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ON THE RADIAL MOTION OF QUARKS BOUND BY A STRING

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1. INTRODUCTION

The calculation of the hadron mass spectrum in the framework of quantum chromodynamics still remains a practically unsolved problem. Therefore the potential [1] and string models [2-11] of hadrons are widely used for this purpose.

In the string model of hadrons the quarks are treated to be tied together by a gluon tube. In the approximation of the vanishing width the gluon tube dynamics is described by the Nambu-Goto action for the relativistic string [12-14]. The string model has the following obvious merits: it ensures the quark confinement into hadrons and from the very beginning it gives the relativistic description of the hadron dynamics. However, even the derivation of classical solutions in the string model proved to be a cumbersome mathematical problem when the quark masses are different from zero.

Two exact solutions to the equations of motion in the model of the relativistic string with massive ends are known. The first solution [15] describes two massive quarks tied together by a straight-line string and vibrating along the string. The string length changes periodically in time, but the system as a whole is at the rest. The second known solution [16,17] describes the uniform rotation in the given plane of the straight string with massive ends. The string length retaines constant during rotation but it depends on the rotation frequency.

In Ref. 18 an attempt was undertaken to construct such new solutions in the string model that unite two exact solutions described above. The straight-like string with massive quarks at its ends has been considered. This system can rotate as a whole in a given plane with an angular velocity dependent in the general case on time. The string length can vary during the motion as well, i.e., quarks can move in the radial direction. In Ref. 18 it has not been argued rigorously that the new solutions are exact or approximate for the string model. If they are approximate, the question arises: what is the criterion of their application?

In the present paper we shall prove strictly that quarks cannot move in the radial direction when they are tied by a straight-line string and the system rotates as a whole with a nonvanishing angular velocity in a given plane. This implies that the new solutions obtained in Ref.18 (the so-called general case) are just approximate from the standpoint of the original string model besides of the two special cases discussed above.

The paper is arranged as follows. In the second section we propose a simple derivation of the equations of motion that lead to new solutions of Ref.18. In the third section it will be argued strictly that the new solutions from Ref. 18 cannot be exact in the string model of hadrons because the string world surface in this case is not minimal and the curvature of quark world trajectories turns out to be not a constant as the string model requires. The nonvanishing mean curvature of the string world surface and the differences of the curvatures of quark world trajectories from the fixed known values can be treated as a measure of the deviation of the new solutions from the exact dynamics in the model of the relativistic string with massive ends. In conclusion (section 4) the implications of the obtained results for phenomenological string models of hadrons are briefly discussed.

2. EQUATIONS OF MOTION

We start with the action of the string connecting two massive quarks. The space-like string coordinates $\vec{x}$ will be parametrized by the time from the Minkowski space and the variable σ numbering the points along the string (the so-called $t = \tau$ gauge/2/) $\vec{x} = \vec{x}(t, \sigma)$. In terms of these variables the action of the system under consideration is

$$S = -\gamma \int_{t_1}^{t_2} dt \int_{\sigma_1(t)}^{\sigma_2(t)} \sqrt{\vec{x}'^2(1-\dot{\vec{x}}^2) + (\dot{\vec{x}}\vec{x}')^2} - \sum_{a=1}^{2} m_a \int_{t_1}^{t_2} dt \sqrt{1-\left(\frac{d\vec{x}_a}{dt}\right)^2}\,, \qquad (2.1)$$

where $\vec{x}_a = \vec{x}(t, \sigma_a(t))$, $a = 1,2$, $\dot{\vec{x}} = \partial_t \vec{x}$, $\vec{x}' = \partial_\sigma \vec{x}$, γ is the string tension with dimension of $(\text{length})^{-2}$. Further we confine ourselves to the consideration of such string motions when it rotates in a given plane keeping the straight-line form. The string length, i.e. the distance between quarks can vary during the motion. Those string motions can be parameterized in the following way

$$\vec{x}(t,\sigma) = r\sigma\vec{n}(t), \quad \sigma_1(t) \le \sigma \le \sigma_2(t),$$
$$\vec{n}^2(t) = 1, \quad \vec{n}(t) = \{\cos\phi(t),\ \sin\phi(t)\}. \qquad (2.2)$$

Here r stands for a constant with dimension of length.

Now we will proceed in the same way as in Ref. 18: we substitute (2.2) into the action (2.1) and vary it. It should be noted that at this point we could go away from the initial string model and the solutions to the new equations of motion would not obey the Lagrange - Euler equations following from (2.1) without any assumptions about the form of the functions $\vec{x}(t, \sigma)$.. Further it will be shown that it is just this situation that takes place really. But first we derive these new equations of motion.

After substituting (2.2) into (2.1) the integration over σ in the last equation can be done exactly

$$S = \int_{t_1}^{t_2} dt\, L\,, \qquad (2.3)$$

$$L = \sum_{a=1}^{2}\left\{\frac{\gamma}{2}(-1)^a \left[r\sigma_a(1-(r\sigma_a\dot{\phi})^2)^{1/2} + \frac{1}{\dot{\phi}}\arcsin(r\sigma_a\dot{\phi})\right] - m_a(1 - r^2\sigma_a^2\dot{\phi}^2 - r^2\dot{\sigma}^2)^{1/2}\right\}, \qquad (2.4)$$
$$\sigma_a = \sigma_a(t), \quad a = 1, 2.$$

The Lagrangian function (2.4) describes the system with three degrees of the freedom $\sigma_a(t)$, $a = 1,2$, and $\phi(t)$. The variables $r\sigma_a(t)$ define the distance of quarks from the origin of coordinated in the rotation plane.

We write the equations of motion following from (2.3) and (2.4) for the symmetric case when $m_1 = m_2 = m$ and $\sigma_1(t) = -\sigma_2(t) = \sigma(t)$ introducing the distance of quarks from the coordinate origin as a dynamical variable. It this case the Lagranfian function (2.4) can be rewritten as follows

$$L = -\gamma[r(1-r^2\dot{\phi}^2)^{1/2} + \frac{1}{\dot{\phi}}\arcsin(r\dot{\phi})] - 2m(1 - r^2\dot{\phi}^2 - \dot{r}^2)^{1/2}\,. \qquad (2.5)$$

The corresponding equations of motion read

$$\frac{dp_r}{dt} = -2\gamma\sqrt{1-r^2\dot{\phi}^2} - \frac{2mr\dot{\phi}^2}{\sqrt{1-r^2\dot{\phi}^2-\dot{r}^2}}\,, \qquad (2.6)$$

$$\frac{dp_\phi}{dt} = 0, \tag{2.7}$$

where

$$p_r = \frac{2m\dot r}{\sqrt{1 - r^2\dot\phi^2 - \dot r^2}}, \tag{2.8}$$

$$p_\phi = -\frac{\gamma}{\dot\phi}[r\sqrt{1 - r^2\dot\phi^2} - \frac{1}{\dot\phi}\arcsin(r\dot\phi) + \frac{2mr^2\dot\phi}{\sqrt{1 - r^2\dot\phi^2 - \dot r^2}}. \tag{2.9}$$

There are two constants of motion

$$p_\phi = J = \text{const}, \tag{2.10}$$

$$E = \frac{\partial L}{\partial \dot r}\dot r + \frac{\partial L}{\partial \dot\phi}\dot\phi - L = \frac{2m}{\sqrt{1 - r^2\dot\phi^2 - \dot r^2}} + \frac{2\gamma}{\dot\phi}\arcsin(r\dot\phi). \tag{2.11}$$

It is easy to verify that E is the energy and p_ϕ is the angular momentum of the string with quarks at the ends calculated with solutions (2.2).

From the standpoint of the Hamiltonian dynamics the model under consideration (2.5) is a completely integrable (according Liouville) Hamiltonian system because it has two constants of motion (2.9) and (2.10) in involution [19]. However, the explicit integration of this system encounters difficulties just in the derivation of the corresponding Hamiltonian as a function of the canonical variables.

From Eqs. (2.6)-(2.9) it is easy to obtain two exact solutions in the string model of hadrons noted above. If one puts $\dot\phi = 0$ in Eqs. (2.6)-(2.9), then only one equation remains

$$\frac{d}{dt}\frac{2m\dot r}{\sqrt{1 - \dot r^2}} = -2\gamma. \tag{2.12}$$

Its integration shows that the massive string ends more along the segments of the hyperbola in the $\{t, r\}$ plane [15]

$$r^2(t) - t^2 = (m/\gamma)^2. \tag{2.13}$$

If we assume in (2.6)-(2.9) that

$$r(t) = R = \text{const}, \quad \dot\phi(t) = \omega = \text{const}, \tag{2.14}$$

then we obtain the second exact solution [16], described in the preceding section. Indeed, Eq. (2.6) with allowance of (2.14) determines the distance quark as a function of the rotation frequency ω

$$m\omega^2 R = \gamma(1 - \omega^2 R^2). \tag{2.15}$$

Substitution of the solutions (2.14) and (2.15) into the integrals of motion (2.10) and (2.11) results in the parametric representation of the Regge trajectory $J = J(M^2)$, $M = E$ which proves to be nonlinear at low M^2 [16,17].

3. RELATIONSHIP WITH THE INITIAL STRING MODEL

It turns out that all other solutions to the equations of motion (2.6)-(2.9), different from (2.13) and (2.14), are not solutions to the initial string model defined by the action (2.1). For verifying this one should substitute the solution (2.2) into the Euler - Lagrange equations and into the boundary conditions that follow from the action (2.1) or from the corresponding covariant action without using the $t = \tau$ gauge

$$S = -\gamma \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \sqrt{(\dot x x')^2 - \dot x^2 x'^2} - \sum_{a=1}^{2} m_a \int_{\tau_1}^{\tau_2} d\tau \sqrt{\dot x^2(\tau, \sigma_a)},$$
$$x = x^\mu(\tau, \sigma), \ \mu = 0, 1, \ldots, D-1; \ \sigma_1 = 0, \ \sigma_2 = \pi. \tag{3.1}$$

Here the metric with signature (+, -, -, ..., -) is used in the ambient D-dimensional space-time. However, it is more simple to consider the invariant geometrical characteristics of the string world surface and the world trajectories of quarks placed at the string ends.

It is well known [20] that the variation of the action (2.1) or (3.1) implies that the string world surface be a minimal surface in the D-dimensional Minkowski space-time, i.e., the mean curvatures h_a of this surface along all its D-2 normals n^μ_a, $a = 1, 2, \ldots D-2$, $\mu = 0, 1, \ldots, D-1$ should vanish

$$h_a = -g^{ij}\, n_a^{\mu}\, \partial_i \partial_j\, x_{\mu} = 0,$$
$$i,j = 0,1;\ \partial_0 x = \dot{x},\quad \partial_1 x = x';\quad g_{ij} = \partial_i x^{\mu} \partial_j x_{\mu},\ g_{ij} g^{jk} = \delta_i^{\ k}. \tag{3.2}$$

The D-2 independent Euler-Lagrange equations following from the action (3.1) reduce really to these requirements.

In the model of the relativistic string with massive ends the equations of motion must be supplemented by the boundary conditions

$$m_a \frac{d}{dt} \frac{\dot{x}^{\mu}}{\sqrt{\dot{x}^2}} = (-1)^a \gamma \frac{(\dot{x}x')\dot{x}^{\mu} - \dot{x}^2 x'^{\mu}}{\sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}}. \tag{3.3}$$

Now we deduce from Eq. (3.3) squared

$$k_a^2 = \frac{(\dot{x}\ddot{x})^2 - \dot{x}^2 \ddot{x}^2}{(\dot{x}^2)^3} = \left(\frac{\gamma}{m}\right)^2, \quad a = 1,2, \tag{3.4}$$

where k_a, a = 1,2 are the curvatures of quarks world lines [21]. It should be noted that at this point the motion of massive quarks tied by a string resembles the classical Delaunay problem[22] of drawing a curve of a constant curvature connecting two given points and having the least length.

Let us consider the consequence of the requirement that the solution of form (2.2) should obey Eqs. (3.2) and (3.4). For the mean curvature squared of the world surface (2.2) one obtains easily

$$h^2 = \begin{vmatrix} \dot{x}^2 & \dot{x}x' & \dot{x}\ddot{x} \\ \dot{x}x' & x'^2 & x'\ddot{x} \\ \dot{x}\ddot{x} & x'\ddot{x} & \ddot{x}^2 \end{vmatrix} = 0. \tag{3.5}$$

Here $x^{\mu}(t,\sigma)$ is a three dimensional Lorentz vector with components $x^{\mu}(t,\sigma) = \{t, r\sigma\, n(t)\}$. Taking this into account we deduce from (3.5)

$$h^2 = \ddot{\phi}^2(t) = 0. \tag{3.6}$$

Thus the condition of minimality for the string world surface of the form (2.2) requires that the string should rotate with a constant angular velocity, i.e., its world surface should

be a helicoid*. Hence, if we would like to choose among all the solutions to the system (2.6)-(2.9) special ones that satisfy the equations of motion in the initial string model (2.1), we should put

$$\dot{\phi}(t) = \omega = \text{const.} \tag{3.7}$$

In this case in the set (2.6)-(2.9) there remains only one degree of freedom the dynamics of which is determined by the Hamiltonian

$$H(r,p) = \left(\sqrt{4m^2 + p^2} + \gamma r\right)\sqrt{1 - r^2\omega^2} + \frac{\gamma}{\omega}\arcsin(\omega r). \tag{3.8}$$

After imposing the conditions

$$\dot{r}(t) = 0,\quad \dot{p}(t) = 0 \tag{3.9}$$

the equations of motion generated by the Hamiltonian (3.8) result in the relation (2.15), i.e., they reproduce the exact solution describing a uniformly rotating string with quarks at its ends. However, the Hamiltonian (3.8), calculated on the solution of the equations of motion, generated by it, and on the solution (2.15) as well, is not an energy for the initial system (2.11). One could reconcile oneself to this fact as there is known the proper formula (2.11) for the energy in the string model. But all the solutions generated by the Hamiltonian (3.8), besides the solutions (2.15) and (3.9), violate the condition (3.4) that should be satisfied by the quark world trajectories in the string model. Indeed, we deduce from (2.2) and (3.4)

$$k^2 = \frac{\dot{r}^2(r\omega^2 + \ddot{r})^2 + (1 - \dot{r}^2 - r^2\omega^2)(\ddot{r}^2 + 4\dot{r}^2\omega^2 + r^2\omega^4 - 2r\ddot{r}\omega^2)}{(1 - \dot{r}^2 - r^2\omega^2)^3} = (\gamma/m)^2. \tag{3.10}$$

But from the Hamiltonian (3.8) it follows that

$$\left(\frac{\gamma}{m}\right)^2 = \frac{[(1 - r^2\omega^2)(r\omega^2 - \ddot{r}) - 2\dot{r}^2\omega^2 r]^2}{(1 - \dot{r}^2 - r^2\omega^2)^3}. \tag{3.11}$$

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* It is really the consequence of the Catalan theorem in the classical surface theory proved as long ago as the last century [27]. The Catalan theorem reads: among all the ruled surfaces there are only two minimal surfaces, a plane and a helicoid. A ruled surface is obtained by a moving straight line in a space, i.e., it is a surface of form (2.2).

Of course, the boundary conditions (3.3) do not only reduce to equation (3.4), nevertheless the relations (3.10) and (3.11) are not coordinated obviously.

A decisive conclusion about the relationship of the quark radial dynamics described by the Hamiltonian (3.8) with the initial string model results from the check of the boundary conditions in each component in the string model with the use of the parametrization (2.2) and of the condition (3.7). It has been done in Ref. 16. The answer reads: the boundary conditions following from (2.1) result in Eqs. (3.9) and (2.15) unambiguously. Thus, the Hamiltonian (3.8) has in general no relationship with the initial string model. The equations (2.15)-(2.11) except the two exact solutions in the string model mentioned above go out of the framework of the string model as well.

The analysis of the dynamical system (2.5)-(2.11) represented above gives us an invariant geomentical measure of the deviation of the solutions to these equations from the initial string model. For this purpose one can use a nonvanishing mean curvature of the string world surface (3.5) and the deviation of the curvature of the quark world trajectories calculated by (3.10) from γ/m.

It should be noted that the dynamical system (2.5)-(2.11) and its simplified version with Hamiltonian (3.8) can be treated independently of the string model as some new phenomenological description of the relativistic rotational and radial motions of quarks inside hadrons. For this purpose equation (3.10) can also be used. It determines the radial motion of quarks in a given rotation plane completely.

4. CONCLUSION

The results obtained above enable us to make an important conclusion concerning the phenomenological string models. We have shown that quarks tied by a string cannot move in the radial direction when the whole system rotates with a nonvanishing angular velocity and there are no transverse string excitations (a straight-line string). This implies that it is impossible to separate the total Hamiltonian of the string model so that one term would describe the transverse string excitations and another term would be responsible for the radial motions of quarks, each of these addends being dependent on its own dynamical variables. In other words in the string model of hadrons it is impossible, in a consistent way, to separate the radial quark motions and the transverse string vibrations. Usually one assumes this separation in phenomenological string models [3-5, 8]. In terms of the interquark potential we can say that all the corrections to the linearly rising potential are due to the transverse string vibrations. This is in accordance with direct calculations of the relativistic static interquark potential in the framework of the string model with fixed ends [23-26]. When the transverse string vibrations are neglected in this model, then the quark interaction is described in the static approximation only by the linear potential $2\gamma r$, which follows from (2.11) when $\dot{r} = \phi = 0$. In quantum theory, this potential is modified due to zero-point fluctuations of the string oscillations.

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О радиальном движении кварков, связанных струной

Строго показано, что кварки не могут совершать радиальные движения, если они связаны прямолинейной струной и вся система вращается как целое с ненулевой угловой скоростью. Это означает, что при последовательной трактовке струнной модели адронов нельзя разделить радиальные движения кварков и поперечные возбуждения струны.

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On the Radial Motion of Quarks Bound by a String

It is shown rigorously that quarks cannot move in the radial direction when they are tied together by a straight-line string and the system as a whole rotates with a nonvanishing angular velocity. This implies that in a consistent string model of hadrons the radial motion of quarks cannot be separated from the transverse string excitations.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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