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CLASSIFICATION OF DOMAINS OF CLOSED OPERATORS

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Классификация областей определения замкнутых операторов

Исследуется структура областей определения замкнутых операторов в гильбертовом пространстве при помощи пространств последовательностей. Окончательная классификация дает три класса этих областей. Получены необходимые и достаточные условия для унитарной эквивалентности данных областей в виде эквивалентности соответствующих последовательностей натуральных чисел. Отмечена связь с теорией возмущений.

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Classification of Domains of Closed Operators

The structure of domains of closed operators in Hilbert space is investigated by means of sequence spaces. A complete classification leads to three classes of domains. We obtained necessary and sufficient conditions for the unitary equivalence of domains expressed by the equivalence of appropriate sequences of naturals. A connection with perturbation theory is mentioned.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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INTRODUCTION

In this paper a complete survey is given about the structure of domains of closed operators in Hilbert space $\mathcal{H}$. Thereby we suppose that the structure of such a domain $\mathcal{D}$ is sufficiently clear described, if there is a space of sequences contained in ℓ^2 which is isomorphic to $\mathcal{D}$.

A sequence of naturals which determines the space of sequences is assigned to any domain in a natural way.

The given classification leads in effect to three types of domains of closed operators. It is naturally to regard, instead of single domains, classes of unitarily equivalent domains (cf. Def. 3). Then, a class of equivalent sequences of naturals, is assigned to any class of unitarily equivalent domains where for the different types of domains different notions of equivalence of sequences are used. In this way one also gets necessary and sufficient conditions for the unitary equivalence of domains of closed operators. This classification is closely connected with perturbation theory and a proposition is obtained which seems to be new.

When our investigations were finished we found by accident two papers of Dixmier[2,3] which are almost completely fallen into oblivion and to which no reference is in the appropriate literature.

Our results have partly coincided in detail with the classification of Dixmier and in the present paper our notions and notations are closely related to his ones.

In these papers of Dixmier the much more complicated class III of domains is not investigated further.

1. PRELIMINARIES

By $\mathbb{N},\mathbb{R},\mathbb{C}$ we denote the set of naturals, real and complex numbers resp. Let H be a separable Hilbert space, $D \subset H$ a dense linear manifold, the domain of a closed operator. Obviously, this is equivalent to: $D = D(A)$, $A = A^* \geq I$, where I is the identity operator on H. From the spectral theorem it follows that on any such a domain there exists a diagonal operator $S = S\{(s_n),(\phi_n)\}$ that is

$$D = \{\phi = \Sigma x_n \phi_n : \Sigma |x_n|^2 (s_n)^2 < \infty\},$$

where (ϕ_n) is an orthonormal basis of H contained in D, (s_n) a sequence of real numbers unbounded if $D \neq H$.

Remark 1: i) Obviously, for any given diagonal operator $A = A\{(a_n),(\phi_n)\}$ one can find a diagonal operator $A' = A'\{(a_n'),(\phi_n)\}$ such that $a_n' > 1$ naturals and $D(A) = D(A')$. ii) Clearly, $D(A) = D(A')$, where $A = A\{(a_n,(\phi_n)\}$, $A' = A'\{(a_n'),(\phi_n)\}$ and (a_n') is obtained from (a_n) by a finite number of changes. In what follows we need some notions about such sequences.

Definition 1: Let $(a_n),(b_n)$ be two sequences of positive numbers C_i suitable positive constant then $(a_n),(b_n)$ are said to be

i) comparable, if there is a permutation π of $\mathbb{N}$ such that

$$C_1 \leq \frac{a_n}{b_{\pi(n)}} \leq C_2, \tag{1}$$

ii) equivalent $(\sim)$ if

$$C_3 \leq \frac{a_n}{b_n} \leq C_4. \tag{2}$$

iii) essentially equivalent $(\tilde{\ell})$, if there are $k,\ell \in \mathbb{N}$ such that $(a_k,a_{k+1},\ldots)$ and $(b_\ell,b_{\ell+1},\ldots)$ are equivalent, that is,

$$C_5 \leq \frac{a_{k+i}}{b_{\ell+i}} \leq C_6 \qquad \text{for } i = 1,2,\ldots \tag{3}$$

iv) weakly equivalent $(\tilde{w})$, if there are equivalent sequences $(\hat{a}_n) = (a_n^\sigma)$, $(\hat{b}_n) = (b_n^\tau)$ obtained from $(a_n),(b_n)$ resp., in the following way: let σ,τ resp. be monotone mappings from $\mathbb{N}$ onto $\mathbb{N}$, then $(\hat{a}_n) = (a_n^\sigma) = (a_{\sigma(n)})$, $(b_n) = (b_n^\tau) = (b_{\tau(n)})$ resp. For example, $(\hat{a}_n)$ could be: $(a_1,a_1,a_2,a_3,a_3,a_3,a_4,\ldots)$.

v) partially equivalent $(\tilde{p})$ if there are equivalent subsequences (a_{k_n}) and (b_{ℓ_n}) of $(a_n),(b_n)$ resp.

Definition 2

i) Let (s_n) be a sequence and (t_n) a subsequence of (s_n), $(t_n) \subset (s_n)$ then we denote by $(a_n) = (s_n) - (t_n)$ the subsequence of (s_n) which we obtain from (s_n) by cancelling the elements (t_n). We also write $(s_n) = (t_n) \cup (a_n)$.

ii) Let (a_n) be a sequence of naturals, then by $(a_n)'$ we denote the set of accumulation points of (a_n) and by (a_n') we mean the sequence $(a_n)' - \{\infty\}$ such that $a_1' \leq a_2' \leq \ldots$.

Definition 3:

i) Two operators A,B are called equivalent, if $A=KBL$ with K,L,K^{-1},L^{-1} bounded; they are called unitarily equivalent, if $A=UBU^{-1}$, U a unitary operator.

ii) Two linear manifolds D_1,D_2 are called linearly equivalent, if $D_2=KD_1$, K,K^{-1} bounded; unitarily equivalent, if $D_2=UD_1$, U unitary.

Because linearly equivalent domains of closed operators are unitarily equivalent, too /2/, we use in this case only the notion "equivalent".

In what follows we make essential use of the following theorem of Köthe /4/:

Theorem: Two bounded operators $A=A\{(a_n),(\phi_n)\}$ and $B=B\{(b_n),(\psi_n)\}$ are equivalent iff the sequences (a_n) and (b_n) are comparable.

A domain D of a closed operator can be equipped with a natural topology t given by the scalar product

$$\langle\phi,\psi\rangle_T = \langle\phi,\psi\rangle + \langle T\phi,T\psi\rangle, \quad \phi,\psi\in D,$$

where $\langle,\rangle$ is the scalar product of H and $D=D(T)$, T a closed operator. The associated norm we denote by $\|\ \|_T$. Equipped with this topology $D[t]$ is a Hilbert space. From the closed graph theorem it easily follows that the topology t is independent of the choice of the closed operator T on D /2/.

2. CLASSIFICATION OF DOMAINS OF CLOSED OPERATORS

We start with a classification of sequences (s_n), $s_n\geq 1$ naturals. Thereby we are

only interested in sequences (s_n) with $\sup s_n = \infty$. Then we give a classification of diagonal operators which is the basis of the classification of domains.

C_1: Classification of sequences

Let (s_n) be a sequence of naturals, as mentioned above, and

$$M_n = \{s_j\in(s_k) : s_j = n\}$$

(s_n) is said to be of

Class I: if all M_n are finite or empty sets, i.e., $\lim s_n=\infty$ or $(s_n)'=\{\infty\}$ (cf. Def. 2),

Class II: if there is a finite number of infinite sets $M_{k_1},M_{k_2},\dots,M_{k_s}$, i.e., $(s_n)'$ contains except ∞ at least one finite point. Therefore we have the following (non-unique) decomposition: $(s_n)=(s_n^b)\cup(s_n^\infty)$ where (s_n^b) is a bounded sequence and $\lim s_n^\infty=\infty$, i.e., (s_n^∞) is of class I.

Example: $(s_n)=(1,2,1,3,2,4,1,2,5,1,2,6,1,2,7,\dots)$. Now we have: $(s_n)'=\{1,2,\infty\}$, $(s_n^b)=(1,2,1,2,1,2,\dots)$, $(s_n^\infty)=(3,4,5,6,\dots)$,

Class III: if there is an infinite number of infinite sets M_{k_j}, $j=1,2,\dots$, i.e., $(s_n)'$ is infinite and $\lim s_n'=\infty$. So, to (s_n) we can assign two sequences: (s_n') and (s_n^0), where (s_n^0) is obtained from (s_n) by cancelling all numbers s_n'. We arrange (s_n^0) such that $s_1^0\leq s_2^0\leq\dots$.

Example: By the well-known diagonal procedure we form (s_n) from the following sequences: $(2,4,6,\ldots), (1,1,\ldots), (3,3,\ldots),\ldots$ .Then we have: $(s_n') = (1,3,5,\ldots)$ and $(s_n^0) = (2,4,6,\ldots)$.

We use also the notation $(s_n) \in I$ and so on. It is clear that this classification is complete and disjoint.

C_2: Classification of (unbounded) diagonal operators

Let $S = S\{(s_n),(\phi_n)\}$ be an unbounded diagonal operator. S is said to be of

Class I iff $(s_n) \in I$
Class II iff $(s_n) \in II$
Class III iff $(s_n) \in III$

According to Remark 1 without loss of generality we restrict ourselves, in this classification, to the case $s_n \geq 1$ naturals. As above, we use the notation $S \in I$ and so on.

C_3: Classification of domains of closed operators

Let D be the domain of a closed operator. D is a said to be of

Class I iff on D there exists a diagonal operator $S \in I$
Class II iff on D there exists a diagonal operator $S \in II$
Class III iff on D there exists a diagonal operator $S \in III$.

Here the same notation as above is used: $D \in I$, and so on. Obviously, this classification has to be justified, i.e., we have to show that the classification is independent of the choice of the diagonal operator S on D. In the following usefull remark we give a first justification based on the Theorem of Köthe (see also /2/). Later a second justification based on a perturbation Theorem is given.

Remark 2: Let S, T be two diagonal operators on $D = D(S) = D(T)$. Then it is easy to see that S and T are equivalent. Therefore, S^{-1} and T^{-1} are equivalent, too. By the Theorem of Köthe their sequences (s_n^{-1}) and (t_n^{-1}) are comparable and, consequently, (s_n) and (t_n) are comparable, too. Now it is easy to show that S and T are of the same class, because comparable sequences must be of the same class (the proof is easy and omitted). Next we give some consequences from the classification which give us some information about the geometry of such domains (for class I and II see also /2/).

Proposition 1

i) $D \in I$ iff S^{-1} is completely continuous for any invertible closed operator S on D.

ii) $D \in I$ iff the imbedding $D[t] \to H$ is completely continuous.

Proof:

i) it is valid obviously for any diagonal operator and therefore for any closed invertible operator on D.

ii) By using i) one shows that $E_T = \{\phi \in D : \|T\phi\| \leq 1, T > I, D(T) = D\}$ is a compact set of H which proves ii).

For class II as an immediate consequence of the definition, we obtain the following proposition:

Proposition 2

$D \in II$ if and only if $D = H_0 \oplus D_0$, H_0 infinitely dimensional and $\| \|$-closed and $D_0 \in I$.

Now we go on to class III which is much more interesting and complicated. Let $S = S\{(s_n), (\phi_n)\}$ be a diagonal operator on $D \in III$. As in the classification C_1 we consider the two sequences (s_n') and (s_n^0) which leads to the following decomposition of D:

$$D = \Sigma_{(s_n')} \oplus H_n \oplus D_0$$

H_n are the infinitely dimensional eigenspaces of the eigenvalues s_n' resp., and $\Sigma_{(s_n')}$ means:

$$\Sigma_{(s_n')} \oplus H_n = \{\phi = \Sigma \phi_n : \phi_n \in H_n, \Sigma (s_n')^2 \|\phi_n\|^2 < \infty\}$$

further

$$D_0 = \{\phi = \Sigma x_n \phi_n^0 : \Sigma |x_n|^2 (s_n^0)^2 < \infty\},$$

where ϕ_n^0 is the eigenvector for the eigenvalue s_n^0. If the sequence (s_n^0) is infinite, then we have $D_0 \in I$. We also will use the notation $D_0 \hat{=} (s_n^0)$. These considerations give us the following proposition:

Proposition 3

$D \in III$ iff $D = \Sigma_{(s_n')} \oplus H_n \oplus D_0$, $\dim D_0 < \infty$ or $D_0 \in I$.

Remark 3: With respect to Remark 1, ii) we always may exclude the case in which D_0 is finite dimensional, i.e., the case, where we have only finite eigenvalues with finite multiplicity. In what follows we need some notions. Let

$$D = \Sigma_{(t_n)} \oplus H_n \oplus D_0, \quad D_0 \hat{=} (a_n).$$

We say that D_0 can be reduced, if there is an infinite subsequence $(b_n) \subset (a_n)$ such that from

$$D = \Sigma_{(t_n)} \oplus H_n \oplus D_1 \oplus D_2, \quad D_1 \hat{=} (b_n), D_2 \hat{=} (a_n) - (b_n)$$

it follows: $D = \Sigma_{(t_n)} \oplus \hat{H}_n \oplus D_2$

with $H_n \subset \hat{H}_n$, $\dim(\hat{H}_n \ominus H_n) < \infty$ for all n (Clearly, a "finite reduction", i.e., (b_n) is a finite subsequence, is always possible). We say that D_0 can be maximal reduced, if there is such a reduction of D_0 that

$$D = \Sigma_{(t_n)} \oplus \hat{H}_n \oplus D_2$$

and D_2 cannot be reduced further. We say that D_0 can be completely reduced, if there is a reduction of D_0 such that

$$D = \Sigma_{(t_n)} \oplus \hat{H}_n \qquad \text{i.e.,} \quad D_2 = (0).$$

By using these notions we are able to give a complete description of class III:

Let $D \in III$, $D = \Sigma_{(s_n')} \oplus H_n \oplus D_0$, $D_0 \hat{=} (s_n^0)$.

Then the following three classes can arise:

Class III_A if $D = \Sigma_{(s_n')} \oplus \hat{H}_n$, i.e., D_0 can be completely reduced.

Class III_B if $D = \Sigma_{(s_n')} \oplus \hat{H}_n \oplus D_1$, $D_1 \hat{=} (a_n)$ and cannot be reduced further, i.e., D_0 can be maximal reduced.

Class III_C if for any reduction of D_0 which leads to $D = \Sigma_{(s_n')} \oplus \hat{H}_n \oplus D_2$, $D_2 \hat{=} (b_n)$, D_2 can be further reduced, i.e., D_0 cannot be maximal reduced.

As usual we write $D \in III_A$, and so on.

Again we have to give a justification, i.e., it must be shown that the class of D is independent of the regarded diagonal operator. Before doing this we give a simple condition under which D_0 can be reduced, and then follow some examples. For the following simple Lemma we omit the proof.

Lemma 1

Let $D = \Sigma_{(t_n)} \oplus H_n \oplus D_0$, $D_0 \hat{=} (a_n)$.
D_0 can be reduced iff there are subsequences $(b_n) \subset (a_n)$ and $(s_n) \subset (t_n)$ such that $(s_n) \underset{w}{\sim} (b_n)$, (cf. Def. 1, iv)

From this Lemma we obtain immediately the following

Conclusion

i) D_0 can be completely reduced iff there is a subsequence $(s_n) \subset (t_n)$ such that $(s_n) \underset{w}{\sim} (a_n)$.

ii) D_0 can be maximal reduced iff we have a decomposition $(a_n) = (b_n) \cup (c_n)$ such that (b_n) is weakly equivalent to a suitable subsequence (s_n) of (t_n) and there is no subsequence (d_n) of (c_n) which is weakly equivalent to a further subsequence of (t_n).

iii) D_0 cannot be maximal reduced iff for any decomposition $(a_n) = (b_n) \cup (c_n)$ such that $(b_n) \underset{w}{\sim} (s_n)$ for a suitable subsequence (s_n) of (t_n) there is a further decomposition $(c_n) = (d_n) \cup (e_n)$ such that $(d_n) \underset{w}{\sim} (u_n)$ for (another) subsequence (u_n) of (t_n).

Now we give some examples to illustrate these classes.

1. Class III_A: If $(s_n^0) = (n)$, $(s_n') = (n^2)$, then $D \in III_A$ because $(n) \underset{w}{\sim} (n^2)$. To see this, regard the equivalent sequences:

$$(\hat{s}_n') = (1,4,4,4,9,9,9,9,9,16,16,16,16,16,16,16,\ldots)$$

$$(s_n^0) = (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,\ldots).$$

2. Class III_B: Now we choose (s_n') such that $\lim (s_{n+1}'/s_n') = \infty$, for example $s_n' = n!$. Let $(*)$ $a_n = \frac{1}{2}(s_{n+1}' + s_n')$, then $\lim(a_n/s_n') = \infty$. Now we regard the following sequence $(s_n^0) = (s_1' + 1, a_1, s_2' + 1, a_2, \ldots)$.
Obviously, we have $(s_n') \sim (s_n' + 1)$, and because of $(*)$ the domain D_1 corresponding to (a_n) cannot be reduced further, i.e., $D_0 \hat{=} (s_n^0)$ can be maximal reduced.

3. Class III_C: To give an example for class III_C we choose:

$$(s_n') = (n!), \quad (s_n^0) = (n).$$

By considerations like in 2 it is easy to see that $D_0 \hat{=} (n)$ cannot be completely reduced. To see that D_0 cannot be maximal reduced we remark the following.
Suppose we have

$$C_1 \le \frac{\hat{s}'_{n_j}}{k_j} \le C_2, \quad (k_j) \subset (n)$$

it is not difficult to see that we can find an infinite subsequence $(\ell_j) \subset [(n) - (k_j)]$ such that

$$\frac{1}{2} \cdot C_1 \le \frac{\hat{s}'_{n_j}}{\ell_j} \le 2C_2.$$

Therefore D_0 cannot be maximal reduced.

By the following proposition we give the mentioned above justification of the notation $D \in III_A$, and so on.

Proposition 4

If $D \in III_A$ (III_B, III_C resp.) with respect to the diagonal operator $S = S\{(s_n), (\phi_n)\}$, then $D \in III_A$) (III_B, III_C resp.) with respect to any other diagonal operator $T = T\{(t_n), (\psi_n)\}$ with $D(T) = D(S) = D$ i.e., the classification of class III is independent of the choice of the operator.

Proof:

1. Let $S \in III_A$, then $D = \Sigma_{(s'_n)} \oplus H_n$.

Suppose $T \in III_B$, then regard besides (t_n) the associated sequences (t'_n) and (t^0_n). By the Theorem of Köthe there is a permutation π of N such that

$$C_1 \leq \frac{s_n}{t_{\pi(n)}} \leq C_2, \quad D_1 \leq \frac{t_n}{s_{\pi^{-1}(n)}} \leq D_2, \text{ resp. } (1)$$

Especially.

$$C_1 \leq \frac{s_{n_j}}{t^0_j} \leq C_2, \qquad (2)$$

Because $\lim t^0_n = \infty$, we have $[(s_n) - (s_{n_j})]' = (s_n)'$ (regarded as sets!), i.e., by (2) none of the eigenvalues of S with infinite multiplicity is exhausted. Therefore we can find a further subsequence $(s_{k_j}) \subset [(s_n) - (s_{n_j})]$ with $s_{k_j} = s_{n_j}$ for all j. By (1) it follows that

$$C_1 \leq \frac{s_{k_j}}{t_{\pi(k_j)}} \leq C_2. \qquad (3)$$

By (2) it follows that $(t_{\pi(k_j)}) \subset [(t_n) - (t^0_n)]$ and because we can suppose that (s_{k_j}) is monotonly increasing, we see that $(t_{\pi(k_j)}) = (\hat{t}'_{\ell_j})$.

But (2) and (3) give now the equivalence of (t^0_j) and $(\hat{t}_{\ell_j})$ which means $D(T) \in III_A$. The same proof gives us that it is impossible that $D(T) \in III_C$.

2. By a slight modification of the considerations above we may prove that from $D(S) \in III_C$ and $D(S) = D(T)$ it follows that $D(T) \in III_C$. Because these are, in principle, the same considerations we omit an explicit proof.

Now we go on with an interesting lemma of perturbation theory which gives us a second proof that the classification is correct.

Lemma 2

Let $D \in II$, A a closed operator on D such that A^{-1} exists and is bounded; further let (λ_n) be a sequence of the spectrum $\sigma(A)$ of A such that $\lambda_n \to \lambda$, $|\lambda_n|$, $|\lambda|$ sufficiently large. If we suppose that there is a sequence $(\phi_n) \subset D$ with $\|\phi_n\| = 1$ and $A\phi_n - \lambda_n\phi_n \to 0$, then (ϕ_n) contains a strongly convergent subsequence $(\tilde{\phi}_n)$.

Proof:

From $D \in II$ there follow the decompositions $H = H_0 \oplus H_1$, , $D = H_0 \oplus D_1$, , $D_1 = D \cap H_1$,, D_1 of type I, $\phi_n = \phi^0_n \oplus \phi^1_n$ and for the identity I we have $I = P_0 \oplus P_1$, P_0, P_1 -projections. Further, A has the representation

$$A = \begin{pmatrix} A_1 & T^*_1 \\ T_1 & B_1 \end{pmatrix} \quad \text{with} \quad \begin{matrix} A_1, T_1, T^*_1\text{-bounded} \\ B_1\text{- unbounded.} \end{matrix}$$

Therefore we obtain $A = B + T$ with the operators

$$B = \begin{pmatrix} 0 & 0 \\ 0 & B_1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} A_1 & T^*_1 \\ T_1 & 0 \end{pmatrix}$$

B_1^{-1} exists and is compact. From

$$(A-\lambda_n)\phi_n-(B+T-\lambda_n)\phi_n\to 0 \quad , \tag{4}$$

multiplying by P_1, P_0 resp., one has

$$B_1\phi_n^1+P_1(T-\lambda_n)\phi_n\to 0\,, \tag{5}$$

$$P_0(T-\lambda_n)\phi_n\to 0\,. \tag{6}$$

From (5) multiplying by B_1^{-1} we obtain

$$\phi_n^1+B_1^{-1}P_1(T-\lambda_n)\phi_n\to 0\,. \tag{7}$$

The boundedness of (ϕ_n) gives the boundedness of $(T-\lambda_n)\phi_n$ and $P_1(T-\lambda_n)\phi_n$. Because B_1^{-1} is compact, we can find a convergent subsequence $\tilde{\psi}_n=B_1^{-1}P_1(T-\tilde{\lambda}_n)\tilde{\phi}_n$ of the compact set $B_1^{-1}P_1(T-\lambda_n)\phi_n$. This and (7) give that $(\tilde{\phi}_n^1)$ is convergent. From (6) we obtain

$$P_0(T-\tilde{\lambda}_n)\tilde{\phi}_n^0+P_0(T-\tilde{\lambda}_n)\tilde{\phi}_n^1\to 0\,,$$

and, consequently, the convergence of $P_0(T-\tilde{\lambda}_n)\tilde{\phi}_n^1$ and $P_0(T-\tilde{\lambda}_n)\tilde{\phi}_n^0$. For sufficiently large $|\tilde{\lambda}_n|$ the bounded inverse of $P_0(T-\tilde{\lambda}_n)$ exists, which gives the convergence of the sequence $(\tilde{\phi}_n^0)$. Thus, we have proved the existence of a convergent subsequence $(\tilde{\phi}_n=\tilde{\phi}_n^0+\tilde{\phi}_n^1)$ of (ϕ_n).

Q.E.D.

Conclusion

Let $D\in \mathrm{II}$ and A a closed operator on D with bounded inverse A^{-1}. If $\lambda\in\sigma(A)$ and $|\lambda|$ sufficiently large, then λ is an eigenvalue of A.

Proof:

From $\lambda\in\sigma(A)$ there follows the existence of a sequence $(\phi_n)\in D(A)$ such that $\|\phi_n\|=1$, $A\phi_n-\lambda\phi_n\to 0$. The Lemma gives now the existence of a strongly convergent subsequence $(\tilde{\phi}_n)$, i.e., $\tilde{\phi}_n\to\phi$. But this means that λ is an eigenvalue of N. Now we are able to prove the following

Theorem 1

The classification C_3 of domains of closed operators is correct, i.e., the classes I, II, III_A, III_B and III_C are disjoint.

Proof:

It is trivial that class I and class II, III resp. are disjoint. The disjointness of the classes III_A, III_B and III_C has been proved in Proposition 4. Thus, it remains to show that classes II and III are disjoint. But if $D\in\mathrm{II}$ and $D=D(A)$, then for sufficiently large $|\lambda|$, $\lambda\in\sigma(A)$, λ is an eigenvalue of A with finite miltiplicity, that follows from the Conclusion. Similar considerations give that for sufficiently large $|\lambda|$ this λ cannot be an accumulation point of eigenvalues. This shows that $D\notin\mathrm{II}$ with respect to an arbitrary closed operator A.

As a consequence of this classification one gets a remarkable perturbation theorem:

Theorem

Let T be an unbounded, positive selfadjoint operator which has, outside of a bounded interval $(0,N)$ only eigenvalues $\lambda_1\le\lambda_2\le\dots$ with finite multiplicity. Any perturbation $T+S$ with a bounded selfadjoint operator S is of the same type.

Let $\mu_1\le\mu_2\le\dots$ the eigenvalues of with finite multiplicity, then

$$C_1 \le \frac{\lambda_{k+\ell}}{\mu_{k+m}} \le C_2$$

for all k with some positive numbers C_1, C_2 and suitable integers ℓ and m.

In other words, the eigenvalues of the two operators T and $T+S$ have the same asymptotic behaviour at infinity. We remark that, of course, we can replace the operator S by an semibounded operator such that on $D = D(T)$ the perturbation $T+S$ is semibounded from below and selfadjoint. By the Theorem of Köthe we arrive now at a complete characterisation of the equivalence of domains of closed operators by equivalence classes of sequences (for classes I,II cf. /2/). In what follows, D and D' are domains of closed operators.

Theorem 2

i) Let $D, D' \in$ I then D and D' are equivalent iff there are two diagonal operators $S = S\{(s_n),(\phi_n)\}$, $D(S) = D$ and $T = T\{(t_n),(\psi_n)\}$ $D' = D(T)$ such that the sequence $(s_n),(t_n)$ are equivalent, i.e.,

$$D \sim D' \qquad \text{iff } (s_n) \sim (t_n).$$

ii) Let $D, D' \in$ II, then D and D' are equivalent iff there are two diagonal operators $S = S\{(s_n),(\phi_n)\}$, $D(S) = D$, and $T = T\{(t_n),(\psi_n)\}$ $D' = D(T)$ such that for the decomposed sequences $(s_n) = (s_n^b) \cup (s_n^\infty)$ and $(t_n) = (t_n^b) \cup (t_n^\infty)$ we have that (s_n^∞) and (t_n^∞) are essentially equivalent, i.e.,

$$D \sim D' \qquad \text{iff } (s_n^\infty) \underset{e}{\sim} (t_n^\infty).$$

The proof is, for example, contained in /2/ and therefore omitted here. The next Theorem characterizes class III.

Theorem 3

Let $D, D' \in$ III, D and D' are equivalent if and only if there are diagonal operators $S = S\{(s_n),(\phi_n)\}$ and $T = T\{(t_n),(\psi_n)\}$ such that we have the decompositions

$$D = D(S) = \Sigma_{(s_n')} \oplus H_n \oplus D_0, \quad D_0 \hat{=} (s_n^0)$$

$$D' = D(T) = \Sigma_{(t_n')} \oplus \hat{H}_n \oplus \hat{D}_0, \quad D_0 \hat{=} (t_n^0),$$

and for the following classes there holds that

III_A: $(s_n'), (t_n')$ are weakly equivalent

III_B: $(s_n'), (t_n')$ are weakly equivalent

$(s_n^0), (t_n^0)$ are essentially equivalent

III_C: $(s_n'), (t_n')$ are weakly equivalent and there are decompositions

$(s_n^0) = (a_n) \cup (b_n)$,

$(t_n^0) = (c_n) \cup (d_n)$,

and subsequences

$(u_n) \subset (s_n')$, $(v_n) \subset (t_n')$

and suitable monotone mappings σ, τ of N onto N such that

$(a_n), (c_n)$ are equivalent

$(b_n), (u_n^\sigma) = (\hat{u}_n)$ are equivalent

$(d_n), (v_n^\tau) = (\hat{v}_n)$ are equivalent.

Before this Theorem will be proved we state the following remark.

Remark 4: i) The condition for case III_C is much more complicated because of the following fact: though we have the same simple situation as in case III_B we can obtain the associated condition only after a suitable reduction of D_0 and $\hat{D}_0$. This is the reason for the mentioned decomposition and for the choice of the subsequences (u_n) and (v_n). ii) Theorems 2 and 3 give us now (in case $D = D'$) a complete description of the structure of domains of closed operators by certain equivalence classes of sequences of naturals.

Proof of Theorem 3:

The weak equivalence of (s'_n) and (t'_n) will be proved at the end. We remark further that it is enough to prove the theorem only for the case $D = D'$, because the general case can easily be obtained from this one.

As usual, we regard the sequences (s_n), (s'_n), (s^0_n) and (t_n), (t'_n), (t^0_n). The Theorem of Köthe gives

$$C_1 \leq \frac{s_n}{t_{\pi(n)}} \leq C_2 \qquad (*)$$

III_B: If in the numerator of $(*)$ the elements (s'_n) stand then we have only a finite number of the elements (t^0_n) in the denominator and conversely, if in the numerator the elements (s^0_n) stand, then in the denominator we have only a finite number of the elements (t'_n) because otherwise we would obtain a contradiction with the assumption that D_0 is maximal reduced. But these considerations lead immediately to the essential equivalence of the sequences (s^0_n) and (t^0_n)

III_C: If in the numerator of $(*)$ there stand the (s^0_n), then not only elements of (t'_n), can be in the denominator

because from this we would obtain that D_0 could be completely reduced. Therefore we have the following decomposition of (s^0_n): $(s^0_n) = (a_n) \cup (b_n)$ such that the (a_n) correspond to the elements of (s^0_n) which stand in $(*)$ together with elements of (t'_n). The (b_n) correspond to those elements of (s^0_n) which stand in $(*)$ together with elements of (t^0_n). Thus, the decomposition of (s^0_n) induces a corresponding decomposition of (t^0_n): $(t^0_n) = (c_n) \cup (d_n)$ in a similar way. Repeating some considerations of the proof of Proposition 4 we arrive at the desired result: (a_n) and (c_n) are equivalent and (b_n), (d_n) are equivalent to suitable sequences (u^σ_n), (v^τ_n) resp. with $(u_n) \subset (s'_n)$, $(v_n) \subset (t'_n)$.

Now we come to the proof of III_A:

Let $M_n = \{s_i : s_i = n\}$, $N_n = \{t_j : t_j = n\}$.

Again, by the Theorem of Köthe we have

$$C_1 \leq \frac{s_n}{t_{\pi(n)}} \leq C_2, \qquad D_1 \leq \frac{t_n}{s_{\pi^{-1}(n)}} \leq D_2. \qquad (1)$$

Hence it follows that

$$s_i \in M_n, \text{ then } t_{\pi(i)} \in \bigcup_{j=1}^{k_n} N_j, \; k_n < \infty, \qquad t_i \in N_n, \text{ then } s_{\pi^{-1}(i)} \in \bigcup_{j=1}^{\ell_n} M_j, \; \ell_n < \infty. \qquad (2)$$

This can be expressed as follows: to any s'_i there corresponds an index-set A_i such that, by the multivalued mapping, which is induced from (1) (i.e., $s_n \to t_{\pi(n)}$) to $(s'_i) \to (t'_i)$, to the element s'_k there correspond the elements t'_j with $j \in A_k$ (Analogously, we could regard the inverse mapping $(t'_i) \to (s'_i)$). By using these notations we construct by induction the desired equivalent sequence $(\hat{s}'_i)$ and $(\hat{t}'_i)$ as follows: Let $k_1 = \max A_1$. Then the first k_1 elements of $(\hat{t}'_i)$ are all equal to t'_1 and the first k_1 elements of $(\hat{s}'_i)$ are: $s'_1, \dots, s'_{k_1}$.

Now suppose that the sequences $(\hat{s}'_i), (\hat{t}'_i)$ are constructed up to the index N and let

$$\hat{s}'_N = s'_j, \quad \hat{t}'_N = t'_m.$$

Now we regard s'_{j+1} and A_{j+1}. If we put $k_{j+1} = \max A_{j+1}$ then there can arise three cases:

1) $k_{j+1} < m$, 2) $k_{j+1} = m$, 3) $k_{j+1} > m$.

In these cases one continues the sequences $(\hat{s}'_i)$, $(\hat{t}'_i)$ in the following way:

ad 1) $\hat{s}'_{N+1} = s'_{j+1}, \quad \hat{t}'_{N+1} = t'_m$,

ad 2) $\hat{s}'_{N+1} = s'_{j+1}, \quad \hat{t}'_{N+1} = t'_m$,

ad 3) $\hat{s}'_{N+1} = s'_{j+1}, \dots, \hat{s}'_{N+(k_{j+1}-m)} = s'_{j+1}$,

$\hat{t}'_{N+1} = t'_{m+1}, \dots, \hat{t}'_{N+(k_{j+1}-m)} = t'_{k_{j+1}}$.

Now we show that by means of such a definition we obtain equivalent sequences with the same constants as in (1) and this will complete the proof. We prove this according to the three cases.

ad 1. If $k_{j+1} < m$, then by definition of A_{j+1} it follows that

$$C_1 \le \frac{s'_{j+1}}{t'_{k_{j+1}}} \le C_2.$$

Consequently, because (t'_n) is monotone increasing,

$$\frac{s'_{j+1}}{t'_m} \le C_2.$$

On the other hand, there must be an s'_h with $h < j+1$ such that

$$C_1 \le \frac{s'_h}{t'_m} \quad \text{and therefore also} \quad C_1 \le \frac{s'_{j+1}}{t'_m}.$$

ad 2. Immediately from the definition of A_{j+1} it follows that

$$C_1 \le \frac{s'_{j+1}}{t'_m} \le C_2.$$

ad 3. Let $k_{j+1} > m$. At first we remark that therefore

$$C_1 \le \frac{s'_{j+1}}{t'_{k_{j+1}}} \le C_2.$$

But this means that $C_1 \le \dfrac{s'_{j+1}}{t'_i}$ for all $i \le k_{j+1}$. i.e.,

$$C_1 \le \frac{s'_{j+1}}{t'_{m+1}}, \dots, C_1 \le \frac{s'_{j+1}}{t'_{k_{j+1}}}.$$

To prove the other inequality (i.e., $\le C_2$) we remark that for the elements t'_n with $m < n \le k_{j+1}$ two cases are possible: a) $n \in A_{j+1}$, then there is nothing to prove. b) $n \in A_q$ with $q > j+1$ but then we have $\dfrac{s'_q}{t'_n} \le C_2$, , $q > j+1$ and consequently $\dfrac{s'_{j+1}}{t'_n} \le C_2$ as desired.

Obviously, the case $n \in A_p$, $p < j+1$ is excluded by construction. By this the proof of the Theorem is completed. The other part of the proof is simple therefore omitted Q.E.D.

From Theorems 2 and 3 we see that in any class I,...,III_C each class $\tilde{D}$ of equivalent domains of closed operators is characterized by one (class I,II,III_A) or two (class III_B, III_C) classes of sequences. Only for different classes there are used different notions of equivalence.

Therefore we can write more precisely:

$\tilde{D} \hat{=} I_\alpha$,

$\tilde{D} \hat{=} II_\beta$,

$\tilde{D} \hat{=} III_{A,\gamma}$, $\tilde{D} \hat{=} III_{B,\delta,\epsilon}$, $\tilde{D} \hat{=} III_{C,\lambda,\mu}$ resp. resp.

and the indices α,β,γ and so on stand for equivalence classes of sequences of naturals described in Theorems 2 and 3. We also write as usual $\tilde{D} \in I_\alpha$ and so on.

Remark 5: In the set of equivalence classes $\tilde{D}$ a semi-order can be introduced: $\tilde{D}_1 \le \tilde{D}_2$ iff there are $D_1 \in \tilde{D}_1$, $D_2 \in \tilde{D}_2$ with $D_1 \subset D_2$. By applying this semi-order we can obtain much more information about the structure of domains and also about some "pathologies" of classes II and III. Because of shortage of place we do not give details of these considerations (cf.[6]).

3. ELEMENTS OF A FUNCTIONAL CALCULUS FOR DOMAINS OF CLOSED OPERATORS

This short section deals with some remarks about a "functional calculus" for domains of closed operators.

As is known,

If S,T are two self-adjoint operators on D then $D(S^\alpha) = D(T^\alpha)$ for $0 \le \alpha \le 1$.

In general, this statement is false for $\alpha > 1$ even if S,T are strictly positive. Therefore in general one cannot speak about the "square" D^2 of a domain D, and so on. But for the classes $\tilde{D}$ one can give a precise definition of a function $f(\tilde{D})$ of a class of domains for a suitable set of functions f.

Definition 4

Let f be a function from R in R which is positive for positive $x \in R$. f is called admissible if f preserves the equivalence, i.e., if from $(a_n) \sim (b_n)$ $\{|f(a_n)|\} \sim \{|f(b_n)|\}$ follows.

It can easily be seen that an admissible function also preserves the other types of equivalence. Now we give the definition of the functional calculus.

Definition 5 (functional calculus)

Let f be admissible and $\tilde{D} \in I_\alpha$ ($\in II_\beta$, $\in III_{A,\gamma}$, $\in III_{B,\delta,\epsilon}$, $\in III_{C,\lambda,\mu}$ resp.) Then by $f(D)$ we mean the following classes of domains: $I_{|f(\alpha)|}$ ($II_{|f(\beta)|}$, ..., $III_{C,|f(\lambda)|,|f(\mu)|}$ resp.) where $|f(\rho)|$ means $\{|f(s_n)|\}$, $(s_n) \in \rho$, and so on.

One can easily give examples of admissible functions. A rather general case is the following:

Let f be monotone and positive for positive x. If for any sequence (b_n), $b_n > 0$, $b_n \to \infty$ $f(Cb_n) \le D_C \cdot f(b_n)$ for all $C > 0$ then f is admissible.

We omit the proof for this statement which says that, for example, all polynomials, log and functions formed of these in a suitable way are admissible functions. An example for a function which is not admissible is given by e^x. On the other hand, it is possible that $\mathcal{D}_1 \neq \mathcal{D}_2$ but $f(\tilde{\mathcal{D}}_1) = f(\tilde{\mathcal{D}}_2)$. It holds, for example, for $f(x) = \log x$.

It can easily be seen that the functional calculus has the following properties

(i) if $h(x) = f(x) + g(x)$, then $h(\tilde{\mathcal{D}}) = f(\tilde{\mathcal{D}}) + g(\tilde{\mathcal{D}})$

(ii) if $h(x) = \lambda \cdot f(x)$, then $h(\tilde{\mathcal{D}}) = f(\tilde{\mathcal{D}})$

(iii) if $h(x) = f(g(x))$, then $h(\tilde{\mathcal{D}}) = f(g(\tilde{\mathcal{D}}))$.

It seems natural to make the following convention: if f is bounded function we put

$$f(\tilde{\mathcal{D}}) = \mathcal{H} \quad \text{for all } \tilde{\mathcal{D}}.$$

Up to now we have not yet given a justification of the functional calculus. This is done by the following proposition.

Proposition 5

Let $\mathcal{D} \in \tilde{\mathcal{D}}$, $S = S^* \geq I$ a positive selfadjoint operator on $\mathcal{D}$ f an admissible function. If $f(S)$ exists, then $\mathcal{D}(f(S)) \in f(\tilde{\mathcal{D}})$.

Proof:

By using the spectral representation of one can construct a diagonal operator T on $\mathcal{D}$ such that $\mathcal{D}(f(T)) \sim \mathcal{D}(f(S))$. Therefore $f(\tilde{\mathcal{D}}) \ni \mathcal{D}(f(S))$. Q.E.D.

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