamiltonian in $\mathcal{H}_{ph}$, we determine the Laplace-Beltrami operator in coordinates (2.8) (see (A.3)) and omit in it the terms containing the derivatives with respect to θ . From (2.8) we can find the new metric tensor:

$$Tr\, dx^T dx = Tr\left(d\rho^T d\rho + d\theta^2 \rho^T\rho + \left(\rho d\rho^T T - T d\rho \rho^T\right) d\theta\right), \tag{2.9}$$

where the matrix $d\rho$ consists of the differentials $d\rho_{ij}$, which we may represent in the form of a 4x4 symmetric matrix

$$g_{\alpha\beta} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} = \begin{pmatrix} 1 & B \\ B^T & Tr\rho^T\rho \end{pmatrix}, \tag{2.10}$$

where A = 1 is the 3x3 matrix (from the first term in (2.9)). Using antisymmetry of T one transforms the last term in (2.8) to the form $2T_{ij}\rho_{jk}d\rho_{iK}d\theta$ to find the column B . So, B_a^T = = (0 , $-\rho_3$, ρ_2) if we write the independent components of ρ as follows $\rho_a = (\rho_{11}, \rho_{12}, \rho_{22})$. Then $\mu \equiv \sqrt{g} = \sqrt{\det g_{\alpha\beta}} = \rho_1$ and the tensor $g^{\alpha\beta}$ inverse to $g_{\alpha\beta}$ is

$$g^{\alpha\beta} = \begin{pmatrix} \delta_{ab} + \rho_1^{-2} B_a B_b & -B_a \rho_1^{-2} \\ -B_a \rho_1^{-2} & \rho_1^{-2} \end{pmatrix}. \tag{2.11}$$

The 3x3 matrix $g_{ph}^{ab} = \delta_{ab} + \rho_1^{-2} B_a B_b$ contributes to the physical part of the Laplace-Beltrami operator.

Therefore

$$\hat{H}_{ph} = -\frac{1}{2\mu}\, \partial_a \circ g_{ph}^{ab}\, \mu\, \partial_b + V(\rho^T\rho), \tag{2.12}$$

where $\partial_a = \partial/\partial\rho_a$. Now one can see how the kinematic coupling of physical degrees of freedom arises. Even if $V = 0$, the variables do not separate in the Schroedinger equation, i.e.,

independent excitations of the degrees of freedom ρ_2 and ρ_3 are impossible. The curvilinearity of physical variables plays a central role in this phenomenon [16]. There exists no kinematic coupling if the quantization is made after the elimination of nonphysical variables through the gauge condition $\chi = \rho$.

Let us turn now directly to the HPI construction. The problem is to find the representation of the evolution operator kernel corresponding to the quantum Hamiltonian (2.12) in terms of a path integral. Since (2.12) contains the part of the total Hamiltonian in curvilinear coordinates (2.8), we may use the method suggested in Appendix for deriving the HPI.

From (2.8) it follows that there exist two types of the symmetry for the new variables:

$$\text{1)} \quad \theta \to \theta + 2\pi n, \quad n = \dots, -1, 0, 1, \dots;$$
$$\text{2)} \quad \rho \to -\rho, \quad \theta \to \theta + \pi. \tag{2.13}$$

So, the fundamental region K is $\theta \in [0, 2\pi)$, $\rho_1 > 0$, $\rho_{2,3} \in \mathbb{R}$. However, the variable θ is unphysical, therefore, the transformations of θ do not influence the form of the unit operator kernel in $\mathcal{H}_{ph}$. The physical variables change under $\mathbb{Z}_2$-group, $\rho \to -\rho$, which should be taken into consideration for the analytic continuation of the unit operator kernel $\langle \rho | \rho' \rangle_{ph}$. Thus

$$\langle \rho | \rho' \rangle_{ph} = \sum_E \varphi_E(\rho)\, \varphi_E^*(\rho') =$$
$$= (\rho_1 \rho_1')^{-1/2}\, \delta(\rho - \rho') + (-\rho_1 \rho_1')^{-1/2}\, \delta(\rho + \rho'), \tag{2.14}$$

where $\rho \in \mathbb{R}^3$, $\rho' \in K$ $(\rho_1' > 0)$ and φ_E present a basis in $\mathcal{H}_{ph}$, hence, $\varphi_E(\rho) = \varphi_E(-\rho)$. The latter allows us to write equality (2.14) for the analytic continuation of $\langle \rho | \rho' \rangle_{ph}$ in unphysical region $\rho_1 < 0$. We may also get (2.14) by the

gauge group averaging of the unit operator kernel in the total Hilbert space

$$\langle \rho | \rho' \rangle_{ph} = \int_0^{2\pi} d\varphi \; \delta^4(x - \exp(\varphi T)x') = \int_0^{2\pi} d\varphi \; \delta^4(\rho - \exp(\varphi T)\rho'), \tag{2.15}$$

where $x = \exp(\theta T)\rho$ and $x' = \exp(\theta' T)\rho'$. Calculating the integral in (2.15), we find (2.14) (compare with deriving (A.6) and (A.7)). Note that the P-transformation changes the kernel (2.14), however this kernel is SO(2)-gauge-invariant in accordance with (2.15). This result corresponds to our consideration of the model in the second quantization representation. There exists the main difference between the $\mathbb{Z}_2$-transformation as RDGG and P-transformation. RDGG $S = \mathbb{Z}_2$ corresponds to the transformation of the unphysical variables $\theta \to \theta + \pi$, while under the P-transformation θ does not change ($x \to Px$, $y \to Py \equiv -y$ in (2.1)).

The obtained group $\mathbb{Z}_2$ reduces the physical PS. Indeed, using the gauge transformations we get the equality $x = \rho$ ($\rho_{21} = 0$). But there exists a residual discrete gauge transformation from $\mathbb{Z}_2$, $\rho \to \pm\rho$ which does not break the condition $\rho_{21} = 0$. Therefore, physical values of ρ lie in K. Of course, the physical states (2.15) should be invariant under this RDGG since it is a subgroup of the gauge group in the present system. The RDGG has a transparent geometrical meaning. The condition picking out the physical variables determines a "line" in the total configuration space. This line can intersect a gauge group orbit several times. The gauge transformations connecting these crossing points on one orbit form RDGG reducing the physical PS. The existence of the RDGG is due to impossibility of choosing the global gauge condition picking out physical variables without an

ambiguity, i.e., the gauge condition "line" always intersects each orbit more than once. In the present model this follows from compactness of the gauge orbits. In the general case of Yang – – Mills fields this results from Singer's theorem [17].

To get the HPI, now it will be sufficient to use formulae from Appendix (A.8)–(A.24) where the operator (2.12) should be taken istead of $\hat{H}$. Then, we conclude

$$U_t^{ph}(\rho,\rho') = \int_{\mathbb{R}^3} \frac{d\rho''}{(\rho_1\rho_1'')^{1/2}} U_t^{eff}(\rho,\rho'')Q(\rho'',\rho'), \qquad (2.16)$$

$$Q(\rho'',\rho') = \delta(\rho''-\rho') + \delta(\rho''+\rho'). \qquad (2.17)$$

The kernel U_t^{eff} is determined by HPI (A.24) in which the effective action has the form

$$S_{eff} = \int_0^t d\tau \left(Tr\, p^T\dot{\rho} - H_{eff}(\rho,p) \right), \qquad (2.18)$$

where p is an upper triangular matrix of momenta and, at least, in accordance with (A.15)

$$H_{eff} = \frac{1}{2} p_\alpha g_{ph}^{\alpha\beta} p_\beta - \frac{\dot{\rho}}{2}(p_2\rho_2 + p_3\rho_3) - \\ - \frac{1}{8}\rho_1^{-2} + V(\rho^T\rho). \qquad (2.19)$$

Here $p_\alpha = (p_{11}, p_{12}, p_{22})$ mean nonzero components of the matrix p.

Note that the part of (2.19) quadratic in momenta contains the metric $g_{ph}^{\alpha\beta}$. Therefore, after integration over p in HPI a determinant arises in the measure on the configuration space, and the effective Lagrangian will contain the nontrivial metric $(g_{ph}^{-1})_{\alpha\beta}$ in the term quadratic in velocities $\dot{\rho}_\alpha$. Moreover, the curvature tensor calculated with $(g_{ph}^{-1})_{\alpha\beta}$ does not vanish (unlike the case of curvilinear coordinates considered in Appendix). Thus, the physical configuration space turns out to be

curved, i.e., it cannot be transformed to the plane one by a global smooth change of variables.

We may use another way for eliminating unphysical variables, for example, $x = \rho = \rho^T$ [13]. In this case RDGG coincides with Z_2 again (the only nontrivial gauge transformation non-breaking the equality $x = x^T$ is $x \to \pm x$). So, the physical configuration space can be picked out as $\rho \in K$ if $\mathrm{Tr}\,\rho > 0$. Assuming ρ to be symmetric matrix in (2.8) we may calculate the metric tensor like (2.9). It has the same form as in (2.10) with A=1, $D = \mathrm{Tr}\,\rho^2$ and column B consists of three independent elements of the matrix $\frac{1}{2}[T, \rho]$. Further, with the help of (A.15) (A.23) and (A.24) we can construct HPI in this case.

Note that HPI for the physical variables $\rho = \rho^T$ differs from (2.16) since the measure and effective action are different in both cases. Nevertheless, the operator $\hat{Q}$ coincides with (2.17). One asks: does $\hat{Q}$ depend on the choice of physical variables? In sect.4 it will be shown that the answer is positive.

In conclusion we study the question about a physical meaning of the operator $\hat{Q}$. As shown above, $\hat{Q}$ symmetrizes the evolution operator kernel over the RDGG. Particularly, $Q(\rho, \rho') = |\rho_1| \langle \rho | \rho' \rangle_{ph}$ in our model. On the other hand, by construction, kernel (2.14) is explicit gauge-invariant (see (2.15)) and we may replace ρ' by x', the result does not depend on the unphysical variable θ', i.e.,

$$\langle \rho | \rho' \rangle_{ph} = \langle x | x' \rangle_{ph} = \langle \rho | x' \rangle_{ph} = \langle x | \rho' \rangle_{ph}. \qquad (2.20)$$

One may also find the explicit form of the gauge-invariant kernel $\langle x | x' \rangle_{ph}$ in the total configuration space of the system

$$\langle x | x' \rangle_{ph} = |\det x \det x'|^{1/2} \delta(x^T x - x'^T x') \left(1 + \frac{\det x}{\det x'}\right) \qquad (2.21)$$

Here δ -function is the product of three 1-dimensional δ -functions for every independent element of the symmetric matrix $(x^T x)_{ij}$ (i.e., ij = 11, 22, 12). One may check that (2.21) coincides with (2.14). The second term in sum (2.21) is P-odd, i.e., it corresponds to the contribution of states from $\mathcal{H}^1_{ph}$. Taking into account both the connection $\hat{Q}$ with $\langle \rho | \rho' \rangle_{ph}$ and equality (2.20), we may replace ρ' by x' in (2.16) and the result is independent of the unphysical variable θ'. Moreover, by construction of kernel (2.16) (see deriving (A.22)), $\hat{Q}\hat{U}^{eff}_{\varepsilon}\hat{Q} = \hat{U}^{eff}_{\varepsilon}\hat{Q}$ ($\varepsilon \to 0$), hence $\hat{Q}\hat{U}^{ph}_{\varepsilon} = \hat{U}^{ph}_{\varepsilon}$. This leads to that ρ can be changed by x in (2.16). Thus, the following equality takes place

$$U^{ph}_t(\rho, \rho') = U^{ph}_t(x, x'). \qquad (2.22)$$

The right-hand side of (2.22) is the explicit gauge-invariant evolution operator kernel of our system, i.e., the function U^{ph}_t as a function of two variables depends only on gauge-invariant combinations $(x^T x)_{ij}$, $(x'^T x')_{ij}$ and $\det x$, $\det x'$. We conclude that taking into consideration of the operator $\hat{Q}$ in the evolution operator kernel (i.e., taking into consideration of the physical phase space reduction) allows to get unique explicit gauge-invariant **continuation** of the evolution operator kernel in the total configuration space and, by the way, it guarantees us a gauge-invariant quantum description.

3. HPI for the Yang-Mills quantum mechanics

The Yang-Mills mechanics with the group $SU(2) \sim SO(3)$ corresponds to the Yang-Mills field theory if all potentials $A^a_\mu = A^a_\mu(t)$ (a=1,2,3 are isotopic indices) depend only on time. The Lagrangian has the form (2.1) where x is a 3x3 real matrix ($x_{ai} = A^a_i(t)$, i=1,2,3) and $y^T \to y$ is a 3x3 real antisymmet-

ric matrix ($y_{ab} = -g\varepsilon_{abc}A_0^c(t)$), g is a coupling constant. Then $V = \frac{1}{4}g^2\left[(\mathrm{Tr}\,x^Tx)^2 - \mathrm{Tr}(x^Tx)^2\right]$, however in what follows the potential form is inessential. We may write the law of gauge transformations as follows

$$x \to \Omega x,\quad y \to \Omega y\Omega^T + \Omega\,\partial_t\Omega^T\ ,\quad \Omega(t)\in SO(3). \qquad (3.1)$$

The Hamiltonian formalism for the present system is analogous to the one in sect. 2. Again y turns out to be unphysical (its momentum vanishes, forming the primary constraint). The secondary constraints are, as it should be, generators of three-dimensional rotating columns of the matrix x . So, any gauge-invariant function of x is a function of the matrix x^Tx , i.e., the system contains six physical degrees of freedom. Of course, det x is also gauge-invariant, but it does not represent new independent physical degree of freedom as it was shown in sect.2.

To construct the quantum theory for this model in terms of HPI, following the logic of sect.2, one should pick out physical variables, find an RDGG reducing a physical PS, and finally take account of the curvilinearity of physical variables, restore the HPI form with the help of Appendix. Upon this procedure we get one-to-one correspondence with a gauge invariant description (coinciding spectra etc.). Ignoring each of these conditions leads to a wrong result, i.e., to rejection of the Dirac quantization scheme for a system with constraints, as it has been shown in sect.2.

Let us realize this programme. Put $x = x^T$. This condition picks out six physical degrees of freedom. For determining RDGG we must solve the equation

$$(u_s x)^T = u_s x\ ,\quad u_s \in SO(3). \qquad (3.2)$$

Note that Lagrangian of the present theory is invariant under transformation (3.1) with $\Omega \in O(3)$. However, as shown in sect.2, transformations of the parity from $O(3)$ are not connected with changing unphysical variables and, so they cannot include in RDGG determined by (3.2).

Putting $x = \omega h \omega^T$ where h is a diagonal matrix, $\omega \in SO(3)$, we get from (3.2) $\omega_S h = h \omega_S^T$, $\omega_S = \omega^T u_s \omega \in SO(3)$. Then $h = \omega_S^T h \omega_S^T$ and $h = \omega_S h \omega_S$ since $h = h^T$. Using two latter equalities one finds easily

$$[h^2, \omega_S] = 0. \tag{3.3}$$

Since h is arbitrary, we conclude that ω_S is a diagonal orthogonal matrix, i.e., $\omega_S = \{I_1 = \mathrm{diag}(1,-1,1), I_2 = \mathrm{diag}(-1,1,-1), I_3 = \mathrm{diag}(-1,-1,1), I_4 = 1\}$. Thus, the transformations ω_S form the group $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ and the group S reducing the physical PS contains also four elements $\hat{S}_\nu x \equiv u_S^\nu x$, $u_S^\nu = \omega(x) I_\nu \omega^T(x)$, $\nu = 1,2,3,4$. S is isomorphic to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ in the neighbourhood of a nondegenerated matrix. Apparently, degenerated matrices x belong to the boundary ∂K, where $K = \mathbb{R}^6 \setminus S$ $(x \in \mathbb{R}^6)$ [13]. Note that the dependence S on a point x is smooth, i.e., $S(x) \sim \mathbb{Z}_2 \otimes \mathbb{Z}_2$ at all nondegenerated x.

To take into consideration the curvilinearity of physical variables it is necessary to make after quantization a change of variables which diagonalizes constraints, i.e., in the new variables constraints must express the equalities to zero of some generalized momentum operators on functions of the physical Hilbert subspace $\mathcal{H}_{ph}$. A momentum operator is always a translation generator of the corresponding canonical coordinate. Thus, choosing new coordinates so that a part of them acquires a displacement under a gauge transformation, one can diagonalize constraints. Based on these notes we introduce the new variables

by analogy with (2.8) where $\mathcal{U} \in SO(3)$ and $\rho = \rho^T$. The following step for the HPI determination is to perform analytic continuation of the unit operator kernel in $\mathcal{H}_{ph}$. For this purpose we calculate the metric tensor. It has the form (2.10) where A is a 6x6 unit matrix corresponding to $Tr d\rho^2$ in $Tr\, dx^T dx$, D is a 3x3 symmetry matrix, $D_{\alpha\beta} = 1/4(\delta_{\alpha\beta} Tr\rho^2 - \rho_{\alpha c}\rho_{c\beta})$ which are multipliers of $d\theta_\alpha d\theta_\beta$ if we put $(u^T du)_{\alpha\beta} = 1/2\, \varepsilon_{\alpha\beta c}\, d\theta_c$, $\bigwedge_c d\theta_c$ is the invariant measure on $SU(2)$. At last, independent elements of the tensor $B_{\alpha\beta,c} = B_{\beta\alpha,c}$ which gives multipliers of $d\rho_{\alpha\beta} d\theta_c$ in $Tr\, dx^T dx$ determine the 6x3 matrix B. By direct calculation we find $B_{\alpha\beta,c} = 1/4(\varepsilon_{adc}\rho_{d\beta} + \varepsilon_{\beta dc}\rho_{d\alpha})$. So, the measure in the scalar product of $\mathcal{H}_{ph}$ is

$$\mu(\rho) = \sqrt{g} = \det(Tr\rho - \rho) \tag{3.4}$$

accurate to a numerical factor, where $(Tr\rho - \rho)_{\alpha\beta} = \delta_{\alpha\beta} Tr\rho - \rho_{\alpha\beta}$. Thus, we can write for the analytic continuation of the unit operator kernel in $\mathcal{H}_{ph}$

$$\langle \rho | \rho' \rangle_{ph} = \sum_{\nu=1}^{4} \left[\mu(\rho)\mu(\hat{S}_\nu \rho') \right]^{-1/2} \delta(\rho - \hat{S}_\nu \rho'). \tag{3.5}$$

Now it only remains to calculate the physical part g_{ph} of the metric tensor inverse to (2.10) which defines the physical addend in the Laplace - Beltrami operator that does not depend on $\partial/\partial\theta_\alpha$. Apparently, it coincides with the 6x6 matrix in the upper left-hand angle of the matrix inverse to (2.10), i.e.,

$$\| g_{ph} \| = 1 + B H^{-1} B^T, \quad H_{cc'} = \frac{1}{8}\left[(Tr\rho - \rho)^2 \right]_{cc'}. \tag{3.6}$$

Now we may derive HPI for the Yang-Mills quantum mechanics with the help of the recipe suggested in Appendix.

As in the previous model, the symmetrization over RDGG $S \sim Z_2 \otimes Z_2$ allows us to continue the evolution operator kernel by explicit gauge-invariant way in the total configuration space. Equations (2.20) and (2.22) remain to be valid. One may also write the analog of (2.21) for the present model. In conclusion it is necessary to emphaze that the present model does not include a condition that piks out physical variables without an ambiguity [13], i.e., RDGG is always nontrivial. We have above suggested the method of solution for this problem according to the Dirac quantization scheme.

4. Gauge fixing and invariant description

Solutions of constraint equations in gauge theories are easily found if we know the basis of gauge group invariants. However, for realistic field theories this problem has no satisfactory solution so far.

That is why for picking out physical degrees of freedom one uses a condition imposed on theory variables, i.e., one fixes a gauge. The elimination of unphysical variables and a subsequent quantization, as shown above, do not always give a correct result corresponding to the Dirac scheme [1], in other words, a quantum theory thus obtained does not always correspond to an initial Lagrangian since it can contain unphysical states [19,7]. So, it is interesting to elucidate the question about the correct HPI form in any gauge which is in a one-to-one correspondence with a gauge-invariant description.

We have seen that it is necessary to take into account two central points for this: the curvilinearity of physical variables (the metric in an effective action) and a physical PS reduction (the operator $\hat{Q}$ in HPI). Here for the simple model the question about the correct HPI entry in an arbitrary gauge will be solved.

The Lagrangian has the form [20,3]

$$L = \frac{1}{2}\left(\dot{x} + y\,T x\right)^2 - V(x^2), \qquad (4.1)$$

where a two-dimensional vector x and a scalar y are dynamical variables. The Lagrangian is invariant under gauge transformations from SO(2) like (2.2), where one should understand x as a two-component column. Apparently, y is an unphysical variable since the corresponding momentum is equal to zero. The circles S^1 in the configuration space $x \in \mathbb{R}^2$ form gauge group orbits, therefore, for the gauge-invariant description it is enough to introduce the polar coordinates. Then the angular variable becomes unphysical, while the gauge invariant $r = (x^2)^{1/2} \geqslant 0$ describes the only physical degree of freedom. The HPI form in invariant variables for the present model was given in [4,5]. It coincides with (2.16) if we put $r = q_1$, $p_r = p_1$ and consider other variables as zeroes (for details see [5]). Note that the physical $PS(r, p_r) = \mathrm{con}(T)$ [3].

A gauge condition means a definition of time evolution of unphysical variables. Let x_1 describe a physical degree of freedom. Then x_2 is an unphysical variable. Define its time evolution by the equality $x_2 = f(x_1)$, where f is an arbitrary function. For the correct dynamic description the gauge condition line $x_2 = f(x_1)$ must cross each gauge group orbit S^1, at least, once. So, $f(0) = 0$ and f is defined for all $x_1 \in \mathbb{R}$. If the latter property does not take place, i.e., the region of definition of f differs from $\mathbb{R}$, the domain of values of f should coincide with $\mathbb{R}$ $(x_2 \in \mathbb{R})$, otherwise there will exist orbits S which do not cross the line $x_2 = f(x_1)$. Therefore, we can always choose a physical variable as changing along the real axis $\mathbb{R}$.

Following the ideology developed above, after quantization $p = -i\,\partial/\partial x$ we introduce new curvilinear coordinates

$$x = \exp T\theta \begin{pmatrix} \rho \\ f(\rho) \end{pmatrix} \tag{4.2}$$

It is easy to check that the secondary constraint operator $\hat{p}\,T\hat{x}$ generating $SO(2)$ gauge transformations [21] coincides with $-i\,\partial/\partial\theta$. So, $\mathcal{H}_{ph}$ contains functions independent of θ.

Using the standard deriving rules of derivation of the Laplace-Beltrami operator in curvilinear coordinates. We find the Hamiltonian in $\mathcal{H}_{ph}$

$$\hat{H}_{ph} = -\frac{1}{2\mu}\,\partial_\rho \circ \frac{\rho^2 + f^2}{\mu}\,\partial_\rho + V(\rho^2 + f^2), \tag{4.3}$$

where $\mu = \rho + f'f$ is the measure (Jacobian) in the new variables. The scalar product in $\mathcal{H}_{ph}$ has the form

$$\langle \varphi | \psi \rangle_{ph} = \int_K d\rho\, \mu(\rho)\, \varphi^*(\rho)\, \psi(\rho). \tag{4.4}$$

Here $K = \mathbb{R} \setminus S$ is a physical configuration space and the group S can be determined as above, i.e., it is necessary to solve the equation

$$U_s \begin{pmatrix} \rho \\ f(\rho) \end{pmatrix} = \begin{pmatrix} s(\rho) \\ f(s(\rho)) \end{pmatrix}, \quad U_s \in SO(2) \tag{4.5}$$

and besides $\hat{s}\rho = s(\rho)$, $\hat{s} \in S$. In fact, all solutions of (4.5) can be found from the equation

$$s^2(\rho) + f^2(s(\rho)) = \rho^2 + f^2(\rho) \tag{4.6}$$

defining all the points of crossing of the line $x_2 = f(x_1)$ with the circle $x_1^2 + x_2^2 = \text{const} > 0$. It is clear that the

number of crossing points depends on ρ . So, RDGG $S = S[\rho]$ has a different number of elements at different ρ . Thus, the region K splits into subregions K_α , and besides, $S[\rho] = S_\alpha$ for $\rho \in K_\alpha$ and has a fixed number of elements, i.e., each subgroup S_α acts smoothly on K_α .

Let us establish the connection of the derived quantum theory with the gauge-invariant description. Note that $\mu^{-1}\partial_\rho = r^{-1}\partial_r$, where $r = (\rho^2 + f^2)^{1/2}$. So, the Hamiltonian (4.3) turns automatically into the radial part of the Hamiltonian in polar coordinates. Moreover, it is easily seen that the equality $\int_K d\rho\,\mu = \int_0^\infty dr\,r$ in (4.4) follows from the change of variables. This proves that the physical Hilbert space of the theory (4.4), (4.3) should be isomorphic to the gauge-invariant Hilbert space of states

$$\psi_{ph}(\rho) = \psi(\rho^2 + f^2) = \psi(r^2) = \psi(x^2). \qquad (4.7)$$

That all physical states depend on r^2 analytically follows from the potential analyticity and parity $\psi(r) = \psi(-r)$ of the wave function with the zero angular momentum $-i\partial/\partial\theta\,\psi(r,\theta) = 0$ [22].

Since physical functions (4.7) are invariant under transformations (4.5), we find for the analytic continuation of the unit operator kernel in $\mathcal{H}_{ph}$

$$\langle \rho | \rho \rangle_{ph} = \sum_{S[\rho']} \left[\mu(\rho)\mu(s(\rho'))\right]^{-1/2} \delta\left(\rho - s(\rho')\right), \qquad (4.8)$$

where $\rho \in \mathbb{R}$, $\rho' \in K$ ($S[\rho'] = S_\alpha$ if $\rho' \in K_\alpha$ (α is fixed). Now we may derive HPI with the help of formulae presented in Appendix. However, one needs take into consideration that the integral over physical region K in (A.21) should be calculated in accordance with a note given at the end of Appendix.

5. Conclusion

A short summary of our results is as follows: to define a correct HPI form for gauge systems, one should take account of the curvilinearity of physical variables and possible physical PS reduction. Moreover, it is not necessary to describe explicitly the physical PS structure, i.e., to find the region K. It is enough to know all crossing points of a gauge condition line with gauge group orbits in the total configuration space, i.e., the RDGG. We may integrate over the physical PS in HPI as over an Euclidean space of an even dimension and then symmetrize the result in the RDGG. The latter procedure guarantees the manifest gauge invariance of the evolution operator kernel, i.e., the physical evolution operator kernel depends only on gauge-invariant combinations of initial variables in a theory. Besides, the symmetrization over RDGG cancels singularities in the evolution operator kernel which arise at zero points of the measure $\mu=0$ [23].

Let x_j be dynamical variables of a theory and p_j be corresponding momenta. Let also a theory have first class constraints $\varphi_a(x,p)$ (primary and secondary). In quantum theory the unitary operator $\hat{U}=\exp i\omega_a\varphi_a(\hat{x},\hat{p})$ is the operator of gauge transformation and $\hat{U}\psi_{ph}(x)=\psi_{ph}(x)$. We also have the law of gauge transformations for canonical variables $\hat{x}_i\to$ $\to\hat{U}\hat{x}_i\hat{U}^+\equiv u_{ij}(\omega)\hat{x}_j$, $\hat{p}_i\to\hat{U}p_i\hat{U}^+$. Note, that these $\hat{U}$ -transformations can be larger than initial gauge transformations in a theory! (for example, in model (2.1) π and G are independent generators!). To derive the HPI, we must introduce in quantum theory new variables $x_i=x_i(\theta,y)=u_{ij}(\theta)y_j$ where $\chi_a(y)\equiv 0$ are extra conditions picking out physical variables. Then $\varphi_a(\hat{x},\hat{p})\sim i\partial/\partial\theta_a$ so, θ_a are nonphysical variables. The HPI for physical variables y_j can be found by using the above suggested recipe. The

RDGG is determined by equations $\chi_a(\hat{s}y) = 0$, where $(\hat{s}y)_i = u^s_{ij} y_j$. It is clearly that this method can be used for the HPI construction in any gauge theory [25] .

APPENDIX

(HPI in an arbitrary curvilinear coordinate system)

Let $\phi_E(x)$, $x \in \mathbb{R}^N$ be a basis in the Hilbert space for some quantum system where E is a spectral parameter set enumerating a basis. Then

$$\sum_E \phi_E(x)\, \phi_E^*(x') \equiv \langle x|x'\rangle = \delta(x-x') . \qquad (A.1)$$

Consider new variables $x = x(y)$, $y \in K \subset \mathbb{R}^N$ where the region K is determined from requiring a one-to-one correspondence between the variables x and new variables y , i.e., the equation $x = x(y)$ should have only one solution if $x \in \mathbb{R}^N$ and $y \in K$. Assume now that there exists an analytic continuation of the function $x(y)$ to the total Euclidean space $\mathbb{R}^N$, in other words, $x(y)$ is a mapping $\mathbb{R}^N$ onto $\mathbb{R}^N$. Then the equation $x = x(y)$ may **have many solutions at fixed** x . For each $y \in K$ and fixed x one may compare $\hat{s}\,y$ so that $x = x(y) = x(\hat{s}y)$. Apparently, the transformations $\hat{s} \in S$ form a group acting on $\mathbb{R}^N$, and $K = \mathbb{R}^N \setminus S$. For example, $x \in \mathbb{R}^2$, $y = (r, \theta)$ (the polar coordinates). The group S contains the following transformations $\theta \to \theta + 2\pi n$, $n \in \mathbb{Z}$; $r \to -r$, and simultaneously $\theta \to \theta + \pi$. If on the plane $(r,\theta) \in \mathbb{R}^2$ the points connected by the transformations from S are identified, then we get the strip $K = \mathbb{R}^2 \setminus S$, $(r,\theta) \in K$, $\theta \in [0, 2\pi)$, $r > 0$.

Let the quantum Hamiltonian of a system have the standard form

$$\hat{H} = -\frac{1}{2}\Delta_{(N)} + V(x)\,,\quad \Delta_{(N)} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} \qquad (A.2)$$

$(i=1,2,\ldots,N)$. Rewriting it in the new variables we find

$$\hat{H} = -\frac{1}{2\sqrt{g}}\,\partial_i \circ g^{ij}\sqrt{g}\,\partial_j + V\,. \qquad (A.3)$$

Here $\partial_i = \partial/\partial y_i$, $g = \det g_{ij}$, $dx^i dx_i = g_{ij}\,dy^i dy^j$ and $g^{ik}g_{kj} = \delta^i_j$. Let in the new variables $\varphi_E(y) = \psi_E(x(y))$ be eigenfunctions of the operator (A.3) [26]. These functions have the symmetry property

$$\varphi_E(\hat{s}y) = \varphi_E(y)\,,\quad \hat{s}\in S \qquad (A.4)$$

After passing to the new variables the scalar product is changed to

$$\int_{R^N} dx\,\psi_E(x)\,\psi^*_{E'}(x) = \delta_{EE'} = \int_K dy\,\mu(y)\,\varphi_E(y)\,\varphi^*_{E'}(y). \qquad (A.5)$$

Here $\mu(y) = g^{1/2}(y) = D(x)/D(y)$. The form of the unit operator kernel follows from (A.5) and (A.4)

$$\sum_E \varphi_E(y)\varphi^*_E(y') \equiv \langle y|y'\rangle = \sum_S \left[\mu(y)\mu(\hat{s}y')\right]^{-1/2}\delta(y-\hat{s}y) \qquad (A.6)$$

if $y'\in K$, $y\in R^N$. Obviously, for $y,y'\in K$ one should limit oneself only to one term in (A.6) with $\hat{s}=1$. Formula (A.6) determines the analytic continuation of the unit operator kernel to the full R^N . For example, we have in the case of polar coordinates [14]

$$\delta^2(x-x') = \frac{1}{\sqrt{rr'}}\,\delta(r-r')\,q(\theta,\theta') + \frac{1}{\sqrt{rr'}}\,\delta(r+r')\,q(\theta,\theta'+\pi)\,, \qquad (A.7)$$

$$q(\theta,\theta') = \sum_{n=-\infty}^{\infty}\delta(\theta-\theta'+2\pi n)\,,$$

where $(\theta, r) \in \mathbb{R}^2$, $r' > 0$, $\theta' \in [0, 2\pi)$. Note that equality (A.6) can be obtained from the rule of changing an argument of a many-dimensional δ-function: $\langle x \mid x' \rangle = \delta(x(y) - x(y'))$ if we assume that $y \in \mathbb{R}^N$, $y' \in K$.

The main difficulty of the HPI construction in curvilinear coordinates is to take into consideration the reduction of a configuration space. Even a finite-dimensional Gaussian integral cannot be calculated explicitly over $K = \mathbb{R}^N \setminus S$. Moreover, the spectrum of some momentum operators becomes discrete, i.e., the integration over it turns into summation. For example, the angular momentum operator $p_\theta = -i\partial/\partial\theta$ has a discrete spectrum. The latter brings an additional difficulty in the HPI definition over PS of a considered system [14].

Consider the kernel of the infinitesimal evolution operator

$$U_\varepsilon(y, y') = \langle y | e^{-i\varepsilon \hat{H}} | y' \rangle = \left[1 - i\varepsilon \hat{H}(y)\right] \langle y | y' \rangle + O(\varepsilon^2), \quad (A.8)$$

where the kernel $\langle y | y' \rangle$ is determined by (A.6). To take into account the operator ordering in (A.3) [19], we rewrite Hamiltonian (A.3) as follows

$$\hat{H} = \frac{1}{2} \hat{p}_i g^{ij} \hat{p}_j + V_q + V, \quad (A.9)$$

where $\hat{p}_j = -i \mu^{-1/2} \partial_j \circ \mu^{1/2}$ are Hermitian momentum operators in curvilinear coordinates (see (A.5)) and V_q is an effective quantum correction to the potential ($\sim \hbar^2$) which can be found from the comparison of (A.9) with (A.3)

$$V_q = \frac{1}{2\sqrt{\mu}} (\partial_i g^{ij}) \partial_j \sqrt{\mu} + \frac{1}{2\sqrt{\mu}} g^{ij} \partial_i \partial_j \sqrt{\mu}. \quad (A.10)$$

For example, in the case of polar coordinates, $\mu = r$ and $V_q = -1/8\, r^{-2}$. Using the equality $\partial_i \circ g^{ij}(y) \partial_j \delta(y - a) =$

$= [g^{ij}(a)\partial_i\partial_j - \partial_i g^{ij}(a)\partial_j]\delta(y-a)$ which is easily proved in the theory of generalized functions, and the representation $\delta(y-a) = (2\pi)^{-N}\int dp \exp ip(y-a)$ one transforms the expression $\frac{1}{2}\hat{p}_i g^{ij}(y)\hat{p}_j \langle y|y'\rangle$ as follows

$$\int_{R^N} \frac{dy''}{(\mu\mu'')^{1/2}} \int_{R^N} \frac{dp}{(2\pi)^N} H_0(p,y'')e^{ip(y-y'')} Q(y'',y'). \qquad (A.11)$$

Here $H_0(p,y) = \frac{1}{2}p_i g^{ij}(y)p_j + i\frac{1}{2}\partial_i g^{ij}(y)\partial_j$, $\mu=\mu(y)$, $\mu''=\mu(y'')$ and

$$Q(y,y') = \sum_S \delta(y-\hat{s}y'),\ y\in R^N,\ y'\in K. \qquad (A.12)$$

In an analogous way one can rewrite other terms in the brackets of (A.8), i.e., 1, V and V_q. On the whole, we get the formula accurate to terms $O(\varepsilon^2)$

$$U_\varepsilon(y,y') = \int_{R^N} \frac{dy''}{(\mu\mu'')^{1/2}} U_\varepsilon^{eff}(y,y'')Q(y'',y'), \qquad (A.13)$$

where

$$U_\varepsilon^{eff}(y,y'') = \int_{R^N} \frac{dp}{(2\pi)^N} \exp i\varepsilon\left[p\dot{y}'' - H_{eff}(p,y'')\right], \qquad (A.14)$$

$$H_{eff}(p,y'') = H_0(p,y'') + V_q(y'') + V(y'') \qquad (A.15)$$

is the effective Hamiltonian of a system and the difference $y-y''$ is changed by $\varepsilon\dot{y}'' + O(\varepsilon^2)$.

Let us pay attention to symmetry properties of the kernel (A.13). By construction, the kernel should satisfy the equalities

$$U_\varepsilon(y,y') = U_\varepsilon(y,\hat{s}y') = U_\varepsilon(\hat{s}y,y')\,,\ \hat{s}\in S. \qquad (A.16)$$

Indeed, the first equality in (A.16) is trivial since S is a group. To prove the second equality, we return to the initial expression (A.8). Now we can see that the second equality in (A.16) will be right if $\langle \hat{s}y|y'\rangle = \langle y|y'\rangle$ since $H(\hat{s}y) = \hat{H}(y) = \hat{H}(x)$. Consider the action of the unit operator with the kernel $\langle y|y'\rangle$ on a smooth function φ

$$\Phi(y) = \int_K dy'\mu(y') \sum_S \left[\mu(y)\mu(\hat{s}y')\right]^{-1/2} \delta(y-\hat{s}y')\varphi(y'). \qquad (A.17)$$

In the general case, $\hat{s}y = S(y)$ is a function of $y\in K$ (for $y\in R^N$ it can be defined as a composition $S(y) = s\circ\tilde{s}(\tilde{y})$, where $\tilde{s}\in S$ and $\tilde{y}\in K$). Since the measure $dx = dx(y)$ is invariant under S, we conclude: $dy\,\mu(y) = dS(y)\,\mu(S(y)) = dy\,J_s(y)\,\mu(S(y))$, where $J_s(y)$ is the Jacobian of transition from $S(y)$ to y, i.e.,

$$\mu(\hat{s}y) = \left[J_s(y)\right]^{-1}\mu(y). \qquad (A.18)$$

With the help of (A.18) we can carry out the integration in (A.17) if the summation is interchanged with integration, and then new integration variables, $y'\to\hat{s}y'$, are introduced for each addend in (A.17)

$$\Phi(y) = \sum_S \theta_{K_s}(y)\,\varphi(\hat{s}^{-1}y)\,, \qquad (A.19)$$

where $\theta_{K_s}(y) = \theta_K(\hat{s}^{-1}y)$ and $\theta_K(y) = 1, 0$ if $y\in K$, $y\bar{\in}K$, respectively. It follows from (A.19) that $\Phi(\hat{s}y) = \Phi(y)$, $\hat{s}\in S$. Therefore, $\langle \hat{s}y|y'\rangle = \langle y|y'\rangle$, and Eq. (A.16) is proved. Note that $\Phi(y) = \varphi(y)$

if $\varphi(y)$ belongs to the initial Hilbert space $\varphi(y)=\psi(x(y))$ since $\varphi(\hat{S}^{-1}y)=\varphi(y)$ and

$$\sum_S \theta_{K_S}(y) = 1 . \qquad (A.20)$$

To get the kernel (A.8) for a finite time interval, it is necessary to iterate (A.13). By definition of the scalar product (A.5), we have

$$U_{2\varepsilon}(y,y') = \int_K dy_1 \mu_1 U_\varepsilon(y,y_1) U_\varepsilon(y_1,y') = \qquad (A.21)$$

$$= \int_{R^N} \frac{dy''dy_2}{(\mu\mu''\mu_2)^{1/2}} U_\varepsilon^{eff}(y,y_2) \int_K dy_1 \mu_1^{1/2} Q(y_2,y_1) U_\varepsilon^{eff}(y_1,y'') Q(y'',y').$$

In the integral over y_1 we substitute (A.12) for $Q(y_2,y_1)$ and interchange the order of summation and integration. Then using (A.18) we change variables $y_1 \to \hat{S}y_1 = S(y_1)$ in each addend after which the integrals over $\hat{S}y_1$ can easily be taken. Further, we take advantage of the second equality (A.16) into which one should substitute the explicit expression (A.13) and transform the measure $\mu^{-1/2}(\hat{S}y)$ in accordance with (A.18). The remaining sum over S disappears because of (A.20). On the whole, we get

$$U_{2\varepsilon}(y,y') = \int_{R^N} \frac{dy''}{(\mu\mu'')^{1/2}} \left[\int_{R^N} dy_2 U_\varepsilon^{eff}(y,y_2) U_\varepsilon^{eff}(y_2,y'') \right] Q(y'',y'). \qquad (A.22)$$

The integral in brackets of (A.22) is identified with $U_{2\varepsilon}^{eff}$ Thus, for a finite time interval t we may write

$$U_t(y,y') = \int_{R^N} \frac{dy''}{(\mu\mu'')^{1/2}} U_t^{eff}(y,y'') Q(y'',y') , \qquad (A.23)$$

where $U_t^{eff}(y,y'')$ is determined by the standard HPI representation

$$U_t^{eff}(y,y'') = \int \prod_{\tau=0}^{t} \frac{dp(\tau)dy(\tau)}{(2\pi)^N} \exp i \int_0^t d\tau \left(p\dot{y} - H_{eff}\right) \qquad (A.24)$$

and $y(t) = y$, $y(0) = y''$.

In conclusion, two notes have to be made. As noticed above, the transformations from S are not alway linear. In the general case, $\hat{S}y = S(y)$ is a function of $y \in K$. Moreover, it is not difficult to invent a change of variables under which the functions $S(y)$ cannot be defined smoothly on the full region K, but this is possible on its subregions K_α, $K = \sum \oplus K_\alpha$. It means that the group S splits into subgroups $S = \prod \otimes S_\alpha$, and besides, each S_α, acts only on K_α and has fixed number of elements. In fact, the split of K into K_α should be performed w.r.t. the number of elements of S in the neighbourhood of a point $y \in K$. An example of such coordinates is given in sect.4. However, the above-suggested derivation of HPI holds valid if in all calculation, $\sum_S \to \sum_\alpha \sum_{S_\alpha}$ and $\int_K \to \sum_\alpha \int_{K_\alpha}$.

Now HPI can be written in the polar coordinates with the help of the recipe (A.23). In this case the integration over new canonical variables $y = (r,\theta)$, $p = (p_r, p_\theta)$ has to be fulfilled within infinite limits and

$$Q(r,r',\theta,\theta') = \delta(r-r')q(\theta,\theta') + \delta(r+r')q(\theta,\theta'+\pi) \qquad (A.25)$$

This result coincides with the one obtained in [14] if one sets: $\mu = r$, $g^{ij} = diag(1, r^{-2})$, i.e., $\partial_i g^{ij} = 0$, $V_q = -1/8\, r^{-2}$.

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16. Note, the variable ρ_1 separates from ρ_2 and ρ_3 in the Schroedinger equation at $V = \frac{1}{2}\mathrm{Tr}\,\rho^T\rho$. The ρ_1 – oscillator spectrum is doubled [3,7]. The doubling of the spectrum of other oscillators follows from their kinematic coupling. One may check it considering the Schroedinger equation for Hamiltonian (2.12) with $V = 1/2\ \mathrm{Tr}\,x^T x$ in new variables $\rho_a \to \rho_1, \rho_2 =$

$= r\cos\varphi$, $\rho_3 = r\sin\varphi$. The variables separate in the Schroedinger equation in this case and spectrum coincides, of course, with (2.7c,d).

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21. The Hamiltonian formalism for (4.1) is similar to that given in sect.2.

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23. An example of HPI quasiclassical calculation with the physical PS reduction was given in [7,24].

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25. The analysis of HPI for Yang-Mills fields with inclusion of the reduction of their physical PS will be done in elsewhere.

26. More correctly it should be written $\varphi(y) = \sum_{E'} C_{EE'} \cdot \psi_{E'}(x(y))$, where $C_{EE'}$ are constants since the spectral parameters denoting bases of operators (A.2) and (A.3) can be of a different structure. However, it will be shown that it is not essential for further consideration.

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