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RELATIVISTIC PARTICLE WITH THE ACTION DEPENDENT ON THE TORSION OF ITS WORLD TRAJECTORY

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I. INTRODUCTION

In the last years interest has been aroused in the models of a relativistic point particle the action of which contains, besides the length of the world curve, new terms that depend on the curvature [1-6] or torsion [7-10] of this curve. Such geometrical invariants of the world trajectory should be taken into consideration, in particular, by introducing the spin factor into the Feynman path integral [11,12]. In this approach there is a possibility of obtaining the spinor propagator by making use of the bosonic path integral without introducing the Grassmannian anticommuting variables at the classical level. The model of the relativistic particle with curvature can be treated as a one-dimensional version of the rigid string [5,13] and its Euclidean setting is used in the polymer theory [14]. In ref. [7] a mechanism of fermion-boson transmutations has been explored. Scalar charged particle was placed in an external Abelian gauge field with the action containing additional Chern-Simons term. In the Euclidean three-dimensional space it was shown that the effective action of the particle acquires an additional term given by the integral along the trajectory of its torsion. By making use of the path integral it was shown that the corresponding effective propagator is the usual three-dimensional Dirac propagator. It is worth-while to construct the canonical quantum theory of this model in the operator form which gives us a more complete description. The present paper is devoted to this problem. The layout of the paper is as follows. In the second section the generalized Hamiltonian formalism for a relativistic particle with torsion in a D-dimensional space-time is constructed. A complete set of the constraints in the phase space is found, their separation into the first-class and the second-class constraints is fulfilled. The third section is devoted to the canonical quantization of this model. At first the general scheme of quantization in D-dimensional space-time is considered. Further the case of a three-dimensional space-time is investigated in detail. In the sector with positive mass squared we obtained a spectrum determined by an equation involving the parameters of the model, the mass and spin of a state. The possibility of describing, in the framework of this model, the states both with integer, half-integer and continuous spins is discussed. By making use of the Casimir operators of the Poincare group we construct the wave equation and the corresponding propagator. In Appendix A a discrete mass spectrum in the model of the relativistic particle with a curvature in the D-dimensional space-time is derived. Different forms of the wave equation are proposed and the corresponding propagator is found. The possibility of describing the states both with integer and with half-integer spins in the framework of this model are discussed.

2. THE HAMILTONIAN FORMALISM

Let us consider the action of a relativistic particle defined by

$$S = -m \int ds - \alpha \int \kappa(s)\, ds\,, \tag{2.1}$$

where ds is a differential of the length of the world trajectory, $\kappa(s)$ is the torsion of this trajectory, m is a constant with a dimension of mass and α is a dimensionless constant*. It $x^\mu(\tau)$, $\mu = 0, 1, \ldots, D-1$ is a parametric representation of the world trajectory, then the action (2.1) can be rewritten in the form [15]

$$S = -m \int d\tau \sqrt{\dot{x}^2} - \alpha \int d\tau \sqrt{\dot{x}^2}\; \frac{\sqrt{d}}{(\dot{x}\ddot{x})^2 - \dot{x}^2\ddot{x}^2}\,, \tag{2.2}$$

where $d = \det(d_{\alpha\beta})$, $d_{\alpha\beta} = \overset{(\alpha)}{x}{}^{\mu}\,\overset{(\beta)}{x}{}_{\mu}$, $\overset{(\alpha)}{x} = d^\alpha x/d\tau^\alpha$, $\alpha, \beta = 1, 2, 3$. The dot means the differentiation with respect to τ. In the D-dimensional space-time the metric with the signature (+, -, ..., -) is used.

The action (2.2) is invariant under the Poincarè transformation in the D-dimensional space-time and under the reparametrization of a world line. As a consequence of the last

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* It should be noted that the integral $(2\pi)^{-1} \oint_C \kappa(s)\, ds$ taken along a closed contour C in the three-dimensional Euclidean space is equal to the linking coefficient of the boundaries of a flat strip connected with this contour [15,16]. In order to get an absolute topological invariant, one should take into consideration the writhing of this strip [17,18].

property the Lagrangian function in (2.2) is singular. We shall construct now the generalized Hamiltonian formalism for this model by making use of the results of papers [1,19]. To begin with we introduce the canonical variables

$$q_1 = x\,, \quad q_2 = \dot{x}\,, \quad q_3 = \ddot{x}\,, \tag{2.3}$$

$$p_1 = -\frac{\partial L}{\partial \dot{x}} - \frac{dp_2}{d\tau}\,, \tag{2.4}$$

$$p_2 = -\frac{\partial L}{\partial \ddot{x}} - \frac{dp_3}{d\tau}\,, \tag{2.5}$$

$$p_3 = -\frac{\partial L}{\partial \dddot{x}}\,, \tag{2.6}$$

where L is the Lagrangian function in (2.2). In eqs.(2.3)-(2.6) the Lorentz indices are omitted for simplicity.

The explicit form of the canonical momentum p_3 will only be required further. It is given by

$$p_3^\mu = \alpha\, \frac{\sqrt{\dot{x}^2}}{(\dot{x}\ddot{x})^2 - \dot{x}^2\ddot{x}^2}\, \sqrt{d}\, \sum_{a=1}^{3} d^{3a}\, \overset{(a)}{x}{}^{\mu}\,, \tag{2.7}$$

where $d^{\alpha\beta}$ is the matrix inverse to $d_{\alpha\beta}$: $d_{\alpha\beta}\, d^{\beta\gamma} = \delta_\alpha^\gamma$. From (2.7) we deduce three primary constraints

$$\overset{(1)}{\phi}_1 = p_3^2 + \alpha^2\, \frac{q_2^2}{(q_2 q_3)^2 - q_2^2 q_3^2} = 0\,, \tag{2.8}$$

$$\overset{(1)}{\phi}_2 = p_3 q_2 = 0\,, \tag{2.9}$$

$$\overset{(1)}{\phi}_3 = p_3 q_3 = 0\,. \tag{2.10}$$

According to Ref. [19] the canonical Hamiltonian is

$$H = -p_1\dot{x} - p_2\ddot{x} - p_3\dddot{x} - L = -p_1 q_2 - p_2 q_3 + m\sqrt{q_2^2}\,. \tag{2.11}$$

The Poisson brackets will be defined as follows

$$(f,g) = \sum_{a=1}^{3} \left(\frac{\partial f}{\partial p^{\mu}_{a}} \frac{\partial g}{\partial q_{a\mu}} - \frac{\partial f}{\partial q^{\mu}_{a}} \frac{\partial g}{\partial p_{a\mu}} \right).$$

The primary constraints (2.8)-(2.10) are in involution between themselves

$$(\overset{(1)}{\phi}_1, \overset{(1)}{\phi}_2) = 0, \quad (\overset{(1)}{\phi}_1, \overset{(1)}{\phi}_3) = 2\overset{(1)}{\phi}_1 \approx 0, \quad (\overset{(1)}{\phi}_2, \overset{(1)}{\phi}_3) = \overset{(1)}{\phi}_2 \approx 0. \tag{2.12}$$

The sign $\approx$ means a weak equality [20].

In Ref. [1] the generalized Hamiltonian formalism developed by Dirac for singular Lagrangians of the first order has been extended to the singular Lagrangians with higher derivatives. We shall follow this approach. The dynamics in the phase space is determined by the equations of motion

$$\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + (f, H) + \sum_{a=1}^{3} \lambda_a (f, \overset{(1)}{\phi}_a), \tag{2.13}$$

where f is a function of the canonical variables and evolution parameter τ.

Let us now look for the secondary constraints by making use of the Dirac prescription. Demanding the stationarity of the primary constraints

$$\frac{d\overset{(1)}{\phi}_a}{d\tau} = (\overset{(1)}{\phi}_a, H_T) = 0, \qquad a = 1,2,3. \tag{2.14}$$

with $H_T = H + \sum_{b=1}^{3} \lambda_b \overset{(1)}{\phi}_b$ we obtain three new constraints

$$\overset{(2)}{\phi}_1 = p_2 p_3 - \alpha^2 \frac{q_2 q_3}{(q_2 q_3)^2 - q_2^2 q_3^2} = 0, \tag{2.15}$$

$$\overset{(2)}{\phi}_2 = p_2 q_2 = 0, \tag{2.16}$$

$$\overset{(2)}{\phi}_3 = p_2 q_3 = 0. \tag{2.17}$$

Further we shall need the Poisson brackets of the constraints (2.15)-(2.17) with the Hamiltonian (2.11) and with

the primary constraints (2.8)-(2-10)

$$(\overset{(2)}{\phi}_1, \overset{(1)}{\phi}_1) \approx 0, \quad (\overset{(2)}{\phi}_1, \overset{(1)}{\phi}_2) = \overset{(1)}{\phi}_1 \approx 0, \quad (\overset{(2)}{\phi}_1, \overset{(1)}{\phi}_3) = \overset{(2)}{\phi}_1 \approx 0,$$

$$(\overset{(2)}{\phi}_1, H) = -\left[p_1 p_3 + p_2^2 + \alpha^2 \frac{q_3^2}{(q_2 q_3)^2 - q_2^2 q_3^2} \right] = -\overset{(3)}{\phi}_1,$$

$$(\overset{(2)}{\phi}_2, \overset{(1)}{\phi}_1) = 0, \quad (\overset{(2)}{\phi}_2, \overset{(1)}{\phi}_2) = \overset{(1)}{\phi}_2 \approx 0, \quad (\overset{(2)}{\phi}_2, \overset{(1)}{\phi}_3) = 0,$$

$$(\overset{(2)}{\phi}_2, H) \approx H = \overset{(3)}{\phi}_2,$$

$$(\overset{(2)}{\phi}_3, \overset{(1)}{\phi}_1) = -2\overset{(2)}{\phi}_1 \approx 0, \quad (\overset{(2)}{\phi}_3, \overset{(1)}{\phi}_2) = \overset{(1)}{\phi}_3 - \overset{(2)}{\phi}_2 \approx 0, \quad (\overset{(2)}{\phi}_3, \overset{(1)}{\phi}_3) = -\overset{(2)}{\phi}_3 \approx 0,$$

$$(\overset{(2)}{\phi}_3, H) = -p_1 q_3 + m \frac{q_2 q_3}{\sqrt{q_2^2}} = \overset{(2)}{\phi}_3. \tag{2.18}$$

The requirement of the stationarity of the constraints (2.15)-(2.17) on the equations of motion

$$\frac{d\overset{(2)}{\phi}_a}{d\tau} = (\overset{(2)}{\phi}_a, H) + \sum_{b=1}^{3} \lambda_b (\overset{(2)}{\phi}_a, \overset{(1)}{\phi}_b) \approx 0, \qquad a = 1,2,3, \tag{2.19}$$

results in the three new constraints with the canonical Hamiltonian (2.11) between them

$$\overset{(3)}{\phi}_1 = p_1 p_3 + p_2^2 + \alpha^2 \frac{q_3^2}{(q_2 q_3)^2 - q_2^2 q_3^2} = 0, \tag{2.20}$$

$$\overset{(3)}{\phi}_2 = H = -p_1 q_2 - p_2 q_3 + m\sqrt{q_2^2} = 0, \tag{2.21}$$

$$\overset{(3)}{\phi}_3 = -p_1 q_3 + m \frac{q_2 q_3}{\sqrt{q_2^2}} = 0. \tag{2.22}$$

On this stage the process of generation of constraints is stopped. The requirement of the stationarity of the last constraints (2.20)-(2.22) enables us to determine the Lagrange multipliers λ_1 and λ_2 in the total Hamiltonian H_T

$$\lambda_1 = m \frac{(q_2 q_3)^2 - q_2^2 q_3^2}{2(p_1 p_3)\, q_2^2 \sqrt{q_2^2}}\,; \qquad \lambda_3 = 3\,\frac{p_1 p_2}{p_1 p_3}. \tag{2.23}$$

Now we have to separate all the constraints into the first and second-class constraints. For this purpose we construauct the matrix Ω with elements

$$\Omega_{ij} \approx (\theta_i, \theta_j), \qquad 1 \le i, j \le 9, \tag{2.24}$$

where $\theta_{3(b-1)+a} = \overset{(b)}{\phi_a}$, a, b = 1, 2, 3. The matrix Ω can be rewritten in a block-form:

$$\Omega = \begin{vmatrix} 0 & 0 & A \\ 0 & B & C \\ -A^t & -C^t & D \end{vmatrix} \tag{2.25}$$

where 0 is the (3x3)-zero-matrix and

$$A = \begin{vmatrix} 0 & 0 & -2p_1p_3 \\ 0 & 0 & 0 \\ -p_1p_3 & 0 & 0 \end{vmatrix}, \qquad B = \begin{vmatrix} 0 & 0 & -p_1p_3 \\ 0 & 0 & 0 \\ p_1p_3 & 0 & 0 \end{vmatrix},$$

$$C = \begin{vmatrix} \alpha^2 \dfrac{p_1p_2}{g} & 0 & -p_1p_2 \\ -2p_1p_3 & 0 & 0 \\ -p_1p_2 & 0 & \dfrac{\alpha^2 m^2}{p_3^2\sqrt{q_2^2}} \end{vmatrix}, \quad D = \begin{vmatrix} 0 & -3p_1p_2 & m^2 - p_1^2 \\ 3p_1p_2 & 0 & \dfrac{mg}{q_2^2\sqrt{q_2^2}} \\ p_1^2 - m^2 & \dfrac{-mg}{q_2^2\sqrt{q_2^2}} & 0 \end{vmatrix} \tag{2.26}$$

Here $g = (q_2 q_3)^2 - q_2^2 q_3^2$, sign t means a transposition. As is known /21/, the number of the first-class constraints equals Dim Ker Ω. If the vector $\xi \in \mathrm{Ker}\,\Omega$, $\xi = \{\xi_1, \ldots, \xi_9\}$, then

$$\begin{gathered} \xi_4 = \xi_6 = \xi_7 = \xi_9 = 0, \\ (p_1p_3)\,\xi_3 + 2(p_1p_3)\,\xi_5 - 3(p_1p_2)\,\xi_8 = 0, \\ 2(p_1p_3)\,\xi_1 - \frac{mg}{q_2^2\sqrt{q_2^2}}\,\xi_8 = 0. \end{gathered} \tag{2.27}$$

Thus we get

Dim Ker Ω = 3.

Therefore in the model under consideration there are three first-class constraints and six second-class constraints. The number of physical degrees of freedom equals obviously 3D-6. The first class constraints can be separated by the formula /21/

$$\Phi_a = \sum_{i=1}^{9} \overset{(a)}{\xi_i}\, \theta_i, \qquad a = 1,2,3, \tag{2.28}$$

where $\overset{(a)}{\xi_i}$, a = 1, 2, 3 are the basis vectors of Ker Ω. By making use of eq. (2.27) one can easily construct these vector up to an arbitrary factor for each $\overset{(a)}{\xi}$. They have the following nonzero components

$$\overset{(1)}{\xi_2} = 1, \quad \overset{(2)}{\xi_3} = -2, \quad \overset{(2)}{\xi_5} = 1,$$

$$\overset{(3)}{\xi_1} = \frac{mg}{2q_2^2\sqrt{q_2^2}\, p_1p_3}, \quad \overset{(3)}{\xi_3} = \frac{2p_1p_2}{p_1p_3}, \quad \overset{(3)}{\xi_8} = 1. \tag{2.29}$$

Taking into account (2.28) and (2.29) we obtain the first-class constraints

$$\Phi_1 = p_3 q_2 = 0, \tag{2.30}$$

$$\Phi_2 = p_2 q_2 - 2p_3 q_3 = 0, \tag{2.31}$$

$$\Phi_3 = -p_1 q_2 - p_2 q_3 + m\sqrt{q_2^2} + \frac{2p_1p_2}{p_1p_3}\, p_3 q_3 + \frac{mg}{2(p_1p_3)\, q_2^2\sqrt{q_2^2}}\left(p_3^2 + \alpha^2 \frac{q_2^2}{g}\right) = 0. \tag{2.32}$$

As the second-class constraints $\omega_s = 0$, $s = 1,\dots,6$ one can take six arbitrary constraints from the set $\{\theta_i,\ i = 1,\dots,9\}$ with $\det \| (\omega_s, \omega_{s'}) \| \neq 0$, $s, s' = 1,\dots,6$. This can be done in many ways. For example, one may put

$$\omega_a = \overset{(2)}{\phi}_a = 0, \quad \omega_{3+a} = \overset{(3)}{\phi}_a, \quad a = 1,2,3. \tag{2.33}$$

In this case the Hamiltonian (2.11) is considered to be the second-class constraint. However we can substitute $H = \omega_5$ in (2.33) by $\overset{(1)}{\phi}_1$. At quantum level we shall consider both these possibilities.

3. QUANTUM THEORY

At first we consider the general scheme of the canonical quantization of this model in the D-dimensional space-time. We are dealing with a generalized Hamiltonian system in the 6D-dimensional phase space with three first-class constraints (2.30)-(2.32) and six second-class constraints (2-33). The state vectors are defined from the conditions

$$\Phi_a | \psi > = 0, \quad a = 1,2,3. \tag{3.1}$$

The commutators of the operators q_a and p_a, $a = 1,2,3$ must be determined by the Dirac brackets constructed by means of the second class constraints ω_s. After this the constraints ω_s will vanish identically at the quantum level, and they can be omitted in conditions (3.1). As a result, the wave equations (3.1) can be rewritten only in terms of the primary constraints

$$\overset{(1)}{\phi}_a | \psi > = 0, \quad a = 1,2,3. \tag{3.2}$$

The number of the wave equations (3.2) can be reduced by introducing the gauge conditions. For example, the condition

$$\chi_1 = q_2 q_3 = 0 \tag{3.3}$$

entails considerable simplification. From (3.3) it follows that

$$q_2^2 = \text{const}. \tag{3.4}$$

Thus eq. (3.3) is in fact, the proper time gauge. This gauge eliminates completely the functional freedom in the equations of motion (3.13), and the last Lagrange multiplier turns out to be

$$\lambda_2 = q_3^2 / q_2^2. \tag{3.5}$$

In principle, we can impose one or two gauge conditions in addition to (3.3) $\chi_c(q_a, p_a, \tau) = 0$, $c = 2, 3$ demanding that

$$\det \| (\chi_a, \Phi_b) \| \neq 0, \quad a,b = 1,2,3, \tag{3.6}$$

$$\frac{\partial \chi_c}{\partial \tau} + (\chi_c, H) + \sum_{a=1}^{3} \lambda_a (\chi_c, \overset{(1)}{\phi}_a) \approx 0, \quad c = 2,3, \tag{3.7}$$

where λ_a, $a = 1, 2, 3$ are defined in (2.23) and (3.5).

Further simplification is achieved when D = 3. In this case a relation between the parameters of the model, α and m, entering into the action (2.1) and the squared mass $M^2 = p_1^2$ of the state and its spin can be obtained. Let us derive this relation. If D = 3, then three vectors $\{q_2, q_3, p_3\}$ form, by virtue of (2.8)-(2.10), a complete orthogonal basis. Taking into account (2.15)-(2.17) and (3.3) we deduce

$$p_2^\mu(\tau) = 0, \quad \mu = 0,1,2. \tag{3.8}$$

The components of the vector p_1^μ in the basis $\{q_2, q_3, p_3\}$ are

$$p_1 p_3 = \frac{\alpha^2}{q_2^2}, \quad p_1 q_2 = m\sqrt{q_2^2}, \quad p_1 q_3 = 0. \tag{3.9}$$

Hence one can write

$$p_1^\mu = p_3^\mu \frac{q_3^2}{q_2^2} + q_2^\mu \frac{m}{\sqrt{q_2^2}}. \tag{3.10}$$

Squaring eq. (3.10) we obtain

$$p_1^2 = M^2 = m^2 + \alpha^2 \frac{q_3^2}{(q_2^2)^2}, \tag{3.11}$$

where M^2 is the mass of the particle with the action (2.1). From (3.11) it follows that

$$M^2 < m^2 \tag{3.12}$$

and M^2 is not positive definite because $q_3^2 < 0$. Thus, in the model under consideration the squared mass is determined by the initial conditions (by the Cauchy data) for variabless q_3^μ and it can be either positive, or negative, or it can vanish. Further we shall confine our consideration to the sector in this model, where $p_1^2 = M^2 > 0$. Obviously, it can be made only in the free case.

Let us examine the angular momentum in this model

$$M_{\mu\nu} = \sum_{a=1}^{3} (q_{a\mu}p_{a\nu} - q_{a\nu}p_{a\mu}) .$$

At the quantum level the algebra of the operators $M_{\mu\nu}$ should be determined by commutators of the operators q_a and p_a, $a = 1, 2, 3$. In their turn these commutators are defined, as mentioned above, by the corresponding Dirac brackets. But the requirement of the Poincare invariance of the theory under consideration determines the algebra of the operators $p_{1\mu}$ and $M_{\mu\nu}$ completely. This algebra must be the same as the algebra of the Poincare group. Without calculating the corresponding Dirac brackets we assume that the Poincare invariance takes place. As the scalar Casimir operators of the Poincare group we take the following ones/22/

$$p_1^2 = M^2, \tag{3.13}$$

$$W = \frac{1}{2} M_{\mu\nu} M^{\mu\nu} p_1^2 - (M_{\mu\sigma} p_1^\mu)^2 . \tag{3.14}$$

It is easy to show that on the surface determined by the constraints and gauge conditions the invariant W is given by*

$$W \approx \alpha^2 \left(p_1^2 - \frac{\alpha^2}{q_2^2} \frac{q_3^2}{q_2^2}\right) . \tag{3.15}$$

Taking into account (3.11) we obtain

$$W \approx \alpha^2 m^2 . \tag{3.16}$$

In the rest reference frame of the particle, where $p_1^\mu = (p_1^0 = M, \vec{p}_1 = 0)$, we have

*In the four-dimensional space-time the invariant W is the squared Pauli-Lubanski vector with sign minus, $W = -w_\mu w^\mu$, where $w_\mu = (1/2)\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} p_1^\sigma$.

$$W = (p_1^0)^2 M_{12} M^{12} = \frac{M^2}{2} C_2(SO(2)) , \tag{3.17}$$

where $C_2(SO(2))$ is the squared Casimir operator of the group SO(2) (see, for example /23/). Hence, there is a general relation

$$2\alpha^2 m^2 = M^2 C_2(SO(2)) . \tag{3.18}$$

Now we have to specify the transformation properties of the state vector $|\psi>$. Here there are three possibilities in accordance with three different representations of the rotation group SO(2) on the plane /24/ with integer, half-odd-integer or continuous values for spin j. In all these cases we have /23/

$$C_2(SO(2)) = 2j^2, \quad j = 0, 1, \ldots, \tag{3.19}$$

The values $j < \alpha$ do not fit condition (3.12). Without loss of generality we can suppose that $\alpha > 0$. Therefore relation (3.18) should be rewritten as follows

$$\left(\frac{M}{m}\right)^2 = \left(\frac{\alpha}{j}\right)^2 , \quad j \geq \alpha . \tag{3.20}$$

If the parameters of the model α, m and the spin j of the state are given, then eq. (3.21) determines the mass of this state. If one assumes that m, M and j are fixed, then relation (3.20) defines the parameter α.

It should be noted here that in the model under consideration we have to deal with the tensor representations of the SO(2)-group only because the initial action (2.1) contains no spin variables at the beginning. And even in the case of the half-odd-integer spin values the corresponding wave function is taken in the Schrödinger coordinate representation instead of the spinor one.

Let us construct the propogator in this theory. After impossing the gauge conditions we have one constraint on the physical state vectors*

$$\overset{(1)}{\phi}_1 |\psi> = 0 \tag{3.21}$$

that can be rewritten as

$$(p_3^2 q_3^2 - a^2)|\psi> = 0 . \tag{3.22}$$

The operators in the left-hand side are expressed in terms of the Casimir operator p_1^2 and W of the Poincare group. As a result, the wave equation becomes

$$\frac{W - a^2 m^2}{p_1^2} |\psi> = 0 . \tag{3.23}$$

The propagator G in the operator form is defined by

$$G = \frac{p_1^2}{W - a^2 m^2} . \tag{3.24}$$

In order to transform the operator equation (3.23) into the partial differential equation for the wave function $\psi(q_1, q_2, q_3)$, one has to construct the exact realization of the Casimir operators p_1^2 and W in terms of $\partial/\partial q_a$, $a = 1,2,3$. It is easy to write this equation in the rest frame by making use of the angular variable ϕ

$$\left[\frac{\partial^2}{\partial \phi^2} + \left(a \frac{m}{M}\right)^2 \right] \psi(\phi) = 0 . \tag{3.15}$$

One may impose in principle, three different boundary conditions

$$\psi(\phi) = \psi(\phi + 2\pi), \quad \psi(\phi) = -\psi(\phi + 2\pi), \quad |\psi(\phi)|^2 = 1 . \tag{3.26}$$

Putting

$$\psi(\phi) \sim \exp(i j \phi) \tag{3.27}$$

*We assume that the constraint $\overset{(1)}{\phi}_3 = 0$ is transformed into the second-class constraint by an appropriate gauge condition that obeys eqs. (3.6) and (3.7).

we obtain from (3.26) integer, half-integer or continuous values for spin j. After substituting (3.27) into (3.25) we arrive at the mass formula (3.20) that gives the position of the propagator poles.

Here it is worthwhile to note that a decisive conclusion about the permissible values of the spin in the model under consideration can be made only after obtaining the connection of angular variable ϕ in (3.25) with the canonical variables q_a, p_a and choosing on this basis the corresponding boundary conditions in (3.26). We would think that it is interesting to remind of the half-integer orbital angular momentum problem in nonrelativistic quantum mechanics that is still being under discussion (see, for example, /26/ and references therein).

As was mentioned in Section 2 one can substitute $H = \omega_5$ in the set of the second-class constraints (2.33) by $\overset{(1)}{\phi}_1$. This entailes, at quantum level, different form of the wave equation. Instead of (3.23) we have

$$\Phi_3 |\psi> = H|\psi> \approx (-p_1 q_2 + m\sqrt{q_2^2}) |\psi> = 0 . \tag{3.28}$$

Probably this equation can be reduced to the Dirac equation more easily in comparison with the equation (3.22) or (3.23) (see, for example, Ref. /25/). The mass formula (3.20) can be derived from the wave equation (3.28) as well.

The relation between mass and spin also takes place in the theory of the relativistic particle whose action depends on the curvature of the world curve (see Appendix A to this paper and Ref. /2/).

4. CONCLUSION

Thus, we have shown that at the quantum level the action (2.1) describes infinite family of states with different spins. The mass of a state cannot be arbitrary but it is determined completely by the model parameters a, m and by the spin of the state. This picture takes place only in the sector with positive mass squared.

In principle this model enables one to describe the states with integer, half-integer and continuous spins. But a decisive conclusion requires more sophisticated investigations here. The extension of this model to include interaction is worthwhile also.

The idea to describe classical relativistic particles with spin by means of the Lagrangian with higher derivatives without introducing anticommuting Grassmannian variables has been proposed long ago /27/. Further development of this approach can be found in Refs. /28–31/.

As far as the investigations of the boson-fermion transmutations in the external Chern-Simons fields are concerned here there is another possibility. One can quantize the whole system, a charged particle and external field, without preliminary construction of the effective partcile action /32, 33/.

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APPENDIX A

Here we obtain the quantization condition for the mass spectrum in the theory of a relativistic particle with the action

$$S = -m \int ds + \alpha \int k(s)\, ds \,, \qquad (A.1)$$

where $k(s)$ is the curvature of the world curve. The D-dimensional space-time is considered. In paper /1/ the complete set of constraints in this model has been obtained as follows:

$$\overset{(1)}{\phi}_1 = p_2^2 q_2^2 + \alpha^2 = 0, \quad \overset{(1)}{\phi}_2 = p_2 q_2 = 0, \qquad (A.2)$$

$$\overset{(2)}{\phi}_1 = p_1 p_2 = 0, \quad \overset{(2)}{\phi}_2 = p_1 q_2 - m\sqrt{q_2^2} = 0. \qquad (A.3)$$

The invariant of the Poincare group W on the surface defined by the constraints reads

$$W = \alpha\,(m^2 - p_1^2) = \alpha^2\,(m^2 - M^2)\,. \qquad (A.4)$$

Further we shall consider only the sector in the model, where $p_1^2 = M^2 > 0$. In the rest frame we have

$$W = \frac{(p_1^0)^2}{2} \sum_{i,j=1}^{D-1} M_{ij} M^{ij} = \frac{M^2}{2} C_2\,(SO(D-1))\,, \qquad (A.5)$$

where $C_2\,(SO(D-1))$is the squared Casimir operator for the SO(D-1) -group. The eigenvalues of this operator are /23/

$$C_2\,(SO(D-1)) = 2j(j+D-3)\,, \quad j = 0, 1, 2, \ldots, \quad D > 3\,, \qquad (A.6)$$

where j is the integral spin of the state. We do not consider here the half-odd-integer values of j in order not to deal with the double-valued eigenfunctions of the angular momentum operator in the coordinate representation. As known, in nonrelativistic quantum mechanics such wave functions are excluded by means of the Pauli criterion /34/. For D = 3 it would be the function $D^j_{m'm}(\alpha, \beta, \gamma)$ with half-odd-integral j (the rotation matrices /26/). The rejection of the Pauli criterion in the model under consideration would be badly undesirable because this admits half-odd-integer values of the ordinary orbital angular momentum*.

Let us return to the Casimir operator $C_2\,(SO(D-1))$. If D=3, then

$$C_2\,(SO(2)) = 2j^2, \quad j \geq 0\,. \qquad (A.7)$$

In this case the spin of the state is arbitrary. From (A.4)-(A.7) we deduce the relation between the spin j and the mass of the state

$$M^2 = \frac{m^2}{1 + \alpha^{-2} j(j+D+3)}\,, \qquad (A.8)$$

and for D = 3

$$M^2 = \frac{m^2}{1 + (j/\alpha)^2}\,, \quad j \geq 0\,. \qquad (A.9)$$

*Nevertheless the authors of paper /31/ considering the analogous problem, proposed to use the wave functions $D^j_{m'm}(\alpha, \beta, \gamma)$ with the half-odd-integral j.

If the spin j in (A.8) and (A.9) increases, then the mass of states descreases. This contradicts the properties of the elementary particles known until now. The same comment concerns the mass spectrum (3.20) too.

For quantization it is important to split the complete set of constraints (A.2) and (A.3) into the first-class and the second-class constraints. As is shown in Ref. /1/ , the first-class constraints are defined by

$$\Phi_1 = (m^2 - p_1^2)\overset{(1)}{\phi_1} + 2p_2^2(p_1q_2)\overset{(2)}{\phi_2} = 0,$$
$$\Phi_2 = p_2q_2 = 0. \qquad (A.10)$$

The second-class constraints can be picked out in two ways. In Ref. /1/ this was made as follows

$$\omega_1 = \overset{(2)}{\phi_1}, \quad \omega_2 = -H = \overset{(2)}{\phi_2}. \qquad (A.11)$$

But one can take as the second-class constraints the following ones

$$\bar{\omega}_1 = \overset{(2)}{\phi_1}, \quad \bar{\omega}_2 = \overset{(1)}{\phi_1}. \qquad (A.12)$$

Equations (A.10) and (A.11) determine the same submanifold of the phase space as the constraints (A.10) and (A.12). However these two sets of the constraints entail, at the quantum level, different wave equations, at least in appearance.

After imposing the gauge

$$q_2^2 = \text{const} \qquad (A.13)$$

the constraints (A.10) and (A.11) result in the following wave equation:

$$(p_1^2 - m^2)(p_2^2q_2^2 + \alpha^2)|\psi> = 0. \qquad (A.14)$$

As usual we assume that the second-class constraints vanish as the operating at the quantum level due to the use of the Dirac brackets instead of the Poisson brackets. The wave

equation (A.1) can be rewritten on the constraint surface in terms of the Poincare group invariants

$$(p_1^2 - m^2 + \alpha^{-2}W)\,|\psi> = 0. \qquad (A.15)$$

Hence it follows that the propagator in this model is

$$G = (p_1^2 - m^2 + \alpha^{-2}W)^{-1}. \qquad (A.16)$$

By making use of the representation where the Casimir operators p_1^2 and W are diagonal we easily obtain the formulae (A.8) and (A.9) that determine the poles of the Green function G.

If we shall quantize the model under consideration using the second-class constraints (A.12), then instead of the wave equation (A.14) we get

$$(p_1^\mu \frac{q_{2\mu}}{\sqrt{q_2^2}} - m)\,|\psi> = 0. \qquad (A.17)$$

On the constraints surface it can certainly be reduced to the form (A.15) with the same poles of the propagator. However, as we think the wave equation (A.17) can be related to the Dirac equation more easily in comparison with the equation (A.14). As the γ-matrices one could probably take $q_2^\mu/\sqrt{q_2^2}$ (see the analogous problem in Ref. /25/).

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