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SINGULAR SOLUTIONS OF RENORMALIZATION GROUP EQUATIONS AND THE SYMMETRY OF THE LAGRANGIAN

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A b s t r a c t

On the basis of solution of the differential renormalization group equations the method is proposed for finding out the Lagrangians possessing some kind of internal symmetry. It is shown that in the phase space of the invariant charges the internal symmetry corresponds to the straight-line singular solution of these equations remaining straight-line when taking into account the higher order corrections. We have studied the model of scalar fields with quartic couplings , as well as the set of models containing scalar, pseudoscalar and spinor fields with Yukawa and quartic interactions. Straight-line singular solutions in the first case correspond to isotopical symmetry only. For the second case they correspond to supersymmetry. No other symmetries have been discovered. For the model containing the gauge fields the solution corresponding to supersymmetry is obtained and it is shown that this is also the only symmetry that can be realized in the given set of fields.

1. Introduction

In recent years the asymptotical properties of renormalizable quantum field theory models have been studied intensively, which was to a certain extent stimulated by the experimentally discovered scaling phenomenon of strong interactions and theoretically discovered property of asymptotical freedom (AF). As a result there arised an impression about the exhaustive character of these investigations. Thus, for instance, the attempt to construct a simple gauge asymptotically free model including scalar particles was not crowned with success. These particles inevitably possess quartic selfinteraction which destroys AF of the theory.

From this point of view new unexpected possibilities arised in the framework of renormalizable supersymmetry models. The simplest supersymmetry model by Wess and Zumino/1/ (WZ model) contains three fields: scalar, pseudoscalar and Majorana spinor (see eq. (11) below) with the interaction of the quartic and Yukawa type. The coupling constants of these interactions are linked by simple algebraical relations which are not destroyed by radiative corrections due to the Ward-Takahashi identities following from the sypersymmetry.

The second renormalizable supersymmetry gauge-invariant model proposed by Salam, Strathdee/2/ and Ferrara, Zumino/3/ (SSFZ model) contains a set of scalar , pseudoscalar and Majorana spinor fields and also Yang-Mills field (see eqs. (18), (19) below). The interaction Lagrangian includes minimal gauge, Yukawa and quartic interactions with the coupling constants connected by simple relations (18).

The models of such a type, when the number of matter fields (scalar, pseudoscalar and spinor) is small enough, possess AF with respect to all couplings.

The essential element here is the presence of supersymmetry Ward identities which lead to the strict relations between the coupling constants of Yukawa, quartic and Yang-Mills interactions.

If now we "unhook" the constants of these interactions from one another (i.e. consider them as independent)in such supersymmetry models, we obtain the models with various coupling constants just like the ones we spoke above. It is evident that they include the corresponding supersymmetry models as a particular (or limiting) case. The question is: Whether it is possible to see these particular cases without knowing the symmetry beforehand. This formulation can be naturally generalized:

Let the Lagrangian be given with a certain number and type of fields and interactions. It needs to find the particular cases possessing the unknown, beforehand, internal symmetry.

It is the very problem we shall consider in the present paper.

We would like to emphasize that we are not going to search for the symmetry itself. The discovery of supersymmetry shows us that this direct way may be very complicated. We want to indicate the method for finding out indirect manifestation of internal (maybe, still unknown) symmetries.

The analysis of ultraviolet asymptotics with the help of the renormalization group (RG) gives us such a method. In phase space of invariant coupling constants internal symmetries correspond to the singular solutions passing through the origin. Such singular solutions in the one-loop approximation are well known/4/ and are typical for a wide class of the Lagrangians with several coupling constants which do not possess any internal symmetry. In the presence of internal symmetry singular solutions in the

space of appropriately determined invariant charges (IC) are straight lines and this straightness retains when taking into consideration the higher (multiloop) contributions.

Thus, the proposed method of detecting the "candidates for internal symmetry" consists in the following: For a given Lagrangian with several coupling constants the phase space of invariant charges is constructed. Then you search for the singular solution passing through the origin and remaining straight in a two-loop approximation.

It is necessary to note that as the theory contains divergences it is not determined completely by the Lagrangian but depends also on the choice of regularization and the accepted renormalization procedure. The cases are known when the regularization destroys the initial symmetry of the Lagrangian. As far as we are searching for the internal symmetry on the basis of investigation of renormalized expressions , it is very important that the regularization-renormalization procedure should not break any possible symmetry.

We use two methods of renormalization: Bogolubov's R-operation and t'Hooft's renormalization method/5/. In the first case the counterterms in the Lagrangian and hence the Gell-Mann-Low-function of Lie equation can depend on the choice of normalization points of three- and four - vertices. That is why in perturbation theory for the validity of the Ward identities dictated by possible symmetry, it is necessary to choose the normalization of vertices in concord. However, since we consider lower approximations such a dependence may not appear. Thus, in the two-loop approximation (see eqs. (2) and (21)) it is not present and though it is available in the three-loop one (see eq. (A1)), it does not affect the final results.

In the case of t'Hooft's method it turns to be easier to work not with the φ-function of the Lie equation but with the β-function of the Ovsiannikov-Gallan-Simanzik one. This function does not depend on the choice of external momenta of the vertex and is distinguished from the φ-function beginning from the second order of perturbation theory/6/. Since the dimensional regularization apparently preserves initial symmetry of the theory, the β-function of O-C-S equation is usefull for finding out the internal symmetry of the Lagrangian in the proposed method.

If there is an internal symmetry in the theory then the Lie equations and the Ovsiannikov ones have the straight singular solutions retaining in higher orders. These singular solutions coincide as they are already found in the one-loop approximation, and in this approximation the functions φ and β are equal.

2. Models with Scalar Fields

Consider first the scalar theory with quartic couplings. Though such a theory is not interesting for application, it can serve as a good illustration of the proposed method and is the simplest theory for calculations. We choose the Lagrangian in the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_i)^2 - \frac{1}{4!} h_{ijkl}\, \varphi_i \varphi_j \varphi_k \varphi_l \,; \quad i,j,k,l = 1,2,\ldots,n. \qquad (1)$$

The mass terms hereafter are neglected due to the reasons which will be discussed below. The RG equations for the invariant charges on the two-loop level are:

$$\frac{d\bar h_{ijkl}}{dL} = \varphi(\bar h_{ijkl}) = \frac{1}{16\pi^2}\frac{1}{2}\left[\bar h_{ijmn}\bar h_{mnkl} + \bar h_{ikmn}\bar h_{mnjl} + \bar h_{ilmn}\bar h_{mnjk}\right]$$
$$- \frac{1}{(16\pi^2)^2}\frac{1}{2}\left[\bar h_{iabm}\bar h_{mjkn}\bar h_{nabl} + \bar h_{jabm}\bar h_{mikn}\bar h_{nabl} + \bar h_{jabm}\bar h_{mlkn}\bar h_{nabi} + \right.$$
$$\left. + \bar h_{iabm}\bar h_{mjln}\bar h_{nabk} + \bar h_{kabm}\bar h_{mjin}\bar h_{nabl} + \bar h_{jabm}\bar h_{miln}\bar h_{nabk}\right] + \qquad (2)$$
$$+ \frac{1}{(16\pi^2)^2}\frac{1}{24}\left[\bar h_{ijkd}\bar h_{abcd}\bar h_{abcl} + \bar h_{ijld}\bar h_{abcd}\bar h_{abck} + \bar h_{ikld}\bar h_{abcd}\bar h_{abcj} + \bar h_{jkld}\bar h_{abcd}\bar h_{abci}\right],$$

where $L \equiv \ln\frac{\kappa^2}{\lambda^2}$, $\bar h_{ijkl} = \bar h_{ijkl}(h_{ijkl}, L)$.

Further we shall confine ourselves only to selfinteraction and binar interactions. The analysis of eq.(2) in this case is considerably simplified and reduces to the analysis on a plane of the variables of two types: $\bar h_{iiii}$ and $\bar h_{iijj} = \bar h_{ijij} = \bar h_{ijji} = \bar h_{jiji} = \bar h_{jjii} = \bar h_{jiij}$, where $i,j = 1,2,\ldots,n$. The RG equations for these two types of variables can be quickly obtained from eq.(2).

Now we shall search for the singular solutions passing through the origin of the phase space. Consider first the one-loop approximation. There arise 3 types of singular solutions:

1) $\bar h_{iiii}$ -independent, $\bar h_{iijj} = 0$; $i,j = 1,2,\ldots,n$; $n = 1,2,\ldots$,
2) $\bar h_{iiii} = \bar h$, $\bar h_{iijj} = \frac{1}{3}\bar h$; $i,j = 1,2,\ldots,n$, $n = 2,3,\ldots$,
3) $\bar h_{iiii} = \bar h$, $\bar h_{iijj} = \frac{1}{n-1}\bar h$; $i,j = 1,2,\ldots,n$, $n = 2,3,\ldots$.

There are also possible different combinations of these solutions when n_1 fields are independent with the couplings corresponding to the first solution, n_2 fields are interacting with the couplings corresponding to the second solution and n_3 fields are interacting with the couplings corresponding to the third solution.

To answer the question whether these solutions really correspond to some symmetry of the Lagrangian, we shall consider the two-loop approximation in eq. (2) and look whether any of these singular solutions retain straight in the phase space.

The analysis shows that the solutions of the first and the second type retain straight and the solution of the third type retains straight only for $n=2$. We have performed also three-loop calculations for $n=2$ and have been convinced that all the solutions retain straight. The Gell-Mann-Low functions in this case are given in Appendix. The behaviour of phase curves for $n=2$ is demonstrated on the phase plane of variables $\bar{h}_1=\bar{h}_2$ and $\bar{h}_{12}=\bar{h}_{21}$ (see Fig.1)

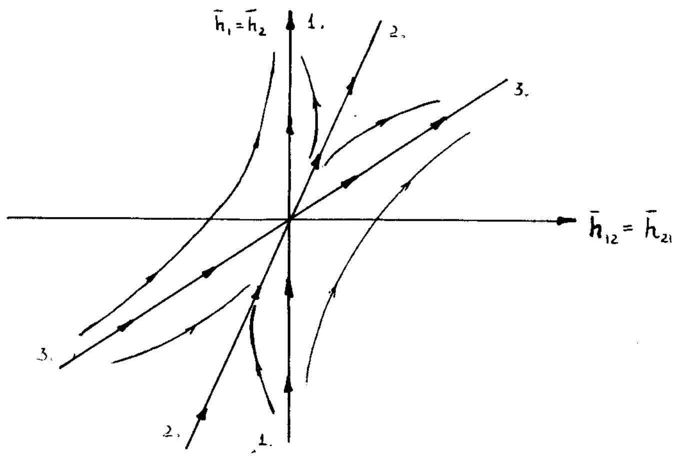

Fig. 1.

Arrows show the dirrection of increasing argument $L=\ln\frac{\kappa^2}{\lambda^2}$.

Thus, we have two types of the Lagrangians "suspicious on internal symmetry". (The solution 1 is trivial as it corresponds to the system of fields noninteracting with each other):

$$\mathcal{L}^{2)}=\frac{1}{2}(\partial_\mu\varphi_i)^2-\frac{1}{4!}h(\varphi_i\varphi_i)^2, \quad i=1,2,\ldots,n,$$

$$\mathcal{L}^{3)}=\frac{1}{2}(\partial_\mu\varphi_1)^2+\frac{1}{2}(\partial_\mu\varphi_2)^2-\frac{1}{4!}h(\varphi_1^4+6\varphi_1^2\varphi_2^2+\varphi_2^4).$$

The Lagrangian $\mathcal{L}^{2)}$ possesses the well known global isotopical symmetry. Singular solution 2 is unstable for $h>0$ and stable for $h<0$. The Lagrangian $\mathcal{L}^{3)}$ at first sight does not correspond to any simple symmetry. However, under the transformation of the fields $\varphi_1=\frac{\varphi_1'+\varphi_2'}{\sqrt{2}}$, $\varphi_2=\frac{\varphi_1'-\varphi_2'}{\sqrt{2}}$ it is diagonalized and coincide with the Lagrangian corresponding to the first solution.

Thus, in a pure scalar theory the proposed method allows us to detect the isotopical symmetry which is realized as a singular solution of the RG equations passing through the origin of phase space. No other such symmetries exist in these models.

3. The Yukawa Type Interactions

Consider the simplest system consisting of scalar and spinor fields with the Yukawa type interaction. The Lagrangian is chosen in the form:

$$\mathcal{L}=\frac{1}{2}(\partial_\mu A)^2+\bar{\Psi}_{(j)}i\hat{\partial}\Psi_{(j)}+g\bar{\Psi}_{(j)}A\Psi_{(j)}-\frac{h}{4!}A^4, \quad j=1,2,\ldots\kappa \qquad (3)$$

In this section we use the t'Hooft's renormalization method. The RG equations for the IC's here are written down in the Ovsiannikov-Callan-Simanzik form:

$$\left(\mu^2\frac{\partial}{\partial\mu^2}+\beta_g\frac{\partial}{\partial g^2}+\beta_h\frac{\partial}{\partial h}\right)\bar{g}^2\left(\frac{\kappa^2}{\mu^2},g^2,h\right)=0,$$
$$\left(\mu^2\frac{\partial}{\partial\mu^2}+\beta_g\frac{\partial}{\partial g^2}+\beta_h\frac{\partial}{\partial h}\right)\bar{h}\left(\frac{\kappa^2}{\mu^2},g^2,h\right)=0 \qquad (4)$$

The β-functions for the Lagrangian (3) in the two-loop approximation are/7/:

$$\beta_g(g^2,h)=\frac{1}{16\pi^2}(3+2\kappa)g^4+\frac{1}{(16\pi^2)^2}g^2\left[-\left(\frac{9}{4}+12\kappa\right)g^4-2g^2h+\frac{h^2}{12}\right],$$
$$\beta_h(g^2,h)=\frac{1}{16\pi^2}\left[\frac{3}{2}h^2+4\kappa g^2h-24\kappa g^4\right]+\frac{1}{(16\pi^2)^2}\left[-\frac{17}{6}h^3-6\kappa g^2h^2+\right. \qquad (5)$$
$$\left.+14\kappa hg^4+192\kappa g^6\right].$$

It should be noted, that in the case when the spinors in the Lagrangian (3) are chosen in the Majorana representation the eqs. (5) remain unchanged if we replace $I \to \frac{I}{2}$ in the Lagrangian and put $k = \frac{n}{2}$, where n is the number of Majorana spinors. That is why we shall search for the singular solutions of eq. (4) for k being integer or halfinteger.

The analysis of eqs. (5) shows that in one-loop approximation there arise 2 singular solutions of the type $\bar{h} = \alpha \bar{I}^2$. However, when taking into account the two-loop contributions, none of these solutions retain straight for any k.

Consider now the Lagrangian possessing $SU(2)$ symmetry:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu A^a)^2 + \bar{\Psi}_{(j)} i\hat{\partial}\Psi_{(j)} + I\bar{\Psi}_{(j)} T^a A^a \Psi_{(j)} - \frac{h}{4!}(A^a A^a)^2, \tag{6}$$

$j = 1,2,\ldots k$; $a = 1,2,3$; T^a – the generator of $SU(2)$ group. Hereafter we shall put k integer or halfinteger supposing that some of the spinors may be taken in the Majorana representation. The two-loop calculations lead to the following results:

$$\beta_I = \frac{1}{16\pi^2}(3T^2 - 2 + 2tk) I^4 + \frac{1}{(16\pi^2)^2} I^2 \Big[-\Big(\frac{9}{4}(T^2)^2 + 12T^2 tk - 6tk - 2\Big) I^4 - \Big(2T^2 - \frac{2}{3}\Big) I^2 h + \frac{5}{36} h^2 \Big],$$

$$\beta_h = \frac{1}{16\pi^2}\Big[\frac{11}{6}h^2 + 4tk I^2 h - \frac{24(3T^2-1)tk}{5} I^4\Big] + \frac{1}{(16\pi^2)^2}\Big[-\frac{23}{6}h^3 - \frac{22}{3}tk I^2 h^2 + 2tk\Big(\frac{19}{5}T^2 - \frac{18}{5}\Big) I^4 h + \frac{48tk}{5}\big(12(T^2)^2 - 14T^2 + 5\big) I^6 \Big], \tag{7}$$

where $T^2 \equiv T^a T^a$, $t\,\delta^{ab} = Sp\, T^a T^b$.

We have analysed these equations for the spinor fields in adjoint and fundamental representations of $SU(2)$. On the one-loop level there exist 2 singular solutions $\bar{h} = \alpha \bar{I}^2$. However, when taking into account the two-loop contributions all singularsolutions are distorted for any value of k.

Thus, in a system consisting of an arbitrary number of spinor (possibly Majorana) fields and a scalar field (or triplet of scalar fields), any symmetry linearly connecting the constants of the Yukawa and quartic interactions cannot be realized.

Consider more complex system consisting of scalar, pseudoscalar and spinor fields. The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \bar{\Psi}_{(j)} i\hat{\partial}\Psi_{(j)} + I_A \bar{\Psi}_{(j)} A \Psi_{(j)} + I_B \bar{\Psi}_{(j)} \gamma^5 B \Psi_{(j)} - \frac{h_A}{4!}A^4 - \frac{h_B}{4!}B^4 - \frac{h_{AB}}{4}A^2 B^2, \quad j = 1,2,\ldots,k \tag{8}$$

There arise 5 ICs and the corresponding functions have the form:

$$\beta_{I_A} = \frac{1}{16\pi^2} I_A^2 \big[I_A^2(3+2k) - I_B^2\big] + \frac{1}{(16\pi^2)^2} I_A^2 \Big[I_A^4\Big(-\frac{9}{4} + 12k\Big) + I_A^2 I_B^2\Big(\frac{15}{2} - k\Big) + I_B^4\Big(\frac{7}{4} + k\Big) - 2I_A^2 h_A - 2I_B^2 h_{AB} + \frac{h_A^2}{12} + \frac{h_{AB}^2}{4} \Big],$$

$$\beta_{I_B} = \beta_{I_A}(A \leftrightarrow B),$$

$$\beta_{h_A} = \frac{1}{16\pi^2}\Big[\frac{3}{2}h_A^2 + \frac{3}{2}h_{AB}^2 + 4k I_A^2 h_A - 24k I_A^4\Big] + \frac{1}{(16\pi^2)^2}\Big[-\frac{17}{6}h_A^3 - \frac{5}{2}h_{AB}^2 h_A - 6h_{AB}^3 - 6k h_A^2 I_A^2 - 6k h_{AB}^2 I_B^2 + 14k h_A I_A^4 - 2k h_A I_A^2 I_B^2 - 24k h_{AB} I_A^2 I_B^2 + 192k I_A^6\Big], \tag{9}$$

$$\beta_{h_B} = \beta_{h_A}(A \leftrightarrow B),$$

$$\beta_{h_{AB}} = \frac{1}{16\pi^2}\Big[2h_{AB}^2 + \frac{h_A+h_B}{2}h_{AB} + 2k\,h_{AB}(I_A^2+I_B^2) - 8k\,I_A^2 I_B^2\Big] + \frac{1}{(16\pi^2)^2}\Big[-\frac{9}{2}h_{AB}^3 -$$
$$- 3h_{AB}^2(h_A+h_B) - \frac{5}{12}h_{AB}(h_A^2+h_B^2) - 4k\,h_{AB}^2(I_A^2+I_B^2) - 2k\,h_{AB}(h_A I_A^2 + h_B I_B^2) -$$
$$- k\,h_{AB}(I_A^4+I_B^4) + 30k\,h_{AB} I_A^2 I_B^2 - 4k(h_A+h_B)I_A^2 I_B^2 + 32k\,I_A^2 I_B^2(I_A^2+I_B^2)\Big].$$

On the one-loop level these equations lead to the existence of 4 pairs of nontrivial singular solutions:

$$\bar I_A^2 = \bar I_B^2 = \bar I^2\,;\quad \bar h_A = \bar h_B = \alpha \bar I^2\,,\quad \bar h_{AB} = \beta \bar I^2\,,$$

$$\text{a)}\ \alpha = 3\beta = 3\,\frac{(1-k)\pm\sqrt{(1-k)^2+40k}}{5}\,;\quad \text{b)}\ \alpha = 3\beta = 3\,\frac{(1-k)-\sqrt{(1-k)^2+40k}}{5}\,, \tag{10}$$

$$\text{c)}\ \beta = \frac{4(1-k)}{3} - \alpha\,;\quad \alpha = \frac{3(1-k)+\sqrt{(1-k)^2+72k}}{3}\,;\quad \text{d)}\ \beta = \frac{4(1-k)}{3} - \alpha\,;\quad \alpha = \frac{3(1-k)-\sqrt{(1-k)^2+72k}}{3}\,.$$

Taking into account the two-loop contributions in (9) we obtain that solution (10a) remains unchanged for $k=\frac{1}{2}$, $\alpha = 3\beta = 3$. All other trajectories are distorted for any k. The obtained singular solution is the unstable singular solution of the system of differential RG equations. The behaviour of the phase curves in the one-loop approximation for $k=\frac{1}{2}$ is demonstrated on Fig.2.

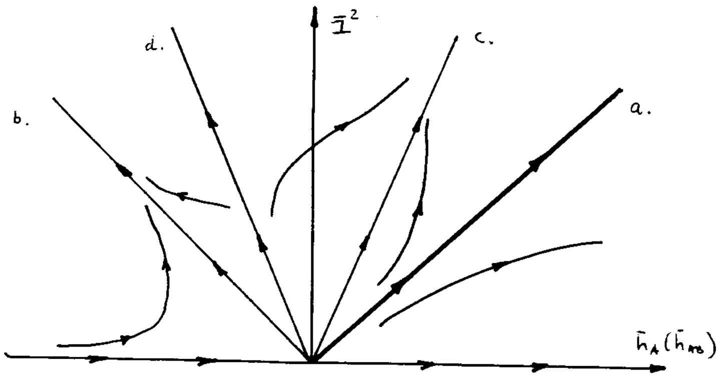

Fig. 2.

Thus, for $k=\frac{1}{2}$ (one Majorana spinor) we obtain the Lagrangian "suspicious of internal symmetry". It has the form:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{1}{2}\bar\Psi i\hat\partial\Psi + \frac{I}{2}\bar\Psi(A+\gamma^5 B)\Psi - \frac{I^2}{8}(A^2+B^2)^2. \tag{11}$$

Indeed, Lagrangian (11) is that of supersymmetry WZ model/1/ up to the mass terms. The supersymmetry in a given set of fields is realized as unstable singular solution of the RG equations. Since all other solutions (10) do not retain straight on the two-loop level, there exist no other symmetry linearly connecting the constants of the Yukawa and quartic interactions in the given set of fields.

Consider the same system of fields with $SU(2)$ symmetry:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu A^a)^2 + \frac{1}{2}(\partial_\mu B^a)^2 + \bar\Psi_{(j)} i\hat\partial\Psi_{(j)} + I_A\bar\Psi_{(j)}T^a A^a\Psi_{(j)} +$$
$$+ I_B\bar\Psi_{(j)}T^a\gamma^5 B^a\Psi_{(j)} - \frac{h_A}{4!}(A^aA^a)^2 - \frac{h_B}{4!}(B^aB^a)^2 - \frac{h_{AB}}{4}A^aA^aB^bB^b - \frac{h}{4}\epsilon^{abc}\epsilon^{ade}A^bB^cA^dB^e. \tag{12}$$

There exist 6 IC's. The corresponding β-functions are given in Appendix (A2). We have analysed the equations (A2) for the spinor fields in adjoint and fundamental representation of $SU(2)$. On the one-loop level there exist some singular solutions, but on the two-loop one all the lines are distorted for any k. Thus, there exist no symmetry linearly connecting the coupling constants in this set of fields.

4. <u>The Model Including Yang-Mills Fields</u>

Consider, at last, a more complete situation including spinor, scalar, pseudoscalar fields and the Yang-Mills field with various interactions. The gauge-invariant Lagrangian has the form:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + i\bar{\psi}\gamma_\mu D_\mu\psi + \frac{1}{2}(D_\mu A^a)^2 + \frac{1}{2}(D_\mu B^a)^2 +$$

$$+ I_A\bar{\psi}T^a A^a\psi + I_B\bar{\psi}T^a\gamma^5 B^a\psi - V(A,B) . \qquad (13)$$

Here, as usual , $F^a_{\mu\nu} = \partial_\mu V^a_\nu - \partial_\nu V^a_\mu + g f^{abc} V^b_\mu V^c_\nu$, $D_\mu A^a = \partial_\mu A^a + g f^{abc} V^b_\mu A^c_\nu$ etc. The fields V^a_μ, A^a, B^a realize adjoint representation of $SU(N)$ group, spinors ψ are transformed according to the arbitrary representation. $V(A,B)$ is a gauge invariant potential of self-interaction of scalar fields. We shall suppose it to be renormalizable.

Under the assumption that $V(A,B) \sim g^2 \sim I_A^2 \sim I_B^2$ it turns out that the differential RG equation for IC's $\bar{g}^2, \bar{I}_A^2, \bar{I}_B^2$ in the one-loop approximation are independent of $V(A,B)$ and can be written in a compact way:

$$16\pi^2 \frac{d\bar{g}^2}{dL} = \varphi_g(\bar{g}^2, \bar{I}_A^2, \bar{I}_B^2, \dots) = -c\,\bar{g}^4 + \cdots , \qquad (14a)$$

$$16\pi^2 \frac{d\bar{I}_A^2}{dL} = \varphi_{I_A}(\bar{g}^2, \bar{I}_A^2, \bar{I}_B^2, \dots) = a\bar{I}_A^4 - b\bar{I}_A^2\bar{g}^2 + d\,\bar{I}_A^2\bar{I}_B^2 + \cdots , \qquad (14b)$$

$$16\pi^2 \frac{d\bar{I}_B^2}{dL} = \varphi_{I_B}(\bar{g}^2, \bar{I}_A^2, \bar{I}_B^2, \dots) = a\bar{I}_B^4 - b\bar{I}_B^2\bar{g}^2 + d\,\bar{I}_A^2\bar{I}_B^2 + \cdots , \qquad (14c)$$

where the coefficient in the r.h.s. of (14) are determined by:

$$c = \frac{11}{3}C_2 - \frac{4}{3}t_\psi - \frac{t_A + t_B}{6}, \quad a = 3T_\psi^2 - C_2 + 2t_\psi, \quad b = 6\left(T_\psi^2 - \frac{C_2}{2}\right) + 3T^2_{A,B}, \quad d = C_2 - T_\psi^2 .$$

Here C_2 is the value of the quadratic Casimir operator of the group for the $SU(N)$ group equals $C_2 = N$; $[T^a, T^b] = if^{abc}T^c$, $T^2 \equiv T^aT^a$, $Sp\,T^aT^b = t\,\delta^{ab}$. In adjoint representation of spinor fields we have: $T^2(adj) = N$, $t(adj) = N$. In this case $c = 2N$, $a = 4N$, $b = 6N$, $d = 0$

Except for trivial zero solutions the system of eqs. (14) possesses one singular solution, namely:

$$\bar{I}_A^2 = \bar{I}_B^2 = \frac{b-c}{a}\,\bar{g}^2 = \bar{g}^{-2} \qquad (15)$$

for any group $SU(N)$.

Generally speaking, there exists the solution when one of the Yukawa coupling equals zero and the other is proportional to g . However, this solution does not lead to any simple symmetry and further we shall not consider it. The situation is illustrated on the phase plane of variables $\bar{g}^2$ and $\bar{I}^2$ on Fig.3.

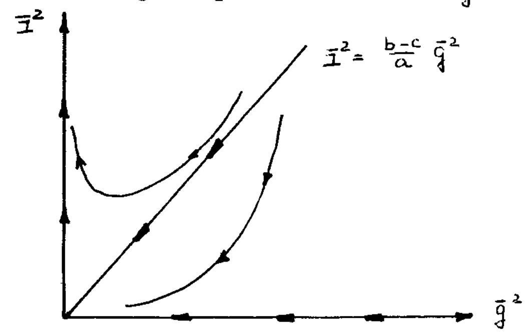

Fig. 3.

The obtained singular solution turns out to be unstable asymptotically free solution. The existence of this solution is defined by the difference $b-c$ (15) and is possible only when $b > c$, which in its turn limits the possible number of spinor and scalar fields in the Lagrangian and their representation/8/.

Consider now the scalar interactions. Confining ourselves, for simplicity, to $SU(2)$ group, we choose the potential in the form:

$$V(A,B)=h_A(A^aA^a)^2+h_B(B^aB^a)^2+2h_{AB}(A^aA^a)(B^bB^b)+h\epsilon^{abc}\epsilon^{ade}A^bB^cA^dB^e. \qquad (16)$$

There arise 4 IC's and the corresponding RG equations in the one-loop approximation are:

$$16\pi^2\frac{d\bar h_A}{dL}=\varphi_{h_A}=44\bar h_A^2+12\bar h_{AB}^2+8\bar h_{AB}\bar h+2\bar h^2-2\bar I_A^4+\frac{3}{2}\bar g^4+8\bar I_A^2\bar h_A-12\bar g^2\bar h_A,$$

$$16\pi^2\frac{d\bar h_B}{dL}=\varphi_{h_B}=44\bar h_B^2+12\bar h_{AB}^2+8\bar h_{AB}\bar h+2\bar h^2-2\bar I_B^4+\frac{3}{2}\bar g^4+8\bar I_B^2\bar h_B-12\bar g^2\bar h_B, \qquad (17)$$

$$16\pi^2\frac{d\bar h_{AB}}{dL}=\varphi_{h_{AB}}=16\bar h_{AB}^2+2\bar h^2+20(\bar h_A+\bar h_B)\bar h_{AB}+4(\bar h_A+\bar h_B)\bar h-2\bar I_A^2\bar I_B^2+\frac{3}{2}\bar g^4+$$
$$+4(\bar I_A^2+\bar I_B^2)\bar h_{AB}-12\bar g^2\bar h_{AB},$$

$$16\pi^2\frac{d\bar h}{dL}=\varphi_h=6\bar h^2+16\bar h_{AB}\bar h+8(\bar h_A+\bar h_B)\bar h-\frac{3}{2}\bar g^4+4(\bar I_A^2+\bar I_B^2)\bar h-12\bar g^2\bar h.$$

The solutions of the system (17) essentially depend on the behaviour of IC's $\bar g^2$, $\bar I_A^2$ and $\bar I_B^2$. The solutions from the top sector of Fig. 3. lead to the rapidly vanishing Yang-Mills constant what is qualitatively equivalent to the situation considered in the previous section. When the solution is chosen in the bottom sector of Fig.3 the system (17) has no untrivial singular solutions. And only when on the plane $(\bar I^2,\bar g^2)$ we choose the unstable singular solution (15), the system (17) has 4 untrivial singular solutions:

$$\begin{aligned}
&1)\ \bar h_A=\bar h_B=\bar h_{AB}=0,\quad \bar h=\tfrac{1}{2}\bar g^2,\quad \bar I^2=\bar g^2,\\
&2)\ \bar h_A=\bar h_B=\bar h_{AB}=0,\quad \bar h=-\tfrac{1}{2}\bar g^2,\quad \bar I^2=\bar g^2,\\
&3)\ \bar h_A=\bar h_B=\bar h_{AB}=\tfrac{1}{14}\sqrt{\tfrac{7}{79}}\,\bar g^2,\quad \bar h=\tfrac{3}{2}\sqrt{\tfrac{7}{79}}\,\bar g^2,\quad \bar I^2=\bar g^2,\\
&4)\ \bar h_A=\bar h_B=\bar h_{AB}=-\tfrac{1}{14}\sqrt{\tfrac{7}{79}}\,\bar g^2,\quad \bar h=-\tfrac{3}{2}\sqrt{\tfrac{7}{79}}\,\bar g^2,\quad \bar I^2=\bar g^2.
\end{aligned} \qquad (18)$$

The situation on the phase plane of variables $\bar h$, and $\bar g^2$ is illustrated on Fig.4.

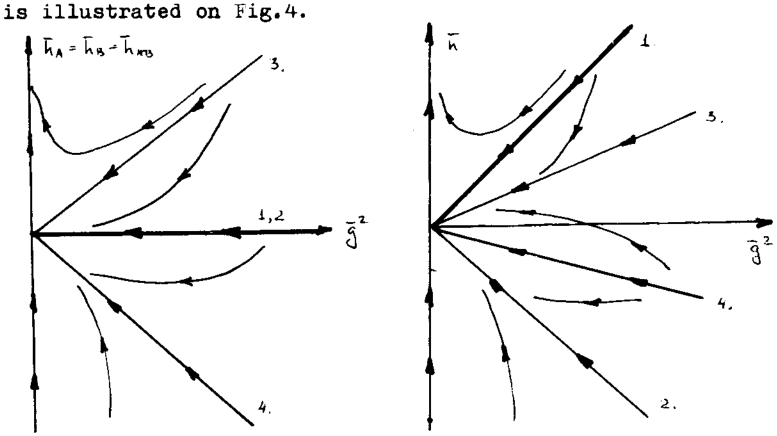

Fig. 4.

The obtained singular solutions are remarkable due to the fact that they lead to AF in all the coupling constants. AF in such a theory is closely connected with the existence of unstable singular solutions of the RG equations and is possible on these solutions only. In this respect it would be very usefull if there existed some kind of symmetry strictly keeping us to unstable singular solution. Consider from this point of view the Lagrangians corresponding to the singular solutions (18):

$$\mathcal{L}^{1,2}=-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}+i\bar\Psi D_\mu\gamma_\mu\Psi+\frac{1}{2}(D_\mu A^a)^2+\frac{1}{2}(D_\mu B^a)^2+$$
$$+g\bar\Psi T^a_\Psi(A^a+\gamma^5B^a)\Psi\mp\frac{1}{2}g^2\epsilon^{abc}\epsilon^{ade}A^bB^cA^dB^e, \qquad (19)$$

$$\mathcal{L}^{3,4}=-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}+i\bar\Psi D_\mu\gamma_\mu\Psi+\frac{1}{2}(D_\mu A^a)^2+\frac{1}{2}(D_\mu B^a)^2+$$
$$+g\bar\Psi T^a_\Psi(A^a+\gamma^5B^a)\Psi\mp\frac{1}{14}\sqrt{\frac{7}{79}}g^2(A^aA^a+B^aB^a)^2\mp\frac{3}{2}\sqrt{\frac{7}{79}}\epsilon^{abc}\epsilon^{ade}A^bB^cA^dB^e. \qquad (20)$$

The Lagrangian $\mathcal{L}^1$ coincides with the supersymmetrical one of the SSFZ model/2,3/ up to the mass terms. Thus, we again obtain that supersymmetry in a given set of fields is realized as unstable singular solution of the RG equations*). As for Lagrangians $\mathcal{L}^2$, $\mathcal{L}^3$ and $\mathcal{L}^4$ one does not know any symmetry with respect to which they are invariant and more over it is not clear if there is such a symmetry at all. To clear up this question we are to examine if any of these solutions retain straight on the two -loop level. It should be noted that in this approximation the function φ_g of eq. (14a) is independent of scalar couplings, while the corresponding functions φ_I of eqs. (14b) and (14c) depend on them. It is evident that the singular solution (15) holds when the functions φ_I and φ_g coincide with each other. Due to the fact that φ_g is independent of h_i this may be realized only for certain values of $\bar{h}_A$, $\bar{h}_B$, $\bar{h}_{AB}$ and $\bar{h}$. Consider the contribution of scalar interactions to the φ_{I_A} function in the second approximation:

$$\varphi_{I_A}^{(h)} = \bar{I}_A^2\left(80\bar{h}_A^2 + 48\bar{h}_{AB}^2 + 12\bar{h}^2 + 32\bar{h}_{AB}\bar{h} - 80\bar{h}_A\bar{I}_A^2 - 48\bar{h}_{AB}\bar{I}_B^2 - 16h\bar{I}_B^2\right), \quad (21)$$

We use the fact that

$$\varphi_{I_A} = \varphi_{I_B} = \varphi_g \quad (22)$$

for $g = I_A = I_B$, $h_A = h_B = h_{AB} = 0$, $h = \frac{1}{2}g^2$. Then $\varphi_{I_A}^{(h)} = \varphi_g = -5g^6$. Consider other solutions (18). From (21) it follows that equality (22) is broken down.

Thus only the first solution of four singular solutions (18) retains straight if one takes into account higher corrections of perturbatuion theory, and realizes the supersymmetry in the given system of fields. The other trajectories on Fig.4 are distorted and

*) This fact for the SSFZ model was for the first time noted by M.Suzuki /9/.

do not lead to such a kind of symmetry. Therefore there exist no other symmetry linearly connecting the coupling constants in the given set of fields.

5. Conclusion

So, the considered different models and types of interactions confirm the efficiency of the proposed method. In all the cases when some kind of the known symmetry (isotopical, supersymmetry) can be realized in the theory, this symmetry was revealed on the phase plane and the corresponding symmetrical Lagrangians were "reconstructed".

It also follows that the accepted regularization-renormalization procedure in all the cases does not destroy the initial symmetry of the Lagrangian.

Unfortunately, in the models under consideration we have not succeeded in obtaining any new symmetry, except for already known isotopical one (§ 2) and supersymmetry (§§ 3,4). There exist no other symmetries linearly connecting the coupling constants in the considered sets of fields. However, while the scalar and Yukawa couplings have been considered rather completely, the incorporation of gauge fields contains wide possibilities.

A general property of arising symmetries is that all of them are realized on unstable singular solutions of the RG equations. The exception forms only scalar interaction (1) with $h_{ijkl} < 0$ (see Fig.1). However, such a theory is usually considered to be unacceptable as it has no ground state in the quasi-classical limit.

In all the Lagrangians under consideration we have not written down the mass terms. This is explained, firstly, by the fact that in logarithmical theories all the masses always become dimensionless by the powers of momenta and vanish in ultra-violet region and, secondly, by the fact that in such theories the renormalization procedure can be formulated and is really formulated in such a way, that the counter terms in a Lagrangian are independent of masses. Therefore, to the obtained Lagrangians we can always add the mass terms which are symmetrical or softly breaking the symmetry. They will not lead to the distortion of the phase trajectories and will not break the symmetry of the interaction Lagrangian.

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Appendix

1. Here we represent the expression for the Gell-Mann-Low functions of eq.(2) for the binar interactions, when $n=2$. The Lagrangian (1) in this case looks like:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi_1)^2 + \frac{1}{2}(\partial_\mu\varphi_2)^2 - \frac{h_1}{4!}\varphi_1^4 - \frac{h_2}{4!}\varphi_2^4 - \frac{h_{12}}{4}\varphi_1^2\varphi_2^2 .$$

The φ-functions in the three-loop approximation are*):

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*) The details of calculations see in /8/.

$$\varphi_{h_1} = \frac{1}{16\pi^2}\,\frac{3}{2}(h_1^2 + h_{12}^2) - \frac{1}{(16\pi^2)^2}\left(\frac{17}{6}h_1^3 + \frac{5}{2}h_1 h_{12}^2 + 6h_{12}^3\right) + \frac{1}{(16\pi^2)^3}\Big[\Big(6\zeta(3) - \frac{9}{2}I + \frac{31}{2} - \frac{1}{4}\ln\frac{3}{4}\Big)h_1^4 + \Big(\frac{81}{4} - \frac{21}{2}I - \frac{3}{4}\ln\frac{3}{4}\Big)h_1^2 h_{12}^2 + \Big(18\zeta(3) - \frac{51}{2}I + \frac{195}{4} - \frac{3}{4}\ln\frac{3}{4}\Big)h_{12}^4 + (24\zeta(3) + 6I + 27)h_1 h_{12}^3 + \Big(\frac{3}{2}I - \frac{7}{4} - \frac{1}{4}\ln\frac{3}{4}\Big)h_2^2 h_{12}^2 + \Big(\frac{9}{4} - 3I\Big)h_1 h_2 h_{12}^2 + 12 h_2 h_{12}^3\Big],$$

$$\varphi_{h_2} = \varphi_{h_1}(1 \leftrightarrow 2), \qquad \text{(A.1)}$$

$$\varphi_{h_{12}} = \frac{1}{16\pi^2}h_{12}\Big(2h_{12} + \frac{h_1+h_2}{2}\Big) - \frac{1}{(16\pi^2)^2}h_{12}\Big(\frac{9}{2}h_{12}^2 + \frac{5}{12}(h_1^2+h_2^2) + 3h_{12}(h_1+h_2)\Big) + \frac{1}{(16\pi^2)^3}h_{12}\Big[h_{12}^3\Big(12\zeta(3) - 13I + 41 + \ln\frac{3}{4}\Big) + h_{12}^2(h_1+h_2)\Big(12\zeta(3) - 5I + \frac{205}{8} + \frac{1}{4}\ln\frac{3}{4}\Big) + h_{12}(h_1^2+h_2^2)\Big(6\zeta(3) - 5I + \frac{65}{6} + \frac{1}{6}\ln\frac{3}{4}\Big) + h_1h_2h_{12}(7 - 3I) + (h_1^3 + h_2^3)\Big(\frac{37}{24} + \frac{\ln 3/4}{12}\Big)\Big],$$

where $\zeta(3) \approx 1.2$, $I \approx 3/4$.

2. The β-functions of the Ovsiannikov equations for the Lagrangian (12) on the two-loop level are:

$$\beta_{g_A} = \frac{g_A^2}{16\pi^2}\Big[g_A^2(3T^2 - 2 + 2tk) + g_B^2(2 - T^2)\Big] + \frac{g_A^2}{(16\pi^2)^2}\Big[g_A^4\Big(-\frac{3}{4}(T^2)^2 + 2 - 12tkT^2 + 6tk\Big) + g_A^2 g_B^2\Big(\frac{15}{2}(T^2)^2 - 20T^2 - T^2tk - 2tk + 12\Big) + g_B^4\Big(\frac{7}{4}(T^2)^2 - 4T^2 + T^2tk - 4tk + 2\Big) - g_A^2 h_A\Big(2T^2 - \frac{2}{3}\Big) - g_B^2 h_{AB}\,2(T^2+1) - 4g_B^2 h + \frac{5}{36}h_A^2 + \frac{3}{4}h_{AB}^2 + h_{AB}h + \frac{3}{4}h^2\Big],$$

$$\beta_{g_B} = \beta_{g_A}(A \leftrightarrow B), \qquad \text{(A.2)}$$

$$\beta_{h_A} = \frac{1}{16\pi^2}\Big[\frac{3}{2}h_A^2 + \frac{9}{2}h_{AB}^2 + 6hh_{AB} + 3h^2 + 4g_A^2 h_A tk - \frac{24tk(3T^2-1)}{5}g_A^4\Big] +$$

$$+\frac{1}{(16\pi^2)^2}\Big[-\frac{23}{6}h_A^3-\frac{15}{2}h_{AB}^2h_A-18h_{AB}^3-10h_{AB}hh_A-\frac{11}{2}h^2h_A-36h_{AB}^2h-42h_{AB}h^2-$$

$$-15h^3-\frac{22}{3}h_A^2I_A^2tk-18h_{AB}^2I_B^2tk-24h_{AB}hI_B^2tk-12h^2I_B^2tk+2I_A^4h_Atk\cdot$$

$$\cdot\left(\frac{19}{5}T^2-\frac{18}{5}\right)-2I_A^2I_B^2h_Atk(T^2+1)-24I_A^2I_B^2h_{AB}tk(T^2-1)-24I_A^2I_B^2htk\cdot$$

$$\cdot\left(\frac{2T^2}{5}-\frac{1}{5}\right)+\frac{48tk}{5}\left(12(T^2)^2-14T^2+5\right)I_A^6+\frac{48tk}{5}(2T^2+1)I_A^4I_B^2\Big],$$

$$\beta_{h_B}=\beta_{h_A}(A\leftrightarrow B),$$

$$\beta_{h_{AB}}=\frac{1}{16\pi^2}\Big[2h_{AB}^2+\frac{5}{6}(h_A+h_B)h_{AB}+\frac{h_A+h_B}{3}h+h^2+2tk(I_A^2+I_B^2)h_{AB}-$$

$$-\frac{8tk(3T^2-1)}{5}I_A^2I_B^2\Big]+\frac{1}{(16\pi^2)^2}\Big[-\frac{11}{2}h_{AB}^3-5h_{AB}^2(h_A+h_B)-2h_{AB}^2h-\frac{17}{36}h_{AB}(h_A^2+h_B^2)$$

$$-\frac{2}{9}h(h_A^2+h_B^2)-4h_{AB}h(h_A+h_B)-\frac{4}{3}h^2(h_A+h_B)-\frac{25}{2}h_{AB}h^2-5h^3-4tkh_{AB}^2(I_A^2+I_B^2)$$

$$-2tkh^2(I_A^2+I_B^2)-\frac{10}{3}tkh_{AB}(h_AI_A^2+h_BI_B^2)-\frac{14}{3}tkh(h_AI_A^2+h_BI_B^2)-$$

$$-(I_A^4+I_B^4)h_{AB}tk(T^2+2)+(I_A^4+I_B^4)htk\left(\frac{8}{5}T^2-\frac{16}{5}\right)-(h_A+h_B)I_A^2I_B^2tk\cdot$$

$$\cdot\frac{4}{15}(11T^2-7)+h_{AB}I_A^2I_B^2tk\left(\frac{86}{5}T^2-\frac{52}{5}\right)-hI_A^2I_B^2tk\frac{16}{5}(2T^2-4)+$$

$$+\frac{48tk}{5}\left(2(T^2)^2-2T^2+1\right)I_A^2I_B^2(I_A^2+I_B^2)\Big],$$

$$\beta_h=\frac{1}{16\pi^2}\Big[\frac{5}{2}h^2+2h_{AB}h+\frac{h_A+h_B}{3}h+2tk(I_A^2+I_B^2)h+\frac{16tk}{5}(T^2-2)I_A^2I_B^2\Big]+$$

$$+\frac{1}{(16\pi^2)^2}\Big[-h^3-14h^2h_{AB}-\frac{1}{2}h^2(h_A+h_B)-\frac{16}{5}h_{AB}h(h_A+h_B)-\frac{27}{2}h_{AB}^2h-$$

$$-\frac{13}{36}h(h_A^2+h_B^2)-3h^2(I_A^2+I_B^2)tk-4h_{AB}h(I_A^2+I_B^2)tk-\frac{4}{3}tkh(h_AI_A^2+$$

$$+h_BI_B^2)+h(I_A^4+I_B^4)tk\frac{7}{5}(T^2+2)-hI_A^2I_B^2tk\left(\frac{42}{5}T^2-\frac{4}{5}\right)-h_{AB}I_A^2I_B^2tk\cdot$$

$$\cdot\frac{32}{5}(2T^2-3/2)+(h_A+h_B)I_A^2I_B^2\frac{16}{5}\left(T^2+\frac{1}{2}\right)-\frac{16tk}{5}\left(2(T^2)^2+T^2-2\right)I_A^2I_B^2(I_A^2+I_B^2).$$

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