ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E2-90-207

L.V.Prokhorov*, S.V.Shabanov

PHASE SPACE OF THE YANG-MILLS FIELDS

Submitted to "Modern Physics Letters A"

*Leningrad State University, Department of Physics,
Stary Petergof, Uljanovskaja, 1, Leningrad, USSR

1990

1. Phase space structure in gauge models

As has been shown by one of the authors [1], the phase space (PS) of physical variables in gauge theories may differ from the conventional plane. In a number of models with both abelian and non-abelian gauge groups and one physical degree of freedom it has been found that its PS turns out to be a cone unfoldable into a half-plane. The reason is simple: a gauge orbit (a circle, a sphere, etc.) has two common points with the physical axis, say x_1, so that points x_1 and $-x_1$ are indistinguishable and the PS is the half-plane $x_1 \geqslant 0$ with identified points p_1 and $-p_1$ at $x_1 = 0$, i.e., a cone. It means, in particular, that after elimination of all unphysical variables there still exists a discrete gauge groups Z_2 with a nontrivial element $x_1 \to -x_1$ acting in the physical space. This phenomenon has some immediate physical consequences. For example, it leads to doubling of an oscillator frequency and as a result to doubling of spacing between energy levels in the quantum case [1]. In quantum theory the path integral approach changes because of the reduction of the physical PS [2,3]. It follows from the latter that the quasiclassical description also changes. In particular, the PS reduction influences energy levels of internal excitations of a quantum soliton [3,4], and the description of quantum-mechanical instantons is altered [3,4].

In the present paper we show that in the Yang-Mills theory with a semisimple gauge groups of a rank l PS of physical degrees of freedom for the field components at any space point differs from the plane. The physical PS reduction arises because after the elimination of all unphysical variables there remains a discrete gauge group acting in the PS of physical variables. This residual discrete gauge group (RDGG) cannot decrease a number of physical degrees of freedom but it decreases their PS identifying some points in it. RDGG is $(\otimes \mathbb{Z}_2)^{\ell} \otimes W$ where W is the Weyl group [5]. The latter is interpreted as the existence, in the system, of ℓ physical field components with PS being a cone equivalent to a half--plane and of the other ℓ components with PS being a 2ℓ-dimensional manifold diffeomorphic to $K^+ \otimes R^{\ell}$, where K^+ is the Weyl camera of the group [5].

2. Gauge fields with SU(2) group

First consider the simplest case of SU(2) group. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a}, \quad a=1,2,3. \qquad (1)$$

where $F^{a}_{\mu\nu}$ is a standard tensor [6]. Canonical momentum is $\pi^{a}_{\mu} = \partial\mathcal{L}/\partial \dot{A}^{\mu}_{a} = F^{a}_{\mu 0}$, so we have primary constraints $\pi^{a}_{0} = 0$ [7]. Define $\underline{\pi}_{k} = \pi^{a}_{k}\tau_{a}$, where τ_a are the Pauli matrices. Under gauge transformations they transform as

$$\underline{\pi}'_{k} = S\,\underline{\pi}_{k}\,S^{-1}, \quad S = \exp\left[i\omega^{a}(x)\tau_{a}\right] \qquad (2)$$

$\omega^{a}(x)$ in (2) are arbitrary function of x. We define the physical variables in the following manner. Choose gauge so that at $x=\mathring{x}$, $\pi^{1}_{1}(\mathring{x}) = \pi^{2}_{1}(\mathring{x}) = 0$ fixing two of three functions. In general, gauge transformations are generated both by primary and secondary constraints, as independent generators [7]. That is why we do not focus attention on unphysical variables A_0, $\underline{\pi}_0 = 0$. Note, however, that we still have gauge transformations changing sign of π^{3}_{1} without touching equalities $\pi^{1,2}_{1}(\mathring{x}) = 0$ (for example, $\tilde{S} = \exp\left[-i\mathring{\omega}(x)\tau_2\right]$, where $\mathring{\omega}(x) = \pi/2, x \in R_{\varepsilon}(\mathring{x})$ and $\mathring{\omega}(x) = 0, x \bar{\in} R_{\varepsilon}(\mathring{x})$, $R_{\varepsilon}(\mathring{x}) = \{x: |\underline{x}-\underline{\mathring{x}}|<\varepsilon, |t-\mathring{t}|<\varepsilon\}, \varepsilon\to 0$. It means that the physical momentum $\pi^{3}_{1} \in [0,\infty)$, while A^{3}_{1} changes

in the interval $(-\infty, \infty)$. According to [1] it means that PS of the canonical pair A^{3}_{1}, π^{3}_{1} is a cone unfoldable into a half-plane. This result follows also from the fact that $\tilde{S}$ changes sign of $A^{3}_{1}(\mathring{x})$ too, i.e. points A^{3}_{1}, π^{3}_{1} and $-A^{3}_{1}$, $-\pi^{3}_{1}$ are indistinguishable. Thus, physical PS of A^{3}_{1}, π^{3}_{1} is the cone con(π).

We are left with gauge transformations connected with $S' = \exp\left[i\upsilon(x)\tau_3\right]$ (stationary group of $\pi^{3}_{1}\tau_3$). The remaining arbitrariness can be fixed by demanding, say $\pi^{1}_{2} = 0$. Again there are discrete gauge transformations preserving this equality but changing sign of π^{2}_{2}, namely $\tilde{S}' = \exp\left[i\mathring{\omega}(x)\tau_3\right]$, $\mathring{\omega}(x)$ being the same as in $\tilde{S}$. So, PS of A^{2}_{2}, π^{2}_{2} is also a con(π).

We conclude that of nine pairs of canonical variables π^{a}_{k}, A^{a}_{k}, there are three unphysical and of the remaining physical ones two have conic PS con(π). It is necessary to say, the choice of concrete physical degrees of freedom with the conic PS is, in a sense, conventional. Indeed, in according to (2) the transformations $\tilde{S}$ and $\tilde{S}'$ from RDGG $= \mathbb{Z}_2 \otimes \mathbb{Z}_2$ act on all physical field components, so they identify points in the total physical PS [8]. However, it is important that such physical degrees of freedom can be picked out. We use this analysis as a pattern in the case of an arbitrary group.

3. The case of an arbitrary gauge group

Consider now the general case of an arbitrary compact simple gauge group G. Its Lee agebra X in Cartan-Weyl basis reads as [5]

$$[e_{\alpha}, e_{-\alpha}] = \alpha, \; [h, e_{\alpha}] = (h,\alpha)e_{\alpha}, \; [e_{\alpha}, e_{\beta}] = N_{\alpha\beta}\, e_{\alpha+\beta}, \qquad (3)$$

where α is a positive root, e_α is the corresponding root vector, $N_{\alpha\beta}$ are numbers and α, h belong to the Cartan subalgebra H. The scalar product (α, h) for elements $x, y \in X$ is defined as $(x,y) = Sp(\hat{x}\hat{y})$, where $\hat{x}$ is an operator in $X: \hat{x}y = [x,y]$. Evidently, canonical momenta $\pi_K \in X$ and we have the expansion

$$\pi_K = \sum_{\alpha>0}\left(\pi_K^{+\alpha} e_\alpha + \pi_K^{-\alpha} e_{-\alpha}\right) + \sum_{\omega\in\Pi} \pi_K^{\omega}\,\omega. \tag{4}$$

where $\pi_K^{\pm\alpha}$, π_K^{ω} are functions of x, Π is a set of simple roots and $\alpha > 0$ stands for summation over positive roots. Gauge transformations of π_K can be written in the form

$$\pi_K' = \hat{\rho}_g \pi_K \,, \quad \hat{\rho}_g = \exp \hat{\chi}(x) \tag{5}$$

equivalent to (2)(according to definition of the operator $\hat{\chi}$). For $\hat{\chi}$ there is an expansion analogous to (4)

$$\hat{\chi} = \sum_{\alpha>0}(\lambda_\alpha^+ \hat{e}_\alpha + \lambda_\alpha^- \hat{e}_{-\alpha}) + \sum_{\omega\in\Pi}\lambda_\omega \hat{\omega} \tag{6}$$

with $N\ (\equiv \dim G)$ arbitrary functions $\lambda_\alpha^{\pm}(x)$, $\lambda_\omega(x)$. After elimination of nonphysical variables A_0, π_0 ($\pi_0 = 0$) we are left with 3N pairs of canonical variables π_K, A_K of which only 2N are physical.

Now let us take π_1. As an element of X it can be represented in the form

$$\pi_1 = \exp\left[\sum_{\alpha>0}\left(\lambda_\alpha^+ \hat{e}_\alpha + \lambda_\alpha^- \hat{e}_{-\alpha}\right)\right]\pi_1^h \,, \quad \pi_1^h = \sum_{\omega\in\Pi}\pi_1^{\omega}\,\omega \in H\,. \tag{7}$$

Eq. (7) states that there are l physical components in π_1 ($l = \dim H =$ number of simple roots. In fact, Eq.(7) gives a general recipe for identification of physical variables in the adjoint representation). The rest of gauge transformations connected with operators

$$\hat{\rho}_h = \exp\left[\sum_{\omega\in\Pi}\lambda_\omega\hat{\omega}\right] \tag{8}$$

do not change π_1^h (they compose a stationary subgroup of π_1^h). But besides this group there is a discrete Weyl group W [5] acting in H, i.e., in the space of physical momenta π_1^h and simultaneously in the configuration space A_1^h. It cannot decrease the number of physical variables but it does reduce their PS. As is well known [5], any π_1^h can be obtained from $\pi_1^{h+} \in K^+$ by transformations from W, where K^+ is the Weyl camera, so that $(\pi_1^{h+}, \omega) > 0$, $\omega \in \Pi$. Thus physical momenta π_1^h belong to K^+ and PS of π_1^h, A_1^h is $\mathbb{R}^{2l}$ with all points $wA_1^h w^{-1}$, $w\pi_1^h w^{-1}$, $w \in W$ identified. PS of the π_1^h, A_1^h is, in fact, a hypercone equivalent to $K^+ \otimes \mathbb{R}^l$.

Now consider the gauge transformations connected with operators (8). According to (3) $\hat{\rho}_h e_\alpha = \exp(\varphi_\alpha) e_\alpha$, $\varphi_\alpha = \sum \lambda_\omega(\omega,\alpha)$, $\omega\in\Pi$. Then, applying $\hat{\rho}_h$ to π_2 one can nullify factors in front of l basic elements, say $e_\omega + e_{-\omega}$. Indeed:

$$\hat{\rho}_h \pi_2 = \pi_2^h + \frac{1}{2}\sum_{\alpha>0}\left\{\left[\pi_2^{+\alpha} e^{\varphi_\alpha} + \pi_2^{-\alpha} e^{-\varphi_\alpha}\right](e_\alpha + e_{-\alpha}) + \left[\pi_2^{+\alpha} e^{\varphi_\alpha} - \pi_2^{-\alpha} e^{-\varphi_\alpha}\right](e_\alpha - e_{-\alpha})\right\} \tag{9}$$

and the factors in front of $e_\omega + e_{-\omega}$ are zero if

$$\pi_2^{+\omega}\exp\left[\sum_{\omega'\in\Pi}\lambda_{\omega'}(\omega',\omega)\right] + \pi_2^{-\omega}\exp\left[-\sum_{\omega'\in\Pi}\lambda_{\omega'}(\omega',\omega)\right] = 0. \tag{10}$$

Matrix (ω_i, ω_j) is nondegenerate [5] (i and j enumerate simple roots), so Eqs. (10) are always solvable relative to λ_ω. Observation that Eqs. (10) are compatible with change of the sign of any basic element $e_\omega - e_{-\omega}$, $\omega \in \Pi$ concludes the consideration. The last statement follows from the solvability of the equations

$$\exp\left[\sum_{\omega''\in\Pi} \eta_{\omega''}\hat{\omega}''\right](e_{\omega}-e_{-\omega})=(1-2\delta_{\omega\omega'})(e_{\omega}-e_{-\omega}). \qquad (11)$$

where ω' is fixed. Eqs.(11) are also always solvable with respect to $\eta_{\omega''}$ because the $\ell\times\ell$-matrix (ω_i, ω_j) is nondegenerate. Therefore operator (8) with $\eta_{\omega''}$ satisfying Eqs. (11) gives for every ω' a nontrivial element of gauge group Z_2 changing sign of physical canonical variables $(\pi_2^{+\omega'}-\pi_2^{-\omega'})/\sqrt{2}$, $(A_2^{+\omega'}-A_2^{-\omega'})/\sqrt{2}$. It means that each of this l pairs of physical variables has a conic PS (π) unfoldable into a half-plane.

In the general case of a reductive group (which is the direct product of simple and abelian groups) the situation for the abelian subgroup is analogous to that of electrodynamics - the PS of the vector field physical variables is standard (full plane) because of the gauge invariance of the corresponding canonical momenta. Thus, for the reductive group of rank l PS of 2 ($\ell-\ell_a$) degrees of freedom reduces, where ℓ_a is a dimension of the invariant abelian subgroup.

Our results concern to the classical Yang-Mills theory. In the present letter, we shall not discuss the quantum theory and physical consequences of the PS reduction for Yang-Mills fields since this problem requires a special investigation. However, in the next point, we shall consider the other gauge field model in this respect.

4. PS structure in the Glashow-Weinberg-Salam model

Let us turn now to one immediate physical consequence of the PS reduction phenomenon. We state that the Higgs field in the Glashow-Weinberg-Salam (GWS) model cannot be elementary. After conveyance of the Higgs doublet $(\varphi^+,\varphi^0)=\varphi$ phase degrees of freedom to lon-

gitudinal components of vector fields we are left with a real scalar field $(0, \rho)$. Its PS is con(π) because ρ is positive $\rho>0$. Equivalently, we may present it as a transition $(\varphi^+,\varphi^0)\rightarrow(0,\tilde{\rho})$, where $\tilde{\rho}$ is a real field (i.e.,$-\infty<\tilde{\rho}<\infty$) with the residual discrete gauge group $\prod_x \otimes Z_2(x)$. Then we immediately get that $\langle\tilde{\rho}\rangle_0=0$ because $\tilde{\rho}$ changes sign under the gauge group Z_2 while vacuum (like any other physical state) is gauge invariant [5] (see Appendix). It means that $\tilde{\rho}$ cannot serve as Higgs field (it cannot develop a nonzero vacuum expectation value (VEV); it is easily seen that all Wightman functions of $\tilde{\rho}$ (i.e., $\langle\tilde{\rho}(x_1)\cdots\tilde{\rho}(x_n)\rangle_0$) also vanish, so $\tilde{\rho}$, in fact, does not exist in a proper sense). The field ρ may have a nonzero VEV but it cannot be considered as a normal scalar field because of its being positive. We could consider ρ as a composite field $\rho=(\bar{\varphi}\varphi)^{1/2}$. For renormalizability of GWS ρ should have a corpuscular manifestation, i.e., matrix elements of even numbers of fields $\bar{\varphi}$, φ should have proper poles in the momentum space. But it is the question of dynamics.

The hypothesis that the Higgs field is composed of two spinor fields look more plausible. In fact, we already have in physics an example of fenomena like that - the Cooper pairs in superconductivity which play the role of the Higgs field. After acquiring masses b vector fields there still exists a residual Z_2 gauge group changing the sign of the electron field ψ [6]. The bilinear combination of ψ (or more accurately a bound state of ψ in the local limit) is invariant under Z_2 and may have a nonzero VEV.

In conclusion we sould like to pay attention to Ref. 10 in which the authors also found for the Higgs expectation value $\langle \varphi \rangle_0 = 0$ in the model on the lattice: the two-point function also vanishes in the continuous limit in full accord with our results.

5. APPENDIX

There is some misunderstanding concerning the notion of "spontaneous breaking of gauge symmetry". Below we argue that in contrast with the global symmetry, the local one cannot be broken anyway.

In the case of spontaneous breaking of global symmetry all the Lagrangian symmetries are left untouched. It is the ground state that becomes "non-symmetric". For the gauge symmetry there is a principle difference - appearance of constraints. They are the conditions of self-consistency of dynamics and they fix unphysical variables. So the constraints neither can be "broken" nor omitted without chaning physics. Even when we break gauge invariance by fixing gauge we must be sure that the physical sector of the theory is left untouched.

All the physical quantities should be gauge invariants, i.e., they should not depend on unphysical variables. In quantum theory it means that all physical quantities should commute with the first class constraints, and the constraints must nullify all the physical states. It is jast because these constraints are generators of gauge transformations. In this sense gauge symmetry never can be violated and the Dirac condition on physical states [7] never can be violated or discarded.

What happens then in the case of the Higgs phenomenon? There are two approaches here: formal and physical one. In the formal approach the phase of the charged field φ becomes a longitudinal component

$A_k^{\parallel}$ of the vector field $A_k = A_k^{\parallel} + A_k^{\perp}$, $\partial_k A_k^{\perp} \equiv 0$. Usually it is tacitly assumed that it is an unphysical degree of freedom (it can be eliminated by gauge transformation). Because the component $A_k^{\parallel}$ of a massive field is physical one everything looks here so as if gauge symmetry were broken. But an unphysical variable cannot become physical one. The thorough analysis [1] shows that in fact both $A_k^{\parallel}$ and the phase of the field φ are linear combinations of physical and unphysical variables. In the "normal" case $\partial_k A_k = 0$ and the physical component of the phase describes the Coulomb field of charged particles. When the Higgs phenomenon takes place formally this physical degree of freedom goes to $A_k^{\parallel}$. Of course, in fact (i.e., in physical approach), it is change of the ground state described by the physical variables - that is all what happens in reality. The superconductivity gives us a clear example in this respect.

References

1. L.V.Prokhorov, Sov.J.Nucl.Phys. 35 (1982) 129
2. L.V.Prokhorov and S.V.Shabanov, Phys.Lett.B 216 (1989) 341.
3. S.V.Shabanov, Phase Space Structure in Gauge Theories (JINR, P2-89-533, Dubna, 1989) (in Russian).
4. L.V.Prokhorov and S.V.Shabanov, in: Topological Phases in Quantum Theory, ed. B.Markovski, and S.Vinitsky (WSPC, Singapore, 1989)
5. O.Loos, Symmetric Spaces (W.A.Benjamen, INC, N.Y.-Amsterdam, 1969).

6. C.N.Yang and RL Mills, Phys.Rev. 101 (1954) 1597
7. P.A.M.Dirac, Lectures on Quantum Mechanics (Yeshiva University, N.Y., 1964).
8. S.V.Shabanov, JINR Preprints No E2-90-23, E2-90-25, Dubna 1990
9. S.Weinberg, Prog.Theor.Phys.Suppl. 86 (1986) 43
10. J.Frohlich, G.Morchio and F.Strocchi, Nucl.Phys. B 190 (1981) 553.

Received by Publishing Department
on March 22, 1990.