ОБЪЕДИНЕННЫЙ
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ДУБНА

E2- 8995

G.Lassner, W.Timmermann

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1975

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Submitted to Reports on Mathematical Physics

Лассснер Г., Тиммерман В. E2 - 8995

Классификация областей определения алгебр операторов

Исследуется структура областей определения алгебр $\mathfrak{A}(D)$. Особое внимание уделено случаю, когда область является пространством Фреше относительно естественной топологии. Приведена полная классификация областей D вида $D = \bigcap_{n \geq 0} D(A^n)$, где $A = A^*$ – самосопряженный оператор.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований
Дубна 1975

Lassner G., Timmermann W. E2 - 8995

Classification of Domains of Operator Algebras

The structure of domains of Op^*-algebras $\mathfrak{A}(D)$ is investigated; especially, there is considered the case where the domain D is a Frechet space with respect to a natural topology. We obtain a complete classification and description of domains D of the form $D = \bigcap_{n \geq 0} D(A^n)$, $A = A^*$ – a self-adjoint operator.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research
Dubna 1975

0. Introduction

In [12] there was given a complete classification of the domains of closed operators in a Hilbert space and a description of the structure of these domains. Now we present a first part of a classification of another type of domains, the domains of operator $*$-algebras (Op^*-algebras). Thereby we restrict ourselves in this paper to the special but very important case, where the domain is a Frechet space with respect to a certain natural topology.

1. Preliminaries

We use the same notations and notions as in [12]. Let H be a separable Hilbert space, $D \subset H$ a dense linear manifold, $\langle , \rangle$, $\| \ \|$, resp., the scalar product, the norm resp. in H. A *kernel* of D is an infinitely dimensional closed subspace $N \subset D$. A kernel N is said to be maximal, if there is no kernel $N' \subset D$, $N \subset N'$ such that $\dim (N' \ominus N) = \infty$. By $\mathfrak{L}^+(D)$ we denote the set of all linear operators A from D into D, $AD \subset D$ such that $D \subset D(A^*)$ and $A^* D \subset D$. $\mathfrak{L}^+(D)$ is a $*$-algebra with respect to the usual operations, and the involution is defined by $A \to A^+$, where A^+ is the restriction of A^* to D. A $*$-subalgebra $\mathfrak{A}(D) = \mathfrak{A}$ of $\mathfrak{L}^+(D)$ containing the identity I will be called Op^*-algebra. An Op^*-algebra $\mathfrak{A}(D)$ is said to be *closed*, if

$$D = \bigcap_{A \in \mathfrak{A}} D(\bar{A}) .$$

Equivalently, an $0p^*$ -algebra $\mathfrak{A}(D)$ is closed, if the domain D equipped with the topology $t_{\mathfrak{A}}$ defined by the seminorms

$$\|\phi\|_A = \|\phi\| + \|A\phi\|, \ A \in \mathfrak{A}, \ \phi \in D$$

is complete (further considerations concerning $0p^*$ -algebras are contained, for example, in /9, 10, 11, 19/). By a *diagonal operator* we mean an operator $S = S\{(a_n), (\phi_n)\}$ with the domain

$$D = D(S) = \{\phi = \Sigma x_n \phi_n : \Sigma |x_n|^2 |a_n|^2 < \infty\},$$

where $\{\phi_n\}$ is an orthonormal basis of H contained in D and $\{a_n\}$ a sequence of real numbers unbounded if $D \neq H$. Two linear manifolds D, D' are called *linearly (unitarily resp.) equivalent* if $D' = KD$, where K is a bounded operator with bounded inverse (a unitary operator resp.).

In what follows we need some notions about sequences. Let (a_n) be a sequence and (b_n) a subsequence of (a_n), $(b_n) \subset (a_n)$ then by $(d_n) = (a_n) - (b_n)$ we denote the subsequence of (a_n) obtained from this by cancelling the elements (b_n). We write $(a_n) = (b_n) \cup (d_n)$. Let (a_n) be a sequence of naturals. By $(a_n)'$ we denote the set of accumulation points of (a_n) and by (a'_n) we denote the sequence $(a_n)' - \{\infty\}$ such that $a'_1 \leq a'_2 \leq \ldots$.

An *iterated* sequence $(\hat{a}_n) = (a_n^\sigma)$ is a sequence obtained from (a_n) in the following way: Let σ be a monotone map from N onto itself, then $(\hat{a}_n) = (a_n^\sigma) = (a_{\sigma(n)})$. For example, $(\hat{a}_n)$ could be $a_1, a_1, a_2, a_3, a_3, a_3, \ldots$. We call σ the "iteration" and, for brevity, we will often use the same sign "^" for any iteration.

Let $(a_n), (b_n)$ be two sequences of positive numbers. One says that (a_n) is *majorized* by (b_n), $((a_n) < (b_n))$, if there is a constant $C > 0$ such that $a_n \leq C \cdot b_n$ for all n .

In /12/ we defined different notions of equivalence of such sequences to obtain a description of the structure of domains of closed operators. Now we deal with systems of sequences and give the following

Definition 1

Let $\mathfrak{F}_1 = \{(a_n)\}$, $\mathfrak{F}_2 = \{(b_n)\}$ be two systems of sequences of positive numbers. $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are said to be *equivalent* ($\approx$) , if any sequence $(a_n) \in \mathfrak{F}_1$ can be majorized by a suitable sequence $(b_n) \in \mathfrak{F}_2$ and conversely, any sequence of $\mathfrak{F}_2$ can be majorized by a suitable sequence of $\mathfrak{F}_1$ *essentially equivalent* ($\overset{\approx}{e}$) , if there are $s, t \in N$ such that the systems $\{(a_{s+n})\}$ and $\{(b_{t+n})\}$ are equivalent; *weakly equivalent* ($\overset{\approx}{w}$), if there are suitable monotone maps σ, τ of N onto itself such that the systems $(\mathfrak{F}_1)_\sigma, (\mathfrak{F}_2)_\tau$ of iterated sequences $\{(a_1^\sigma)\}$, $\{(b_2^\tau)\}$ obtained from σ and τ are equivalent.

If the systems $\mathfrak{F}_1$ and $\mathfrak{F}_2$ contain only one sequence, we obtain the same notions as in /12/, Definition 1.

2. Classification of the Domains of Op^* -Algebras

In this section we give a classification of the domains of $0p^*$ -algebras and note some properties of the structure of these domains.

First we remark that these considerations are proper extensions of those of /12/ because of the following fact /9, 18, 19/:

Let D be the domain of an $0p^*$-algebra containing at least one unbounded operator. Then D annot be the domain of a closed operator, i.e., these two types of domains are essentiall different.

With the following classification we continue the enumeration begun in /12/.

C_4: *Classification of the domains of closed Op^*-algebras*

Let $\mathfrak{A}(D)$ be a closed $0p^*$ -algebra. D is said to be of

Class I iff D does not contain a kernel,
Class II iff D contains a maximal kernel,
Class III iff D contains kernels but not maximal one.

We write $D \in I$, $\in II$ and $\in III$.

For the domains of closed operators in /12/ there was given the same definition and this geometrical picture was translated in the language of equivalence of sequences. The classification C_4 is complete and disjoint. No class is empty, as can easily be seen (cf. also section 3).

The following remark shows roughly speaking that the class to which a domain D belongs is not determined by the class to which $D(\bar{A})$ belongs, for all $A \in \mathcal{L}^+(D)$ in an invariant manner.

Remark 1

Let D be the domain of closed Op^*-algebra. Then there can arise the following cases:

i) $D \in I$, then there are $A, B \in \mathcal{L}^+(D)$ such that $D(\bar{A}) \in II$, $D(\bar{B}) \in III$.

ii) $D \in II$, then there is an $A \in \mathcal{L}^+(D)$ such that $D(\bar{A}) \in III$, but there is no $B \in \mathcal{L}^+(D)$ such that $D(\bar{B}) \in I$.

iii) $D \in III$, then there is an $A \in \mathcal{L}^+(D)$ such that $D(\bar{A}) \in II$, but there is no $B \in \mathcal{L}^+(D)$ such that $D(\bar{B}) \in I$.

Proof

All statements can be proved in a similar way by examples constructed from diagonal operators. We regard for example i) and construct the operator B.

Let $H = \ell^2$, $D = s = \{(a_n) : \Sigma n^l |a_n|^2 < \infty \ \forall\, l \in N\}$.

Let $\phi_n = (0, \dots, 1, 0, \dots)$ where the 1 stands as the n th element. Now regard an arbitrary decomposition of the set of naturals: $N = \cup M_n$, M_n infinite sets, $M_i \cap M_k = \phi$ for $k \neq i$. Put $m_n = \min M_n$ and let C be the following diagonal operator:

$$C = C\{(b_n), (\phi_n)\} \quad \text{with } b_j = m_n \quad \text{for all } j \in M_n .$$

It can easily be seen that $D \subset D(C)$ and $B = C|_D \in \mathcal{L}^+(D)$ and $D(\bar{B}) = D(C) \in III$, but D obviously $\in I$. Q.E.D.

For a detailed classification and description of the structure of the domains it is important to investigate the following

Problem

Under which conditions the domain D of a closed Op^*-algebra $\mathfrak{A}(D)$ can be represented in the form

$$D = \{\phi = \Sigma x_n \phi_n : \Sigma |x_n|^2 (a_n^{(\alpha)})^2 < \infty \quad \alpha \in A\}$$

$(\phi_n) \subset D$ an orthonormal basis, $\{(a_n^{(\alpha)}), \alpha \in A\}$ a suitable system of positive sequences. In other words, D is isomorphic to the space of sequences $D^s \subset \ell^2$:

$$D^s = \{(x_n) : \Sigma |x_n|^2 (a_n^{(\alpha)})^2 < \infty \ \ \forall\, \alpha \in A\} .$$

Then with the diagonal operators $T_\alpha = T_\alpha\{(a_n^{(\alpha)}), (\phi_n)\}$, $\alpha \in A$

$$D = \bigcap_{\alpha \in A} D(T_\alpha) .$$

The crucial point of this problem is the fact that the basis (ϕ_n) must be the same for all operators T_α.

This problem leads to the following

Statement

If the domain D of a closed Op^*-algebra $\mathfrak{A}(D)$ can be realized as a space of sequences, then the maximal Op^*-algebra $\mathcal{L}^+(D)$ is selfadjoint, that is,

$$D = \bigcap_{A \in \mathcal{L}^+(D)} D(\bar{A}) = \bigcap_{A \in \mathcal{L}^+(D)} D(A^*)$$

(cf. also /1, 8, 11, 19/).

In general it is an open question whether $\mathcal{L}^+(D)$ is selfadjoint in any case where D is the domain of an arbitrary closed Op^*-algebra.

In what follows we regard such algebras $\mathfrak{A}(D)$ in which the topology $t_{\mathfrak{A}}$ is metrizable, i.e., $D[t_{\mathfrak{A}}]$ is an F-space because we regard only closed algebras. Therefore one has

$$D = \bigcap_n D(A_n), \quad A_n|_D \in \mathcal{L}^+(D) . \tag{1}$$

A special case is the following

$$D = D^\infty(T) = \bigcap_{n \geq 0} D(T^n) . \quad T = T^*, \quad T|_D \in \mathcal{L}^+(D) . \tag{2}$$

Without loss of generality one always may assume that $T = T^* \geq I$, $I \leq A_n \subset A_n^*$ and $D(A_{n+1}) \subset D(A_n)$ for all n. Domains of form (2) were regarded by many authors from other points of view as we do it here. These were regarded, for example, in connection with differential operators or in the interpolation theory as centers of Hilbert- or Banach-scales. (Cf. for example, /8, 13 – 16, 20/ and the references there). We here only remark that an important example for a space of form (2) is the Schwartz-space $\mathcal{S}$ of rapidly increasing functions.

Remark 2

Obviously, the class of domains of form (2) is contained in the class of domains of form (1), but the two classes do not coincide, i.e., there are domains $D = \bigcap_{n \geq 0} D(A_n)$ such that there is no closed operator T with $D = \bigcap_{n \geq n} D(T^n)$

We give an example:

Let $\{\phi_n\}$ be an orthonormal basis in a Hilbert space H, (t_n) a sequence of naturals with $\lim t_n = \infty$. Further let $\{M_n\}$ be a sequence of infinite sets with the following properties:

1. $M_n \subset N$, i.e., M_n is an infinite set of naturals,
2. $M_{n+1} \subset M_n$,
3. the sets $N_1 = N - M_1$, $N_j = M_{j-1} - M_j$ are infinite for all $j > 1$.

At last we use the notion (t_{j+n}) for the sequence $(t_j, t_{j+1}, \ldots)$. Now we define a system of diagonal operators $A_n = A_n\{(a_l^{(n)}), (\phi_l)\}$ as follows:

$$a_l^{(1)} = \begin{cases} t_l & \text{if } l \in N_1, \\ 1 & \text{if } l \in M_1, \end{cases}$$

and for $n > 1$:

$$a_l^{(n)} = \begin{cases} t_l & \text{if } l \in N_1, \\ t_{j+1} & \text{if } l \in N_j \text{ for } 2 \leq j \leq n, \\ 1 & \text{if } l \in M_n. \end{cases}$$

For the operators A_n constructed above there hold:

1. $D(A_n) = H_n \oplus D_n$, $D_n \in I$, H_n closed, infinitely dimensional,
2. $D(A_{n+1}) \subset D(A_n)$ more precisely:

$$H_{n+1} \subset H_n, \quad D_n \subset D_{n+1},$$

$$H_n = H_{n+1} \oplus \hat{H}_{n+1}, \; D_{n+1} = D_n \oplus \hat{D}_{n+1}, \; \hat{D}_{n+1} \subset \hat{H}_{n+1}.$$

Roughly speaking the operators A_n are formed in such a way that, if A_n is bounded on D_n, A_{n+1} is bounded only on an infinitely dimensional subspace of H_n and unbounded on the also infinitely dimensional orthogonal complement.

3. The representation

$$D = \bigcap_{n \geq 0} D(T^n), \qquad T \text{ closed}$$

cannot hold, were D is defined to be the domain of the algebra generated by the operators A_n, i.e.,

$$D = \bigcap_{m,n \geq 0} D(A_n^m).$$

In fact, from the closed graph theorems it would otherwise follow that

$$\|T\phi\| < C(\|A_l^k \phi\| + \|\phi\|) \qquad \text{for suitable } k, l \in N, \; C > 0 \text{ and all } \phi \in D.$$

But then the boundedness of A_l^k on the dense subspace $H_l \cap D$ of H_l leads to

$$\|T\phi\| \leq C(\|A_l^k \phi\| + \|\phi\|) \leq D\|\phi\| \qquad \text{for all } \phi \in H_l \cap D.$$

This means that T is bounded on $H_l \cap D$ and because T is closed we have $H_l \subset D(T)$ therefore $H_l \subset D$ which is a contradiction with 2. This concludes the example. Because the topology $t_{\mathcal{G}}$ on the domains of form (1) can be given by a system of scalar products $\{\langle , \rangle_n\}$ these domains are countable Hilbert spaces /6/. With respect to this and in connection with the construction of the space D^s of sequences isomorphic to D we recall the following

Definition /2, 5/

A system (x_n) of elements of a linear topological space E is called a basis of E, if there is a system

(f_n) of linear functionals $f_n \in E'$ such that for any $x \in E$ there holds the unique representation

$$x = \sum_{n=1}^{\infty} f_n(x)\, x_n. \tag{3}$$

The basis (x_n) is called *unconditional* if $(x_{\pi(n)})$ is a basis for any permutation π of N, or equivalently, if the series (3) is unconditionally convergent for any $x \in E$. In [15] the following fact is mentioned:

Proposition

Let D be a countable Hilbert space $D = \cap H_n$ with the topology τ defined by the scalar products $\{<,>_n\}$. If (x_j) is an unconditional basis in D, then there is a system $\{<,>_n'\}$ of scalar products defining the same topology τ such that (x_j) is an orthogonal system with respect to any $<.>_n'$. Hence we obtain the following

Conclusion

If the domain $D = \cap D(A_n)$ contains an unconditional basis (ψ_n) then there are diagonal operators $T_1 = T_1\{(t_n^{(1)}),(\psi_n)\}$ such that $D = \cap D(T_1)$ namely, we choose $t_n^{(1)} = \|\psi_n\|_1'$, where $\| \ \|_1'$ denotes the norm corresponding to the new scalar product $<,>_1'$. In the special case $D = D^{\infty}(T) = \cap D(T^n)$, $T = T^*$ from the spectral theorem it can be easily obtained: there is an unconditional basis $(\psi_n) \subset D$ such that $D = D^{\infty}(A)$, where $A = A\{(a_n),(\psi_n)\}$ is a suitable diagonal operator.

Thus, the existence of an (unconditional) basis gives the possibility to regard D as a sequence space. But to speak about *the* sequence space associated with D it must be shown that this space is independent of the choice of the (unconditional) basis. This is the "problem of quasi-equivalence of unconditional bases" which can be formulated as follows.

Let $D[\tau]$ be a locally convex space, (ϕ_n), (ψ_n) two unconditional bases of $D[\tau]$. Is there a permutation π of N and a sequence of positive numbers (r_n) such that the operator T defined by $T\phi_n = r_n \psi_{\pi(n)}$ is a homeomorphism of $D[\tau]$ onto itself?

For the domains in which we are interested (namely (F)-spaces) a positive answer to this question can be given, in essence, only in two cases. The first one deals with the so-called "regular" bases introduced by Dragilev [24]. The results along this line concern nuclear (F)-spaces and countable Hilbert spaces and were obtained by Crone, Robinson [22], Kondakov [23] and Djakov [25]. The second case deals with centers of Hilbert scales and has been regarded by Mityagin [25]. In the sequel we make essential use of his result which can be summarized as follows

Theorem [15]

Let $D = \bigcap_{n\geq 0} D(T^n) = \bigcap_{n\geq 0} D(S^n)$ with $S = S\{(s_n),(\phi_n)\}$ and $T = T\{(t_n),(\psi_n)\}$, , $S, T \geq I$. Then there are constants $R > 1$, $C > 0$ such that with a suitable permutation π of N

$$\frac{1}{C}(s_n)^{1/R} \leq t_{\pi(n)} \leq C(s_n)^R \quad \text{holds for all } n \in N. \tag{M}$$

This Theorem can be regarded as a generalization of the Theorem of Köthe [7] essentially used in [12].

All the results known up to now about bases and quasi-equivalence of bases make clear that it is very hard to give a detailed description of the structure of domains of Op*-algebras by means of spaces of sequences in quite general case $D = \cap D(A_\alpha)$, $\alpha \in A$. Therefore we restrict ourselves to the case $D = D^{\infty}(T) = \bigcap_{n \geq 0} D(T^n)$, $T = T^* \geq I$ which will be investigated in the next section.

3. Classification of Domains $D = D^{\infty}(T)$

For the domains of this form we have: $D \in$ I ($\in$ II, $\in$ III, , resp.) if and only if $T \in$ I ($\in$ II, $\in$ III, resp.) with respect to the classification of operators given in [12]. Recalling the facts from the conclusion above, we find a suitable diagonal operator $A = A\{(a_n),(\psi_n)\}$ on $D(T)$ such that

$$D = \{ \psi = \Sigma x_n \psi_n : \Sigma |x_n|^2 (a_n)^{2j} < \infty \quad \forall \; j = 1, 2, \dots \}$$

and the system of sequences $\mathcal{F} = \{(a_n^j), j = 1, 2, \dots\}$, $a_n \in N \; \forall \; n$, describes the associated sequence space. That is, we have an isomorphism between D and the space of sequences

$$D^s = \{ (x_n) : \Sigma |x_n|^2 a_n^{2j} < \infty \quad \forall \; j \in N \}.$$

It is clear that we have to justify that the system $\mathcal{F}$ in fact characterizes the domain D up to unitary equivalence. By using Theorem of Mityagin we obtain the desired justification and a complete description of the regarded domains. We remark that it is sufficiently to restrict oneself in the following Theorems to the case $D = D'$ and to prove only the direction that from $D = D'$ something follows about the systems $\mathcal{F}, \mathcal{F}'$ of sequences. The remaining parts of the Theorems follow by simple considerations or are obvious.

Theorem 1

Let $D, D' \in$ I. D and D' are unitarily equivalent if and only if there are diagonal operators $S = S\{(s_n), (\phi_n)\}$ and $T = T\{(t_n), (\psi_n)\}$ with

$$D = D^\infty(S), \qquad D' = D^\infty(T)$$

such that the systems of sequences

$$\mathcal{F} = \{(s_n^l), \; l = 1, 2, \dots\}, \quad \mathcal{F}' = \{(t_n^l), \; l = 1, 2, \dots\}$$

are equivalent.

Proof

With respect to the remark before Theorem 1 we use the inequalities (M) and can prove, as it is done in /3/, Lemma 5.4 that (M) holds with the identical permutation. But this means just the equivalence of $\mathcal{F}$ and $\mathcal{F}'$. We note that the assertion of this Theorem can also be obtained from the closed-graph theorem and some considerations like in /12/ (cf. for such a proof /19/). Now we go on to the domains of class II.

Theorem 2

Let $D, D' \in$ II. D and D' are unitarily equivalent if and only if there are diagonal operators

$$S = S\{(s_n), (\phi_n)\}, \qquad T = T\{(t_n), (\psi_n)\}$$

with $(s_n) = (s_n^b) \cup (s_n^\infty)$, $(t_n) = (t_n^b) \cup (t_n^\infty)$ and $D = D^\infty(S)$, $D' = D^\infty(T)$ such that the systems of sequences

$$\mathcal{F} = \{(s_n^{\infty j}), \; j = 1, 2, \dots\}, \quad \mathcal{F}' = \{(t_n^{\infty j}), \; j = 1, 2, \dots\}$$

are essentially equivalent.

Before proving the Theorem we repeat that from $D, D' \in$ II it follows that the diagonal operators S, T are also of the class II (cf. /12/ and the remark at the beginning of this section). From the considerations of /12/ it follows that the sequences $(s_n), (t_n)$ resp., can be decomposed in the above mentioned way, where (s_n^b) is a bounded sequence and for (s_n^∞) : $\lim s_n^\infty = \infty$ (analogously for (t_n)). Repeat that the decomposition is unique up to a finite number of elements in any subsequence.

Proof of the Theorem

From (M) we obtain especially

$$\frac{1}{C} (s_n^b)^{1/R} \le t_{k_n} \le C (s_n^b)^R , \tag{1}$$

$$\frac{1}{C} (s_n^\infty)^{1/R} \le t_{j_n} \le C (s_n^\infty)^R , \tag{2}$$

for suitable subsequences (t_{k_n}) and (t_{j_n}) of (t_n) (Clearly, (1) means, if $s_n^b = s_{l_n}$ then $t_{k_n} = s_{\pi(l_n)}$ and an analogous interpretation for (2)).

From (1) and (2) it follows that (t_{k_n}) is a bounded sequence and for (t_{j_n}) : $\lim t_{j_n} = \infty$.

But this means: (t_{k_n}) coincides, up to a finite number of elements, with (t_n^b) and (t_{j_n}) coincides, up to a finite number of elements, with (t_n^∞). Consequently, (2) gives the equivalence of $\{(s_n^{\infty j})\}$ and $\{(t_{j_n}^j)\}$, but this means the essential equivalence of $\{(s_n^{\infty j})\}$ and $\{(t_n^{\infty j})\}$ Q.E.D.

Now we go on the class III. Like in /12/, we shall distinguish three subclasses described below. Let $D \in$ III, $D = D^\infty(S)$, $S = S\{(s_n), (\phi_n)\}$ with the associated sequences (s'_n) (Cf. Preliminaries) and (s^0_n), the sequence of all eigenvalues of S with finite multiplicity, $s^0_1 \le s^0_2 \le \dots$ Put $\mathcal{J} = \{(s'^j_n), j = 1, 2, \dots\}$ It can easily be seen, that

$$D = \Sigma_{\mathcal{J}} \oplus H_n \oplus D_0,$$

where H_n is the infinitely dimensional eigenspace associated with the eigenvalue s'_n and $\Sigma_{\mathcal{J}} \oplus H_n$, D_0 resp. mean

$$\Sigma_{\mathcal{J}} \oplus H_n = \{\chi = \Sigma \chi_n : \chi_n \in H_n, \Sigma \|\chi_n\|^2 (s'_n)^{2j} < \infty, \forall j \in N\},$$

$$D_0 = \{\psi = \Sigma x_n \psi_n : \Sigma |x_n|^2 (s^0_n)^{2j} < \infty, \forall j \in N\},$$

where ψ_n is the eigenvector of S associated with the eigenvalue s^0_n. We also use the notation $D_0 \hat{=} \{(s^{0j}_n), j = 1, 2, \dots\}$ Like in [12], we use the notation of "*reduction*" of D (or D_0). Let $D = \Sigma_{\mathcal{J}} \oplus H_n \oplus D_0$. We say that D (or D_0) can be reduced, if D also has a representation

$$D = \Sigma_{\mathcal{J}} \oplus \hat{H}_n \oplus \hat{D}_0$$

with $H_n \subset \hat{H}_n$, $\dim(\hat{H}_n \ominus H_n) < \infty$, , $\dim(\overline{D_0} \ominus \overline{\hat{D}_0}) = \infty$.

(Because finite dimensional reductions, i.e., $\dim(\overline{D_0} \ominus \overline{\hat{D}_0}) < \infty$ are trivial and always possible, we regard only infinite dimensional reductions such as just defined).

More informally but roughly speaking, the possibility of reduction of $D = \Sigma_{\mathcal{J}} \oplus H_n \oplus D_0$ means that we can find an infinitely dimensional submanifold of D_0 and can "add" it to $\Sigma_{\mathcal{J}} \oplus H_n$. Now we give the following definition of three cases which can arise in class III.

Class III_A if $D = \Sigma_{\mathcal{J}} \oplus \hat{H}_n$ we say D_0 can be *completely reduced*.

Class III if $D = \Sigma_{\mathcal{J}} \oplus \hat{H}_n \oplus D_1$, $D_1 \hat{=} \{(a^l), l = 1, 2, \dots\}$ and D_1 cannot be reduced further, we say can be *maximal reduced*.

Class III if for any reduction of D_0 which leads to $D = \Sigma_{\mathcal{J}} \oplus \hat{H}_n \oplus D_2$, D_2 can be further reduced, we say D_0 can be reduced but not maximal.

Like in [12] one can easily obtain the following statement which we give here without proof.

Statement

i) D_0 can be completely reduced iff there is a subsequence $(s'_{n_l}) \subset (s'_n)$ such that with a suitable iteration $\hat{}$ (cf. preliminaries) $\{(\hat{s}'^j_{n_l})\}$ and $\{(s^{0j}_n)\}$ are equivalent.

ii) D_0 can be maximal reduced iff there is a decomposition $(s^0_n) = (a_n) \cup (b_n)$ and a subsequence $(s'_{j_n}) \subset (s'_n)$ such that with a suitable iteration $\hat{}$

$$\{(a^l_n), i = 1, 2, \dots\} \quad \text{and} \quad \{(\hat{s}'^l_{j_n}), l = 1, 2, \dots\}$$

are equivalent, and $D_2 \hat{=} \{(b^l_n)\}$ cannot be reduced.

iii) D_0 can be reduced but not maximal iff, for any decomposition $(s^0_n) = (a_n) \cup (b_n)$ such that $\{(\hat{s}'^j_{j_n})\}$ and $\{(a^j_n)\}$ are equivalent for a suitable subsequence of (s'_n) and a suitable iteration, there are a further decomposition $(b_n) = (c_n) \cup (d_n)$ and subsequence $(s'_{h_n}) \subset (s'_n)$ and an iteration such that $\{(\hat{s}'^j_{h_n})\}$ and $\{(c^j_n)\}$ are equivalent.

The next proposition deals with the proof that the membership of a domain D to one of classes III_A, III_B or III_C does not depend on the choice of the diagonal operator S in the representation $D = D^\infty(S)$

Proposition 1

Let $D = D^\infty(S) = \cap D(S^n)$. If $D \in III_A (\in III_B, \in III_C)$ with respect to the representation $D = D^\infty(S)$, so $D \in III_A$ $(\in III_B, \in III_C)$ with respect to any representation $D = D^\infty(T)$ where

$$S = S\{(s_n), (\phi_n)\}, \; T = T\{(t_n), (\psi_n)\}.$$

Proof:

1. Suppose, $D^\infty(S) \in III_A$. Regard again the sequences (s_n), (s'_n) and (t_n), (t'_n), (t^0_n). Note that by the assumption $D^\infty(S) \in III^n_A$ the set (s^0_n) is void. From the estimations

$$\frac{1}{C}(s_n)^{1/R} \le t_{\pi(n)} \le C(s_n)^R, \tag{M}$$

it follows especially for $(t_n^0) = (t_{l_n})$ that

$$\frac{1}{C}(s_{\pi^{-1}(l_n)})^{1/R} \le t_{l_n} \le C(s_{\pi^{-1}(l_n)})^{R} : \qquad (3)$$

Let $(s_{\pi^{-1}(l_n)}) = (s_{h_n})$ Since any s'_n has infinite multiplicity, we can find a further subsequence (s_{u_n}) with:

$$s_{u_n} = s_{h_n} \text{ for all } n; \quad (s_{u_n}) \subset \{(s_n) - (s_{h_n})\}.$$

The latter and (3) lead to

$$\frac{1}{C} s_{u_n}^{1/R} \le t_{\pi(u_n)} \le C s_{u_n}^{R}, \; (t_{\pi(u_n)}) \subset \{(t_n) - (t_n^\infty)\}, \qquad (4)$$

that is $(t_{\pi(u_n)}) = (\hat{t}'_{j_n})$ for a suitable subsequence $(t'_{j_n}) \subset (t'_n)$ and a suitable iteration. From (3) and (4) we obtain

$$t_n^0 \le C(s_{h_n})^{R} \le C \cdot C^{R^2} t_{\pi(u_n)}^{R^2} = E(\hat{t}'_{j_n})^{R^2}, \qquad (5)$$

$$t_n^0 \ge \frac{1}{C} s_{\pi^{-1}(l_n)}^{1/R} \ge \left(\frac{1}{C}\right)^{1+\frac{1}{R^2}} (t_{\pi(u_n)}^{1/R^2}) = D(\hat{t}'_{j_n})^{1/R^2} : \qquad (6)$$

Consequently,

$$D(\hat{t}'_{j_n})^{1/R^2} \le t_n^0 \le E(\hat{t}'_{j_n})^{R^2},$$

i.e., the systems $\{(t_n^{01})\}$ and $\{(\hat{t}'^{\,1}_{j_n})\}$ are equivalent.
Hence $D^\infty(T) \in III_A$.

2. Suppose $D^\infty(S) \in III_B$, $D^\infty(T) \in III_C$ and regard $(s_n), (s'_n), (s_n^0)$ and $(t_n), (t'_n), (t_n^0)$.
From (M) it follows that

$$\frac{1}{C}(s_n^0)^{1/R} \le t_{k_n} \le C(s_n^0)^{R}, \qquad (7)$$

for all n.

$$\frac{1}{C}(s_{l_n})^{1/R} \le (t_n^0) \le C(s_{l_n})^{R}, \qquad (8)$$

This leads to $(t_{k_n}) \subset (t_n^0)$ up to a finite number of elements, because, if it were not the case one could obtain a contradiction with $D^\infty(S) \in III_B$ by similar consideration like in 1. (Because finite dimensional reductions are always possible, we may, without loss of generality assume that $(t_{k_n}) \subset (t_n^0)$ for all n)
From (7) and (M) it follows that for $\{(t_n^0) - (t_{k_n})\} = (u_n)$ $\{(u_n^k)\}$ is equivalent to a suitable $\{(\hat{s}'^{\,k}_{r_n})\}$ and, again, by (M) and the considerations of 1: $\{(u_n^k)\}$ is equivalent to a suitable $\{(\hat{t}'^{\,k}_{h_n})\}$. From this and $D^\infty(T) \in III_C$ it follows that there is a further decomposition $(t_{k_n}) = (a_n) \cup (b_n)$ such that the system $\{(a_n^k)\}$ is again equivalent to a suitable $\{(\hat{t}'^{\,k}_{j_n})\}$. But this means after all that the subsequence $(s_{k_n}^0)$ of (s_n^0) the elements of which stand in (7) together with the elements of (a_n) "could be reduced" which is a contradiction with the assumption $D^\infty(S) \in III_B$: Q.E.D.

The next Theorem gives us information about unitary equivalence of domains of class III.

Theorem 3

Let $D, D' \in III$: D and D' are unitarily equivalent if and only if the following requirements hold:

1. there are operators $S = S\{(s_n), (\phi_n)\}$ and $T = T\{(t_n), (\phi_n)\}$ such that $D = D^\infty(S)$ and $D' = D^\infty(T)$:
2. both D and D' are of the same class III_A, III_B, III_C, resp.
3. for the sequences $(s'_n), (t'_n), (s_n^0), (t_n^0)$ one has
   i) (s'_n) and (t'_n) are weakly equivalent, that is, there are suitable iterations σ and τ such that $(s'_{\sigma(n)}) \approx (t'_{\tau(n)})$
   ii) for the classes III_B and III_C in addition to condition i) one has:
   III_B: the systems $\{(s_n^{01})\}$, $\{(t_n^{01})\}$ are essentially equivalent
   III there are decompositions $(s_n^0) = (a_n) \cup (b_n)$, $(t_n^0) = (c_n) \cup (d_n)$ subsequences $(u_n) \subset (s'_n)$ $(v_n) \subset (t'_n)$ and suitable iterations such that the three pairs of systems of sequences

$\{(a_n^1)\}$, $\{(c_n^1)\}$,

$\{(b_n^1)\}$, $\{(\hat{u}_n^1)\}$,

$\{(d_n^1)\}$, $\{(\hat{v}_n^1)\}$.

are equivalent.

Remark 3

Let us note that the conditions for classes III_B and III_C must be formulated in the language of systems of sequences (as it is done) while the condition for class III_A is requirement only for the sequences themselves.

Proof of the Theorem

Let $M_k = \{ s_j \in (s_n) : s_j = k \}$,
$N_k = \{ t_j \in (t_n) : t_j = k \}$.

From the estimations

$$\frac{1}{C} s_n^{1/R} \leq t_{\pi(n)} \leq C s^R \tag{M}$$

it follows: if $s_i \in M_k$, then $t_{\pi(i)} \in \bigcup_{n=1}^{n_k} N_j$ $\quad m_k, n_k < \infty \quad k$

if $t_i \in N_k$, then $s_{\pi^{-1}(i)} \in \bigcup_{j=1}^{m_k} M_j$.

But this is the same situation as in the proof of Theorem 3 of /12/ from which it follows that (s_n') and (t_n') are weakly equivalent.

III_B: Like in the Proof of Proposition 1 we obtain from (M) for the sequences (s_n^0) and (t_n^0) the following:
i) if on the left-hand side and on the right-hand side of (M) the elements s_n^0 stands, then the corresponding $t_{\pi(n)}$ must belong to (t_n^0) up to a finite number elements.
ii) if for the $t_{\pi(n)}$ the elements of (t_n^0) stand, then on the right-hand side and on the left-hand side of (M) elements of (s_n^0) must stand up to a finite number of elements (if it is not the case, we obtain in both cases a contradiction with $D^\infty(S) \in III_B$, i.e., D maximal reduced).

i) and ii) mean that in (M) the elements of (s_n^0) and (t_n^0) "stand together" up to a finite number of elements. But this shows the essential equivalence of $\{(s_n^{01})\}$ and $\{(t_n^{01})\}$.

III_C: Regard the following decomposition of (t_n^0):

$$(t_n^0) = (c_n) \cup (d_n)$$

where c_n are those elements of (t_n^0) which stand in (M) together with elements of (s_n^0), denote these elements of (s_n^0) by (a_n). Let $(d_n) = \{(t_n^0) - (c_n)\}$; $(b_n) = \{(s_n^0) - (a_n)\}$. The elements of (s_n) which stand in (M) together with the elements d_n form a sequence $(\hat{s}_{j_n}')$. From our "standard argumentation" it follows that $\{(\hat{s}_{j_n}'^1)\}$ is equivalent to a suitable system

$$\{(\hat{t}_{h_n}'^1)\} = \{(\hat{v}_n^1)\}, \qquad \text{i.e., } \{(\hat{d}_n^1)\}$$

is equivalent to $\{(\hat{v}_n^1)\}$. Analogously, one finds that $\{(b_n^1)\}$ is equivalent to $\{(\hat{s}_{p_n}'^1)\} = \{(\hat{u}_n^1)\}$ for a suitable subsequence $(s_{p_n}') = (u_n) \subset (s_n')$ and a suitable iteration. This concludes the proof of the Theorem.

Acknowledgement

We thank A.Uhlmann for reading the manuscript and his interest in this work and B.S.Mityagin for discussions.

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Received by Publishing Department on June 18, 1975.