объединенный
институт
ядерных
исследований
дубна

E2-90-205

D.I.Kazakov

CRITICAL EXPONENTS IN MATRIX MODELS

Submitted to "Journal of Physics A"

1990

1 Introduction

There are two main motivations to study matrix models. The first one is the existence of real physical objects such as liquid crystals [1] or liquid Helium-3 [2], which can be described within the models with a 3×3 matrix order parameter. In both cases, different phases could exist and a phase transition occurs. Though its nature is not yet clear, several characteristics have a singular behaviour which presumably can be described by the standard renormalization group approach.

The second motivation is connected with lattice formulation of 2-dim. quantum gravity. In the discrete approach based on dynamically triangulated random surfaces the theory can be regarded as $N \times N$ matrix field theory, where the sum over various genuses is simply the large N expansion [3]. Continuum limit actually corresponds to the critical point of the underlined field theory. For a number of models corresponding to conformal field theories with central charge $c \leq 1$ the critical exponents have been calculated from the discrete approach [3], while for the string theory $c > 1$, and in this regime calculations have so far failed.

In the present paper we write down matrix models and treat them according to the standard field theory approach and renormalization group method. Critical behaviour in the infrared region is studied and all the critical points are found. We show that the critical point, if it exists, corresponds to an n-vector model with an appropriate number of parameters, where the critical exponents have been calculated already with great accuracy.

2 The Model. Relation to Critical Phenomena

We consider the model with a single order parameter (field) which is an $N \times N$ real (Hermitian) matrix $\hat{\Phi}$. Symmetry properties are dictated by the form of this matrix and could be different in different phases. We concentrate below on three particular cases being irreducible representations of $SO(N)$ and $SU(N)$. Namely, we consider $\hat{\Phi}$ to be real traceless symmetric, antisymmetric and hermitian traceless matrices.

In the field theoretical approach to critical phenomena a crucial role is played by a Lagrangian rather than by free energy. To construct a Lagrangian, which is invariant under an appropriate symmetry group, we consider all possible invariants restricted by the renormalizability requirement. Having in mind ε-expansion, where dimension is $D = 4 - 2\varepsilon$, we are left with quadratic and quartic terms. Thus, we come to the Landau-Ginzburg type Lagrangian for a traceless field $\hat{\Phi}$:

$$\mathcal{L} = \frac{1}{2} Tr\, \partial\hat{\Phi}\partial\hat{\Phi} - \frac{m^2}{2} Tr\, \hat{\Phi}^2 - \frac{h_1}{4!} Tr\, \hat{\Phi}^4 - \frac{h_2}{4!} (Tr\, \hat{\Phi}^2)^2. \quad (1)$$

Three different choices of the matrix $\hat{\Phi}$ are distinguished by the form of the propagator. We have respectively :

$$\begin{aligned} symmetric \quad \widehat{\hat{\Phi}^{ab}\hat{\Phi}^{cd}} &= \frac{i}{p^2 - m^2}\frac{1}{2}(\delta^{ad}\delta^{bc} + \delta^{ac}\delta^{bd} - \frac{2}{N}\delta^{ab}\delta^{cd}), \\ antisymmetric \quad \widehat{\hat{\Phi}^{ab}\hat{\Phi}^{cd}} &= \frac{i}{p^2 - m^2}\frac{1}{2}(\delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd}), \\ hermitean \quad \widehat{\hat{\Phi}^{ab}\hat{\Phi}^{cd}} &= \frac{i}{p^2 - m^2}(\delta^{ad}\delta^{bc} - \frac{1}{N}\delta^{ab}\delta^{cd}), \end{aligned}$$

where $a, b, c, d = 1, 2, \ldots, N$.

Critical phenomena are associated with the infrared properties of the model. Scaling behaviour in the vicinity of the critical point caused by the appearance of a long-range order can be described in terms of Euclidean quantum field theory possessing an infra-red stable fixed point [4].

A systematic approach to the description of infra-red asymptotics is based on the renormalization group. In the presence of infra-red stable fixed points defined by the vanishing of RG β-functions the dimensionless Green functions obey the scaling laws for small p^2

$$\Gamma_R \stackrel{p^2\to 0}{\sim} (p^2)^{-\gamma_\Gamma(h^*)}$$

with the powers γ_Γ equal to the anomalous dimensions at $h = h^*$. There exist direct relations between the anomalous dimensions and critical exponents, which characterize the scaling behaviour of various quantities in the neighbourhood of a second order phase transition. For example, the critical exponents η (correlation function) and ν (correlation length) can be expressed through the anomalous dimension of the field and mass, respectively [5,6]

$$\eta = 2\gamma_2(h^*), \quad \nu = \frac{1}{2(1 - \gamma_m(h^*))}. \quad (2)$$

All other critical exponents are not independent and can be evaluated via the scaling laws

$$\gamma = (2 - \eta)\nu, \quad \alpha = 2 - \nu D, \quad \beta = \frac{\gamma}{\delta - 1}, \quad \delta = \frac{D + 2 - \eta}{D - 2 + \eta}.$$

Critical point h^* is the infra-red stable fixed point of the renormalization group equation. Within the ε-expansion method it is a power series of ε calculated in perturbation theory.

3 Renormalization Group Equations. Fixed Points

In this section we consider RG equations for the effective couplings of the model at hand in $4 - 2\varepsilon$ dimensions. Remind that in the MS-scheme the β-functions in $4 - 2\varepsilon$ dimensions are connected with those in 4 dimensions by the equation [7]

$$\beta_{4-2\varepsilon}(h) = -\varepsilon h + \beta_4(h).$$

Fixed points correspond to the r.h.s equal to zero. They are all the power series of ε :

$$h_i^* = \varepsilon u_i^1 + \varepsilon^2 u_i^2 + \cdots,$$

where the coefficients u_i^k are determined in k-th order of perturbation theory.

Having this in mind we get the following RG equations written to one-loop order in symmetric, antisymmetric and Hermitian cases, respectively:

3.1 Symmetric matrix, SO(N)

$$\begin{aligned} \dot{h}_1 &= -\varepsilon h_1 + \frac{2N^2 + 9N - 36}{12N} h_1^2 + 2h_1 h_2, \\ \dot{h}_2 &= -\varepsilon h_2 + \frac{N^2 + 6}{4N^2} h_1^2 + \frac{2N^2 + 3N - 6}{6N} h_1 h_2 + \frac{N^2 + N + 14}{12} h_2^2. \end{aligned} \quad (3)$$

According to the general analysis [4, Sect XI] there are four types of fixed points of eq.(3):

1. $u_1 = u_2 = 0$;
2. $u_1 = 0, u_2 = \frac{12}{N^2+N+14}$;
3. $u_1 = \begin{cases} 12/85 & N = 2 \\ -4/39 & N = 3 \\ absent & N > 3 \end{cases}, \quad u_2 = \begin{cases} 9/17 & N = 2 \\ 20/39 & N = 3 \\ absent & N > 3 \end{cases}$;
4. $u_1 = \begin{cases} -12/17 & N = 2 \\ -4/3 & N = 3 \\ absent & N > 3 \end{cases}, \quad u_2 = \begin{cases} 6/17 & N = 2 \\ 2/3 & N = 3 \\ absent & N > 3 \end{cases}$.

The situation is illustrated in Fig.1. Before analysing the stability properties of these fixed points it is useful to note that real symmetric $N \times N$ matrices for $N = 2$ and 3 obey the following equation:

$$Tr\, \hat{\Phi}^4 = \frac{1}{2}(Tr\, \hat{\Phi}^2)^2. \quad (4)$$

Hence for $N = 2, 3$ there exists only one independent coupling in eq.(1) equal to $(h_1 + 2h_2)(Tr\, \hat{\Phi}^2)^2$. Looking for the value of the coupling $h_1 + 2h_2$ at the critical points

(1 - 4) we find out that it is the same for the points 1 and 4 as well as for 2 and 3, respectively. Thus, the presence of four different points for $N = 2,3$ is just an artefact, and there are only two relevant fixed points for any value of N.

Stability properties of the fixed points can be investigated in a standard way. The fixed point 1 is absolutely infra-red unstable and the fixed point 2 is a saddle point. The phase portrait of the trajectories is shown in Fig.2. One can see that the fixed point 2 can be reached only when the coupling $h_1 = 0$. In this case, the fixed point is infra-red stable and according to a general belief corresponds to a second order phase transition one. Otherwise, there is no fixed point solution of eq.(3).

3.2 Antisymmetric matrix, SO(N)

$$\begin{aligned} \dot{h}_1 &= -\varepsilon h_1 + \frac{2N-1}{12}h_1^2 + 2h_1h_2, \\ \dot{h}_2 &= -\varepsilon h_2 + \frac{1}{4}h_1^2 + \frac{2N-1}{6}h_1h_2 + \frac{N^2-N+16}{12}h_2^2. \end{aligned} \tag{5}$$

The situation here is exactly the same as in the previous section. There are four types of the fixed points:

1. $u_1 = u_2 = 0$;

2. $u_1 = 0, u_2 = \frac{12}{N^2-N+16}$;

3. $u_1 = \begin{cases} 2/9 & N=2 \\ -12/77 & N=3 \\ absent & N>3 \end{cases}, \quad u_2 = \begin{cases} 4/9 & N=2 \\ 36/77 & N=3 \\ absent & N>3 \end{cases};$

4. $u_1 = \begin{cases} -4/3 & N=2 \\ -12/7 & N=3 \\ absent & N>3 \end{cases}, \quad u_2 = \begin{cases} 2/3 & N=2 \\ 6/7 & N=3 \\ absent & N>3 \end{cases}.$

Here again equation (4) is valid for $N = 2,3$, i.e. again only two points are relevant. Qualitative picture repeats that shown in Figs.1, 2.

3.3 Hermitian matrix, SU(N)

$$\begin{aligned} \dot{h}_1 &= -\varepsilon h_1 + \frac{N^2-9}{3N}h_1^2 + 2h_1h_2, \\ \dot{h}_2 &= -\varepsilon h_2 + \frac{N^2+3}{2N^2}h_1^2 + \frac{2N^2-3}{3N}h_1h_2 + \frac{N^2+7}{6}h_2^2. \end{aligned} \tag{6}$$

The situation here is only slightly different from those of the previous sections. The fixed points are:

1. $u_1 = u_2 = 0$;

2. $u_1 = 0, u_2 = \frac{6}{N^2+7}$;

3. $u_1 = \begin{cases} 6/121 & N=2 \\ -1/4 & N=3 \\ absent & N>3 \end{cases}, \quad u_2 = \begin{cases} 63/121 & N=2 \\ 1/2 & N=3 \\ absent & N>3 \end{cases};$

4. $u_1 = \begin{cases} -6/11 & N=2 \\ -1 & N=3 \\ absent & N>3 \end{cases}, \quad u_2 = \begin{cases} 3/11 & N=2 \\ 1/2 & N=3 \\ absent & N>3 \end{cases}.$

In this case equation (4) is also valid for $N = 2,3$. Hence, qualitatively we have the same picture as before. There are essentially two fixed points for any value of N.

So far, we have considered the leading approximation. However, the obtained results are stable with respect to higher order corrections. In any loop order the infra-red fixed point will lie on the h_2 axis being the power series of ε

$$h_1^* = 0, \quad h_2^* = u_2^1\varepsilon + u_2^2\varepsilon^2 + \cdots.$$

It is a saddle point in the coupling constant space. This will be true for all three models considered above.

It should be stressed that the conclusion is valid for any value of ε , i.e. for any value of D.

4 Critical Exponents

To find the critical exponents one has to calculate the anomalous dimensions at the infra-red fixed point. The results will be expressed via the power series of ε. However, there is no necessity to perform any new calculation. Indeed, if one looks at the Lagrangian, eq.(1), at the fixed point, one finds out that the only coupling which survives is $(Tr\ \hat{\Phi}^2)^2$. Then, expanding the matrix field $\hat{\Phi}$ over the irreducible set of matrices in an appropriate representation $\hat{\Phi} = \sum T^i\phi^i$, we get

$$(Tr\ \hat{\Phi}^2)^2 \sim (\phi^i\phi^i)^2,$$

where we have taken into account that $Tr\ T^iT^j \sim \delta^{ij}$.

Thus, what we finally get is the n-vector model with the number of components equal to that of the original matrix. For the three cases of interest we have respectively

$$n = \begin{cases} \frac{(N-1)(N+2)}{2} & symmetric\ SO(N) \\ \frac{(N-1)N}{2} & antisymmetric\ SO(N) \\ N^2-1 & hermitian\ SU(N) \end{cases}.$$

Critical exponents in the n-vector model have been calculated with high accuracy. Recent most accurate estimates have been achieved in the framework of ε-expansion [8]. We present below the results of five-loop calculations for ν and η for arbitrary value of n [6,9]:

$$
\begin{aligned}
\eta &= (2\varepsilon)^2\frac{n+2}{2(n+8)^2} + (2\varepsilon)^3\frac{n+2}{8(n+8)^4}(-n^2+56n+272)\\
&- (2\varepsilon)^4\frac{n+2}{32(n+8)^6}(5n^4+230n^3+1183.949n^2+10698.571n+35095.814)\\
&+ (2\varepsilon)^5\frac{n+2}{128(n+8)^8}(6.2336n^6-282.5728n^5+15724.800n^4\\
&+ 561446.40n^3+5088128n^2+21985280n+38204160),
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\frac{1}{\nu} &= 2-(2\varepsilon)\frac{n+2}{n+8}-(2\varepsilon)^2\frac{n+2}{2(n+8)^3}(13n+44)\\
&+ (2\varepsilon)^3\frac{n+2}{8(n+8)^5}(3n^3+124.9873n^2+4482.6427n+14997.953)\\
&- (2\varepsilon)^4\frac{n+2}{32(n+8)^7}(-60.6987n^5+3967.5948n^4\\
&+ 111115.32n^3+997507.91n^2+4907350.4n+9646986.2)\\
&- (2\varepsilon)^5\frac{n+2}{128(n+8)^9}(-91.1731n^7+13165.389n^6+502397.44n^5\\
&+ 12698829n^4+184391780n^3+1361732900n^2+5154359300n\\
&+ 7846792400).
\end{aligned} \tag{8}
$$

As can be seen from eqs.(7) and (8), the coefficients grow very fast which is a manifestation of asymptotical character of ε-expansion. This means that to get a numerical result for $D=2$ or 3 (i.e. $\varepsilon = 1$ or $1/2$) one needs a special summation procedure. The latter was proposed in a number of papers [5,6,8]. The results obtained for small values of n are in very good agreement with experiment as well as with other approaches. The procedure can be repeated for any value of n.

In a recent paper [10] we have proposed an empirical expression for the correlation length critical exponent ν. It is valid for arbitrary n and D and fits all known exact and numerical values. Even if it is not an exact solution, the advocated result can serve as a very accurate approximation to the true value. It has the following form:

$$
\nu = \frac{(D-2)(3x+2)-x}{2(D-2)(3x+2)-2x-(4-D)(x+2)}, \tag{9}
$$

where the parameter x is connected with n by the equation

$$
x = \begin{cases} \frac{n-6}{4} & for\ even\ n \\ \frac{n-[\frac{n}{4}]-5}{3} & for\ odd\ n \end{cases} \tag{10}
$$

For $D=3$ eq.(9) gives a smooth curve

$$
\nu = \frac{3+x}{4+x}, \tag{11}
$$

shown in Fig.3.

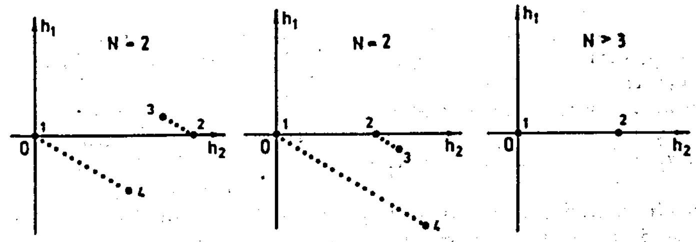

Figure 1: Fixed points for various values of N.

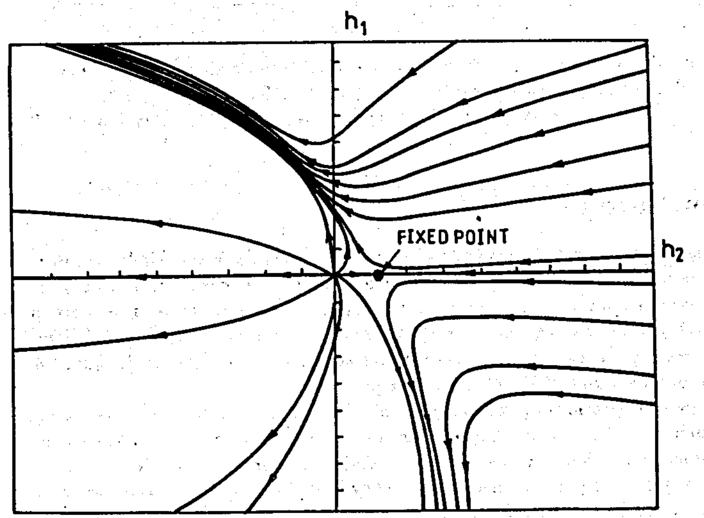

Figure 2: Phase portrait of solutions for $N > 3$. The arrows show the direction of decreasing argument t corresponding to the infra-red limit.

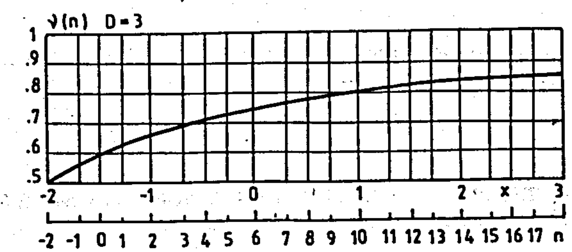

Figure 3: The critical exponent ν as a function of n for $D=3$.

Thus, to get the values of the critical exponents for the matrix model one has to substitute an appropriate value of $n = n(N)$ into eqs.(7) and (8) or directly into eqs.(9),(10) and (11).

Special attention is paid to the $N = \infty$ case. For the matrix model, eq.(1), it corresponds to taking into account of planar diagrams in all orders of perturbation theory. However, at the critical point due to the absence of the $Tr\ \hat{\Phi}^4$ term the situation is drastically simplified. For the $(Tr\ \hat{\Phi}^2)^2$ interaction (or equivalently for the n-vector model) in the large N limit only one-loop diagrams survive. As can be seen from eqs.(7) and (8) the results for $n = \infty$ are

$$\eta = 0, \quad \nu = \frac{1}{D-2}. \tag{12}$$

This corresponds to the so-called spherical model which admits an exact solution [11]. Strictly speaking, eq.(12) is valid only for $D = 3$ or 4 as far as for $D \leq 2$ the phase transition for large $n \geq 2$ disappears. For $D = 3$ we get $\eta = 0$, $\nu = 1$ in accordance with eq.(11).

5 Conclusion

We have shown above that the critical behaviour in matrix models of eq.(1) is determined by the presence of nontrivial infra-red fixed point. Our conclusion is true in all orders of perturbation theory for any value of ε and N. This fixed point is believed to be associated with the second order phase transition with the matrix order parameter. As we have seen, it exists only when the interaction is essentially $(Tr\ \hat{\Phi}^2)^2$. Addition of arbitrary small amount of $Tr\ \hat{\Phi}^4$ destroys a phase transition.

At this point, our conclusion contradicts that of ref.[12], where different phases were obtained with critical exponents having negative values. From our point of view the negative values of exponents mean that the formulas are out of the range of applicability. A way out of this descrepancy is probably in different meaning attached to a phase transition and connection between the field theory model and statistical system.

Acknowledgements

The author would like to thank A.L.Kuzemsky, O.V.Tarasov, V.B.Priezzhev and V.A.Zagrebnov for numerous useful discussions.

References

[1] P.G. de Gennes, *The Physics of Liquid Crystals*, Oxford Clarendon, 1974; N.Angelescu and V.Zagrebnov, *Journ. Phys. A* **15** (1982) L639.

[2] G.Barton and M.A.Moore, *Journ. Phys. C* **7** (1974) 4220.

[3] V.A.Kazakov and A.A.Migdal, *Nucl. Phys.* **B20** (1989) 654.

[4] E.Brezin, J.C.Le Guillou and J.Zinn-Justin in *Phase Transitions and Critical Phenomena* Vol.6, Ed. C.Domb and M.S. Green, Academic Press 1976, p.125.

[5] J.Zinn-Justin, *Phys. Rep.* **70** (1981) 109.

[6] A.A.Vladimirov, D.I.Kazakov and O.V.Tarasov, *Zh. Eksp. Teor. Fiz.* **77** (1979) 1035; (*Sov. Phys. JETP* **50** (1979) 521).

[7] G.'t Hooft, *Nucl.Phys.* **B61** (1973) 455.

[8] J.C.Le Guillou and J.Zinn-Justin, *J.Physique Lett.* **46** (1985) L137; *J.Physique* **48** (1987) 19; ibid **50** (1989) 1365.

[9] K.G.Chetyrkin, A.L.Kataev and F.V.Tkachov, *Phys. Lett.* **99 B** (1981) 147, Er. ibid **101 B** (1981) 457; S.G.Gorishny, S.A.Larin and F.V.Tkachov, *Phys. Lett.* **101 A** (1984) 120.

[10] D.I.Kazakov, *Exact critical exponent for n-vector model with arbitrary n and D*, JINR Preprint E17-89-79, Dubna, 1989.

[11] M.E.Fisher, *The Theory of Critical Point Singularities* in 1970 Enrico fermi Summer School of Critical Phenomena, Italy, 1970.

[12] S.R.Das et al, *New critical behaviour in $d = 0$ large N matrix models*, Preprint TIFR/TH/89-70, Bombay, 1989.

Received by Publishing Department
on March 21, 1990.