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THE ROLE OF GAUGE INVARIANTS IN PATH INTEGRAL CONSTRUCTION

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1. In the description of systems with constraints [1] by a path integral (PI) it is necessary to eliminate unphysical degrees of freedom [2] . In this way, it is supposed a priori that the phase space (PS) of remaining physical variables is a plane, i.e., it is isomorphic to an Euclidean space of an even dimension. Typical examples of systems with constraints are gauge theories. It is shown [3,4] that PS of some physical degrees of freedom in gauge models can differ from the ordinary plane. It leads to a modification of the Hamiltonian PI [4] .

It is also known that there are restrictions on values of gauge invariant variables in the gauge system description with these variables [5] , i.e., physical coordinates can take values only on a part of the real axis (see also [6]). However this fact is ignored in the PI method (note that a finite-dimensional Gaussian integral cannot be calculated explicitly over a part of a real axis).

In the present work, for the finite-dimensional model with an arbitrary gauge group the method of PI construction is given for the gauge-invariant kernel of the evolution operator. This way does not require us to pick out physical variables explicitly. The problem of integration over physical variables having reduced PS in PI turns into analysis of the change of the variables in an ordinary integral. It is shown that restrictions on physical PS lead to a PI modification according to results [4] .

2. Consider a model with the Lagrangian [4]

$$L = \frac{1}{2} \operatorname{Tr} (D_t x)^2 - V(x), \tag{1}$$

where $D_t x = \dot{x} + [y, x]$, y, x are elements of a Lie algebra X and are dynamical variables of the theory. The basis in X is chosen in the following way $x = \lambda_i x_i$, $\operatorname{Tr} \lambda_i \lambda_j = \delta_{ij}$, $[\lambda_i, \lambda_j] = f_{ijk} \lambda_k$, f_{ijk} are completely antisymmetric structural constants of X $i, j, k = 1, 2, \dots, N$, $N = \dim G$, G is a semi-simple compact gauge group. Lagrangian (1) is invariant under the gauge transformations

$$x \to \Omega x \Omega^{-1}, \; y \to \Omega y \Omega^{-1} + \Omega \frac{\partial}{\partial t} \Omega^{-1}, \; \Omega = \Omega(t) \in G . \tag{2}$$

Passing to the Hamiltonian formalism we find canonical momenta $p = \partial L / \partial \dot{x} = D_t x$, $\pi = \partial L / \partial \dot{y} = 0$ (are the primary constraints [1]) so the Hamiltonian for our system is

$$H = \frac{1}{2} \operatorname{Tr} p^2 + V(x) - y_i G_i , \tag{3}$$

where $G_i = \dot{\pi}_i = \{\pi_i, H\} = f_{ijk} p_j x_k = 0$ are the secondary constraints [1] , $\{ , \}$ are Poisson brackets. All constraints are of the first class [1] $\{G_i, G_j\} = f_{ijk} G_k$, $\{H, G_i\} = f_{ijk} y_j G_k$ so the quantization of the system is realized by the change of canonical variables to the operators $[x_j, p_k] = i\delta_{jk}$, $[y_j, \pi_k] = i\delta_{jk}$ and operators of constraints must nullify all physical states [1] . Therefore we have the quantum problem in the coordinate representation

$$\begin{cases} \left[-\frac{1}{2} \Delta_{(N)} + V(x) \right] \psi_E = E \psi_E , & (4) \\ G_j \psi_E = 0 , & (4a) \end{cases}$$

where $\Delta_{(N)}$ is an N -dimensional Laplacian (the constraints $\pi_j \psi_E = 0$ are easily solved– ψ_E is independent of y_j . Below we shall not consider these degrees of freedom).

Let $\mathcal{H}$ be a total Hilbert space which is a set of all functions satisfying Eq.(4) and let $\mathcal{H}_{ph}$ be a physical subspace in $\mathcal{H}$, $\psi_E \in \mathcal{H}_{ph}$ if ψ_E satisfies (4a). Since the Hamiltonian in Eq.(4) commutes with G_j , $\mathcal{H}$ is an orthogonal sum of two subspaces (physical and nonphysical)

$$\mathcal{H} = \mathcal{H}_{ph} \oplus \mathcal{H}_{nph} . \tag{5}$$

Let both $\psi_E \in \mathcal{H}_{ph}$ and $\tilde{\psi}_E \in \mathcal{H}_{nph}$ be basis functions in $\mathcal{H}_{ph}$ and $\mathcal{H}_{nph}$, respectively, i.e., $\langle \psi_E | \psi_{E'} \rangle = \delta_{EE'}$, $\langle \tilde{\psi}_E | \tilde{\psi}_{E'} \rangle = \tilde{\delta}_{EE'}$, $\langle \tilde{\psi}_E | \psi_{E'} \rangle = 0$, where both $\delta_{EE'}$ and $\tilde{\delta}_{EE'}$ are kernels of unit operators in spectral-parameter spaces both for $\mathcal{H}_{ph}$ and $\mathcal{H}_{nph}$ respectively. The scalar product is defined in a standard way $\langle \varphi | \psi \rangle = \int d^N x \, \varphi^*(x) \psi(x)$, where the integration region is the whole $X \sim \mathbb{R}^N$.

To solve Eq.(4a) explicitly, we pass to new variables [4]

$$x = e^{z} h e^{-z}, \tag{6}$$

where $h \in H$, H is the Cartan subalgebra, $z \in X \setminus H$. Any element $x \in X$ can always be directed into H by a group transformation. However, there is a Weyl group W in H [7] and any element of H can be directed into the Weyl camera K^+ [7] by a transformation from W . If $h \in K^+$ then $\operatorname{Tr} h\omega > 0$, ω takes values from the set of simple roots in H [7] . It follows from (2) that z are unphysical variables. The number of physical degrees of freedom is really $\ell = \operatorname{rank} G = \dim H$ here.

Since the group W is a subgroup of the gauge group, elements h and $\hat{w}h \equiv whw^{-1}$, $w\in W$ are gauge equivalent. Therefore permissible values of physical variables h are K^+. For example, K^+ for the group SU(3), $\ell=2$ is a sector with the angle $\pi/3$. Thus the physical configurational space differs from the plane $\mathbb{R}^2$. We must take into account this reduction of the configurational space in PI. For variables (6) this has been made in [4]. Below we shall find a fit technique for any separation of physical variables.

The change of variables (6) means the splitting of the algebra X = orbit $\oplus$ K^+, where K^+ is the transversal space [8]. Thereby the equality $\psi_E(x) = \psi_E(h)$ takes place, for the generators G_j are generators of movement along orbits. Hereafter, to determine the basis in $\mathcal{H}_{ph}$, the transversal part of the Laplace-Beltrami operator must only be used. According to [8, Theor. II.5.33], Eq.(4) in $\mathcal{H}_{ph}$ can be written as

$$\left[-\frac{1}{2}\mu^{-1}(h)\Delta_{(\ell)}\circ\mu(h)+V(h)\right]\psi_E = E\,\psi_E\,, \tag{7}$$

where $\mu(h)=\prod_{\alpha>0}\mathrm{Tr}\,\alpha h$, α enumerates all positive roots of X. The scalar product in $\mathcal{H}_{ph}$ gets the measure $\int d^Nx = \int_{orbit} d\tilde{\mu}(z)\int_{K^+} d^\ell h\,\mu^2(h) = v\int_{K^+} d^\ell h\,\mu^2(h)$, in fact, $v\,\mu^2(h)$ is a volume of the orbit for the element h.

Since vectors ψ_E are gauge invariants, we have $\psi_E(\hat{w}h)=\psi_E(h)$, $w\in W$. There is an opposite statement (Chevalley theorem) [9,10] that any analytic function invariant under the Weyl group transformations on H can be continued to X in an invariant way with respect to the adjoint action of the group G. This theorem gives

us the possibility to restore solutions of (4), (4a) provided solutions of (7) are known.

Eq.(7) can be transformed to a standart Schroedinger equation in $\mathbb{R}^\ell$ by the substitution $\psi_E = \mu^{-1}\varphi_E$. Since $\mu(\hat{w}h)=\det w\,\mu(h)=\pm\mu(h)$, we have $\varphi_E(\hat{w}h)=\det w\,\varphi_E(h)$. The latter condition guarantees that wave functions are finite on the boundary of the Weyl camera ∂K^+. So far as $\mu(h)=0$, $h\in\partial K^+$, $\varphi_E\sim \mathrm{const}\cdot\mu$ in the neighbourhood of ∂K^+. It is really right, as any polynomial odd with respect to W is divided by $\mu(h)$ [8, con. III.3.8] and φ_E is an analytic function.

Let us suppose that φ_E are known, then they are normalized by the equality

$$\int_{K^+} d^\ell h\,\varphi_E^*(h)\,\varphi_{E'}(h) = \frac{1}{v}\,\delta_{EE'}\,. \tag{8}$$

Thereby we can easily find the expansion of the unit in $\mathcal{H}_{ph}$ (or the kernel of a projector on $\mathcal{H}_{ph}$).

$$Q(x,x') = \sum_E \psi_E(x)\psi_E^*(x') = \sum_E [\mu(h)\mu(h')]^{-1}\varphi_E(h)\varphi_E^*(h') =$$
$$= \frac{1}{v}\,[\mu(h)\mu(h')]^{-1}\,\delta^\ell(h-h')\,,\quad h,h'\in K^+. \tag{9}$$

To write (9) in a gauge invariant way, we use the formula [8, lem. III.3.7]

$$\det\left\|\frac{\partial j_\alpha(h)}{\partial h_\beta}\right\| = c\,\mu(h)\,,\quad \alpha,\beta=1,2,\dots,\ell\,,\quad c=\mathrm{const}, \tag{10}$$

where $j_\alpha(h)=\mathrm{Tr}\,h^{\tau_\alpha}=\mathrm{Tr}\,x^{\tau_\alpha}$, τ_α are degrees of independent Casimir operators in the algebra X. In fact, $j_\alpha(x)$ make the basis in space of polynomials which are invariant under the adjoint action of G. So, any gauge invariant function $\psi_E\in\mathcal{H}_{ph}$ is a function of ℓ variables $j_\alpha(x)$ in full accordance with that the considered system contains ℓ physical degrees of freedom. Now we shall find an analytic continuation to X

of the measure $\mu(h)$. Since $\mu(h)>0$ if $h\in K^+$ ($\mathrm{Tr}\,\alpha h>0$ if $h\in K^+$ and $\alpha>0$ [7]), then $\mu(h) = [\mu^2(h)]^{1/2} = [\mu^2(x)]^{1/2}$. Because of invariance of $\mu^2(h)$ with respect to W we can replace h by x according to the Chevalley theorem. The polynomial $\mu^2(h)$ is, in fact, a function of $j_\alpha(h)$. Now from (9),(10) and the rule for a replacement of arguments in multidimensional δ-functions there follows the manifestly gauge-invariant form of the unit operator in $\mathcal{H}_{ph}$

$$Q(x,x') = \frac{c}{v}\left(\mu^2(x)\mu^2(x')\right)^{-1/4}\prod_{\alpha=1}^{\ell}\delta\left(j_\alpha(x)-j_\alpha(x')\right). \quad (11)$$

Consider the evolution operator in the total Hilbert space. Using the Feynman-Kac formula we write

$$U_t(x,x') = \sum_{E\in\mathcal{H}_{ph}}\psi_E(x)\psi_E^*(x')e^{-iEt} + \sum_{E\in\mathcal{H}_{nph}}\tilde\psi_E(x)\tilde\psi_E^*(x')e^{-iEt}. \quad (12)$$

The first sum in (12) is the kernel of the evolution operator in $\mathcal{H}_{ph}$. We shall denote it by $U_t^{ph}(x,x')$. There is a standard representation for the kernel (12) by PI

$$U_t(x,x') = \int \mathcal{D}x \exp i\left\{\int_0^t d\tau\left(\frac{1}{2}\mathrm{Tr}\,\dot x^2 - V(x)\right)\right\}, \quad (13)$$

where $\mathcal{D}x$ is a standard measure in an N-dimensional configurational space of the system, $x = x(t)$, $x' = x(0)$. Since the operator Q is a projector on $\mathcal{H}_{ph}$, it follows from (5) and (12) that

$$U_t^{ph}(x,x') = \int_X d^N x''\, U_t(x,x'')\,Q(x'',x'). \quad (14)$$

Formula (14) gives a manifestly gauge-invariant transition amplitude by PI. Into representation (14), all degrees of freedom enter on an equal footing, i.e., there is

no elimination of unphysical variables from the beginning. However, strictly speaking, the degree of freedom y is inled out, i.e., it is the "temporal" gauge $y=0$.

3. Now we shall discuss the question of gauge fixation. We use the technique from [2] for PI construction in terms of physical degrees of freedom. Let us put $x = x_a\lambda_a + h_\alpha\lambda_\alpha$, where both $a,b,c = 1,2,\dots,N-\ell$ and $\alpha,\beta = 1,2,\dots,\ell$ enumerate basis elements in both $X\setminus H$ and H, respectively. We assume extra conditions in the following form: $\chi_a(x) = x_a = 0$, $y_i = 0$. The constraints $G_i = 0$ are not independent. In fact, there is a stationary subgroup (Cartan subgroup) for any vector $x = h$ so we can pick out independent constraints as $G_a = f_{abc}p_b x_c + f_{ab\alpha}p_b h_\alpha + f_{a\alpha b}p^h_\alpha x_b = 0$, p^h_α is a momentum canonical conjugated to h_α (note that $f_{a\alpha\beta} = 0$). Then PI for system (1) has the following form

$$U_t(h,h') = \int \mathcal{D}(p,x,p^h,h,\pi,y)\,e^{iS}\prod_t\left(\Delta\,\delta^N(\pi)\delta^N(y)\prod_a\delta(x_a)\delta(G_a)\right), \quad (15)$$

where $\Delta = \det M$, $M_{ab} = \{G_a, x_b\}|_{x_a=0}$, the measure in PI (15) is defined in a usual way, $\mathcal{D}(p,x,p^h,h,\pi,y) = \prod_t\left((2\pi)^{-2N}\times d^N y\, d^N\pi\, d^\ell h\, d^\ell p^h\prod_a dx_a dp_a\right)$, $S = \int_0^t d\tau\left(\pi_i\dot y_i + p_a\dot x_a + p^h_\alpha\dot h_\alpha - H\right)$ is the system action, the Hamiltonian H is given by (3). We can easily carry out integrations over π_i, y_i and x_a. To integrate over p_a, we should transform the delta-function of constraints $\prod_a\delta(G_a)|_{x_a=0} = \prod_a\delta(M_{ab}p_b) = \Delta^{-1}\prod_a\delta(p_a)$. Thereby the determinant Δ cancels out in the measure (15). After integrations over p_a and p^h_α we get PI which coincides with (13) if the substitution $x = h \in H$ is done in (13).

Now consider the amplitude which follows from (14) in gauge $x = h$. Take the infinitesimal kernel (14)

$U_\varepsilon^{ph}(h,h')$, $\varepsilon \to 0$ (since $U_t^{ph}(x,x')$ is a gauge invariant, both x and x' can be replaced by h and h' , respectively). Then passing to variables (6) in the integral over x'' we get

$$U_\varepsilon^{ph}(h,h') = \int_{K^+} d^\ell h'' \mu^2(h'') \int_{orbit} d\tilde{\mu}(z) \exp\left[\frac{i}{\varepsilon} Tr\left(h - e^{z} h'' e^{-z}\right)^2 - i\varepsilon V(h'')\right] Q(h'',h') \tag{16}$$

(the potential V is gauge invariant). The integral over orbit is proportional to the integral over the whole group G because of the stationarity of the Cartan subgroup for h'' . So, using the formula [8, th. II. 5.35] we find

$$U_\varepsilon^{ph}(h,h'') = \sum_{w \in W} \frac{v}{\mu(h)} \int_{K^+} d^\ell h'' \mu(\hat{w}h'') \exp\left[\frac{i}{\varepsilon} Tr\left(h - \hat{w}h''\right)^2 - i\varepsilon V(h'')\right] Q(h'',h'). \tag{17}$$

Substitution of (11) into (17) and a subsequent integration over h'' and iteration of the kernel $U_\varepsilon^{ph}(h,h')$ lead to the final result for a finite-time interval

$$U_t^{ph}(h,h') = \sum_{w \in W} \left[\mu(h)\mu(\hat{w}h')\right]^{-1} U_t(h,\hat{w}h'), \tag{18}$$

where the kernel $U_t(h,h')$ is determined by (15). The obtained form for U_t^{ph} coincides with the result of [4].

Why is there a difference between (16) and (15)? In the formula (15) finding it has been taken into account a priori that PS of physical variables p^h and h is $\mathbb{R}^{2\ell}$. However, it is shown in [4] , this PS is a hypercone isomorphic to $K^+ \oplus \mathbb{R}^\ell$ (we have found above that physical values of h lie in K^+). So the integration in (15) should be carried out over that manifold. Note also one more peculiarity. We have for the determinant in (15) $\Delta = const\ \mu^2(h)$ since $M_{ab}(h) = f_{ab\alpha} h_\alpha$ [4]. Therefore $\Delta = 0$ if $h \in \partial K^+$ so the integration region should be restricted to K^+ (the additional conditions $x_a = 0$ become inadmissible for $h \in \partial K^+$).

But then we arrive at the problem of calculation of PI over $h \in K^+ \subset \mathbb{R}^\ell$. In fact, (18) gives the receipt of solution of this problem.

Physical variables can certainly be picted out by the definition of any ℓ -dimensional smooth manifold isomorphic to $\mathbb{R}^\ell$ in the total configurational space $\mathbb{R}^N$. It is equivalent to $N-\ell$ conditions $\chi_a(x) = 0$. However, PS of new physical variables should not be isomorphic to $\mathbb{R}^{2\ell}$ although it seems to be quite arbitrary at any χ_a . It follows already from that the gauge-group orbits in this model are closed compact manifolds with dimension $N-\ell$. Choice of physical variables by the gauge fixation is the choice of a "line" (ℓ -dimensional surface) along which the physical variables change in the total configurational space. The "line" cannot cross a closed manifold only once like a line passing through an interior of a sphere crosses the sphere twice at least (by the way, this case is realized for groups of rank $\ell = 1$, i.e., $SU(2) \sim SO(3)$, where orbits are spheres S^2). Therefore there are always points on the "line" which belong to the same orbit, in other words, they are gauge equivalent. So, the reduction of both configurational and phase spaces always takes place. We have seen the manifestation of this general fact in the analysis of variables (6). Thus, formulae (14) and (17) give the recipe of calculation of PI over reduced PS for any choice of physical variables. To determine the connection between PI for a system with reduced PS and PI for a corresponding system with plane PS, it is enough to change the variables in (17) $h''_\alpha = \tilde{\chi}_\alpha(\tilde{h}'')$, where $\tilde{\chi}_\alpha$ are restored from χ_a and to carry out integration over $\tilde{h}''$.

It is essential here that except the symmetrization with respect to the Weyl group, there is a measure in (17) which depends, in general, on the choice of physical variables.

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