объединенный
институт
ядерных
исследований
дубна

E2-89-687

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PATH INTEGRAL IN HOLOMORPHIC REPRESENTATION WITHOUT GAUGE FIXATION

Submitted to "Letters in Mathematical Physics"

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1989

1. It is well known that a gauge symmetry leads to constraints of dynamical variables in a theory [1]. Therefore, the evolution of unphysical degrees of freedom should be given when working with gauge theories, which is equivalent to a gauge fixation. There is another way here to pass to gauge-invariant variables. In this case constraints in a theory become diagonal, i.e., some of generalized momenta are equal to zero. The passage to gauge-invariant variables is, in general, an introduction of curvilinear coordinates, so changes of a configurational space of physical variables take place [2,3]. In other words, physical coordinates take values not on the whole real axis but only on its part (a half-line or a segment). Moreover physical degrees of freedom (in any way of their determination) can have a phase space which differs from a plane [4,5]. It leads to a modification of PI [5], and as a result, the quasi-classical description is changed [6].

According to the above remarks the following question can be asked: is there any way of PI construction which does not require elimination of unphysical degrees of freedom, and the evolution operator determinated by such PI would be manifestly gauge-invariant? It is shown in the present letter for finite-dimensional models with a gauge group (including the Yang-Mills quantum mechanics [7]) that this question is not deprived of sense, and the receipt of PI finding without any condition in a holomorphic representation is suggested.

2. We shall explain the main idea of the note by a simple example where there is only one physical degree of freedom. The Lagrangian of the model is [4]

$$L = \frac{1}{2}\left(\dot{x} - y_a T^a x\right)^2 - V(x^2) \qquad (1)$$

here both an N-dimensional vector $x = (x_1, x_2, \dots, x_N)$ and y_a (a=1,2,...N) play roles of dynamical variables of the theory, T^a are NxN antisymmetric matrices which are generators of the group SO(N), $[T^a, T^b] = f_{abc} T^c$, f_{abc} are structural constants of SO(N), $(T^a x)_i = T^a_{ij} x_j$ and V is a potential. Lagrangian (1) remains invariable with respect to gauge transformations

$$x \to \Omega x \Omega^T, \; y \to \Omega y \Omega^T - \Omega \partial_t \Omega^T, \; y = y_a T^a, \qquad (2)$$

where $\Omega = \exp \omega_a(t) T^a$, ω_a are arbitrary functions of time, Ω^T is the transposed matrix Ω.

Passing to the Hamiltonian formalism we find canonical momenta $\pi_a = \partial L/\partial \dot{y}_a = 0$ (primary costrants[1]), $p = \partial L/\partial \dot{x} = \dot{x} - y_a T^a x$ then Hamiltonian is

$$H = \frac{1}{2} p^2 + V(x^2) - y_a G^a, \qquad (3)$$

where $G_a = \dot{\pi}_a = \{\pi_a, H\} = p_i T^a_{ij} x_j = 0$ are secondary constraints ($\{,\}$ are Poisson brackets) which follow from the requirement of the consistency of the theory[1]. All constraints are of the first class $\{G_a, G_b\} = f_{abc} G_c$, $\{G_a, H\} = -f_{abc} y_b G_c$. Thereby the quantization of the theory is carried out by the change of both the momenta and coordinates to operators with the commutation relations $[x_j, p_k] = i\delta_{jk}$, $[y_a, \pi_b] = i\delta_{ab}$ and constraints operators pick out physical states [1]

$$G_a |\psi_{ph}\rangle = 0, \quad \pi_a |\psi_{ph}\rangle = 0. \qquad (4)$$

The second equality in (4) means that wave functions do not depend on y_a; so below we shall not take these degrees of freedom into consideration. Other equations of (4) can be easily solved in the holomorphic representation which is introduced by the definition of operators [8] $\hat{a}_j = (x_j + i p_j)/\sqrt{2}$ and $\hat{a}^+_j \psi(a^*) = a^*_j \psi(a^*)$, $\hat{a}_j \psi(a^*) = \partial/\partial a^*_j \, \psi(a^*)$. The scalar product will be defined in a standard way

$$\int d^N(a^*, a) \left(\psi_1(a^*)\right)^* \psi_2(a^*) = \langle \psi_1 | \psi_2 \rangle, \qquad (5)$$

where $d^N(a^*, a) = (2\pi i)^{-N} d^N a^* d^N a \exp(-a^*_j a_j)$. Any state in the holomorphic representation decomposes over the basis $\langle a^* | n_1, \dots, n_N \rangle = \prod_{i=1}^N (a^*_i)^{n_i}/\sqrt{n_i!}$ here $n_i = 0, 1, \dots$. This basis is orthonormal with respect to the scalar product (5). Constraints operators become $G_a = T^a_{ij} \hat{a}^+_i \hat{a}_j$. Note that there is no problem of the operator ordering here as T^a are antisymmetric matrices.

Obviously, the vacuum $\langle a^* | 0 \rangle = 1$ satisfies (4), so any physical state is determined by the application of a function of operators $\hat{a}^+_j$ which commutes with all constraints G_a. It is easy to understand that any function of that sort depends only on the operator $\hat{a}^+_j \hat{a}^+_j$. Actually, gauge transformations (2) are rotations of an N-dimensional vector and G_a are generators of these rotations, on the other hand, any quantity invariant with respect to the rotation group in $\mathbb{R}^N$ is formed out of the only variable, the square of a vector in $\mathbb{R}^N$. Consequently, we find the basis in the physical subspace

$$\langle a^* | n \rangle_{ph} = C_n \left(a^*_j a^*_j\right)^n, \; n = 0, 1, \dots . \qquad (6)$$

The normalization factors C_n can be calculated both from the equality $\langle n|n'\rangle_{ph} = \delta_{nn'}$ and (5):

$$C_n^{-2} = \left(\frac{\partial}{\partial a_j^*}\frac{\partial}{\partial a_j^*}\right)^n (a_j^* a_j^*)^n = \frac{4^n n!\,\Gamma(n+\frac{N}{2})}{\Gamma(\frac{N}{2})} . \qquad (7)$$

Non-negative integers n_i (i=1,2,...,N) enumerate the total basis as the system contains N degrees of freedom. Basis (6) is defined by the only integer n, i.e., there is only one physical degree of freedom in our system. Really, it follows from the law of gauge transformations (2) that the role of a physical variable plays the absolute value of the position vector $r = (x^2)^{1/2} \geqslant 0$, and besides, the phase space of r and canonical conjugated momentum p_r is a cone [4]. Thereby for PI finding one should integrate over the semiaxis $r \geqslant 0$. However, even a Gaussian finite-dimensional integral over the semiaxis cannot be calculated explicitly. The latter problem is usually ignored in the PI construction for gauge theories. Incidentally, as it has been shown in [5], it leads to a PI modification, and as a result, the quasiclassical description is changed [6]. This problem, nevertheless, can be avoided if one uses the receipt of PI finding suggested below in which unphysical degrees of freedom are not eliminated explicitly.

Using the Feynman-Kac formula we write the kernel of the evolution operator in the physical subspace

$$U_t^{ph}(a^*,a) = \sum_E \psi_E^{ph}(a^*)\left(\psi_E^{ph}(a^*)\right)^* e^{-iEt} , \qquad (8)$$

where summation runs over the whole physical spectrum of Hamiltonian H (Hamiltonian (3) without the last term), i.e., $\psi_E^{ph}(a^*)$ satisfy Eq.(4). If in Eq.(8) we sum over all eigenstates of H, we get the kernel of the evolution operator $U_t(a^*,a)$ in the total Hilbert space of

states. Our purpose is to establish a connection between U_t and U_t^{ph} without explicit elimination of unphysical degrees of freedom by a gauge fixation.

Note that at t=0 $U_t^{ph}(a^*,a) = Q(a^*,a)$ is the projector on the physical subspace, for the functions $\psi_E^{ph}(a^*)$ compose a complete orthonormal system in the physical subspace because of hermiticity of H and commutativity of G_a and H, i.e., the eigenvector space of H can be expanded into the orthogonal sum of both physical and unphysical subspaces. According to this remark we deduce the equality

$$U_t^{ph}(a^*,a) = \int d^N(b^*,b)\, U_t(a^*,b)\, Q(b^*,a), \qquad (9)$$

i.e., the projective operator Q removes contributions of unphysical states to the evolution operator. There is a standard representation for the kernel $U_t(a^*,a)$ by PI [8]

$$U_t(a^*,a) = \int \prod_{\tau=0}^{t} \frac{d^N a^* d^N a}{(2\pi i)^N} \exp\left[\frac{1}{2}\left(a_j^*(t)a_j(t) + a_j^*(0)a_j(0)\right)\right] \exp iS , \qquad (10)$$

where $a^*(t) = a^*$, $a(0) = a$ are standard boundary conditions for PI in the holomorphic representation, $S = \int_0^t d\tau \left[\frac{1}{2i}\left(\dot{a}_j^* a_j - a_j^* \dot{a}_j\right) - H(a^*,a)\right]$ is the action of the system including unphysical degrees of freedom too, $H(a^*,a)$ is the kernel of Hamiltonian H to be obtained from the operator H by the change of operators $\hat{a}_j^+$ and $\hat{a}_j$ to complex numbers a_j^* and a_j , respectively, after an arrangement of all $\hat{a}_j$ to the right from $\hat{a}_j^+$.

Thus, the task is reduced to finding the kernel $Q(a^*,a)$. Since Q is a projector on a physical subspace and vectors (6) form a complete orthogonal system in it, we can use the decomposition of the unity in the physical subspace

$$Q(a^*,a) = \sum_{n=0}^{\infty} c_n^2 (2\xi)^{2n} = \Gamma\left(\frac{N}{2}\right) \xi^{1-\frac{N}{2}} I_{\frac{N}{2}-1}(2\xi), \quad (11)$$

where $\xi = \frac{1}{2}\left(a_j^* a_j^* a_i a_i\right)^{1/2}$, I_ν is a modified Bessel function.

Formulae (9)-(11) solve the above task. However, the standard form for U_t^{ph} can be given:

$$U_t^{ph}(a^*,a) = \int \prod_\tau \left(d^N(a^*,a)\mu(a^*,a)\right) \exp\Phi \exp iS_{ef} \quad (12)$$

here $\mu(a^*,a)$ is some measure in the total phase space of the system, S_{ef} is an effective action in it, Φ is a phase connected with a choice of boundary conditions like the phase in (10). Accordingly to (8) the kernel of U_t^{ph} satisfies the equation $i\partial_t U_t^{ph}(a^*,a) = H(\hat{a}^+,\hat{a})\, U_t^{ph}(a^*,a)$ with the initial condition $U_{t=0}^{ph} = Q$ (note that kernel (10) also satisfies the same equation but with the other initial condition: $U_{t=0}(a^*,a) = \exp a_j^* a_j$ which is the unit operator kernel in the total Hilbert space). From this equation we obtain the infinitesimal kernel of U_ε^{ph}, $\varepsilon \to 0$

$$U_\varepsilon^{ph}(a^*,a) = Q(a^*,a)\exp\left[-i\varepsilon H_{ef}(a^*,a)\right] + O(\varepsilon^2), \quad (13)$$

where

$$H_{ef}(a^*,a) = \frac{1}{Q(a^*,a)} H\left(a^*, \frac{\partial}{\partial a^*}\right) Q(a^*,a) \quad (14)$$

Iterating kernel (13) in accordance with the scalar product (5) we find the kernel of U_t^{ph} for a finite time region in the form of PI (12) in which

$$\mu(a^*,a) = Q(a^*,a) \quad (15)$$

$$S = \int_0^t d\tau \left[\frac{1}{2iQ}\left(\dot{a}_j^* \frac{\partial}{\partial a_j^*} - \dot{a}_j \frac{\partial}{\partial a_j}\right) Q - H_{ef}(a^*,a)\right] \quad (16)$$

$$\Phi = a_j^*(t)a_j(t) - a_j^*(0)a_j(0) - \frac{1}{2}\ln\frac{Q(a^*(t),a(t))}{Q(a^*(0),a(0))} \quad (17)$$

and $a^*(t) = a^*$, $a(0) = a$. Note, if there is no gauge symmetry, then $Q(a^*,a) = \exp a_j^* a_j$ and Eq.(12) reduces to (10).

Thus, to avoid explicit separation of physical variables in PI, there are two ways: either to use the projective formula (9) or to change both the measure and action according to formulae (12),(14)-(17) in the ordinary PI over the total phase space. However, the main problem in both cases is to find the operator Q .

3. Now consider systems with several physical degrees of freedom. Let us find the operator Q for the Yang-Mills quantum mechanics [7] with the group SU(2). The model is obtained from the Yang-Mills theory [9] with the additional condition that all fields depend only on time, i.e., they are homogeneous in space. The Lagrangian is [10]

$$L = \frac{1}{2}\,\mathrm{Tr}\,(\dot{x} + yx)^T(\dot{x} + yx) - V(x) \quad (18)$$

here x is a real 3x3 matrix, y is an antisymmetric matrix. If in the Yang-Mills Lagrangian we identify potentials $A_i^a = A_i^a(t)$ with x_{ai} , where i,a=1,2,3 enumerate spatial and isotopic coordinates, respectively, and $y_{ab} = -g\varepsilon_{abc} A_0^c$, g is a coupling constant, we get Lagrangian (18) in which $V = g^2/4\left[(\mathrm{Tr}\,x^T x)^2 - \mathrm{Tr}\,(x^T x)^2\right]$, however, our consideration does not depend on the potential form.

Lagrangian (18) is invariant with respect to gauge transformations like (2) where the vector x should be replaced by a matrix x and Ω is considered an orthogonal 3x3 matrix. The Hamiltonian formalism for this model is also analogous to that of model (I). The momentum canonical conjugated to y vanishes too, so we shall not take this degree of freedom into consideration. The secondary constraints are generators of isotopic rotations of columns of a matrix x . Any real matrix x can be written in the polar representation $x = u\rho$, where u is an orthogonal matrix and ρ is a positive definite symmetric matrix. Clearly, u contains only unphysical degrees of freedom (they can be eliminated by the gauge transformation $x \to u^T x$). If the PI is constructed only for physical variables ρ (their number is six $\rho = \rho^T$), the problem of integration over positive definite matrices arises. It is not equivalent to integration over $\mathbb{R}^6$ [10]. The last observation is connected with the phase space reduction of physical variables [4,5]. So it is convenient to use the above given receipt for PI finding.

Note that after a passage to the holomorphic representation for each component of the matrix x_{ai} all physical state vectors should be gauge-invariant $\psi_{ph}(\Omega a^*) = \psi_{ph}(a^*)$, where $a^*_{aj} = (x_{aj} - i p_{aj})/\sqrt{2}$, p_{aj} are momenta canonical conjugated to x_{aj} . It is easy to understand that for any vector $\psi_{ph}(a^*)$ a series expansion must exist over elements of the matrix $(a^{*T}a^*)_{ij} = a^*_{ia} a^*_{aj}$ which describe six physical degrees of freedom in this model. So the orthonormal basis in the physical subspace has the form

$$\langle a^*|n\rangle = c(n_{ij}) \left[(a^{*T}a^*)_{ij}\right]^{n_{ij}}, \quad n_{ij} = 0, 1, \ldots \qquad (19)$$

here $i<j$. States (19) are normalized by the scalar product (5) where N=9 is a total number of degrees of freedom and $-\mathrm{Tr}\, a^{*T}a = -a^*_{ia} a_{ai}$ is to be placed in the measure in the exponential argument instead of $-a^*_j a_j$. The normalization factors $c(n_{ii})$ (no summation over i) and $c(n_{ij})$, $i<j$ are obtained from (7) by the change $n = n_{ii}$, N=3 and by omitting factor 4^n with a subsequent change $n=n_{ij}$, N=6 respectively. Now we use again the decomposition of the unity in the physical subspace for finding Q . Analogously to the calculation of (11) we have

$$Q(a^*, a) = \pi^{3/2} \prod_{i=1}^{3} \xi_{ii}^{-1/2} I_{1/2}(\xi_{ii}) \prod_{i<j=1}^{3} \xi_{ij}^{-2} I_2(2\xi_{ij}), \qquad (20)$$

where $\xi_{ij} = \left[(a^{*T}a^*)_{ij}(a^T a)_{ij}\right]^{1/2}$. Further by formula (14)-(17) we restore the physical (gauge-invariant) evolution operator (12) or we can apply (9).

4. In the conclusion we shall show the group method of the operator Q calculation for any model with a finite number of degrees of freedom and a gauge symmetry. Let the brackets $\langle\, ,\rangle$ mean the scalar product in a representation space of a compact gauge group G and T_g be a group element in this representation. Then

$$Q(a^*, a) = \mu_G^{-1} \int d\mu(g) \exp \langle a^*_i, T_g a_i \rangle, \qquad (21)$$

here μ_G is a volume of the group space, $d\mu(g)$ is a right- and left-invariant Haar measure on G , the index i enumerates "particles" in a representation space, i.e., degrees of freedom are enumerated by i and the group index on which operators T_g act . It is also assumed that operators T_g are unitary with respect to the scalar product $\langle T_g a_i^*, T_g a\rangle = \langle a_i^*, T_g^+ T_g a_i\rangle = \langle a_i^*, a_i\rangle$, i.e., $T_g^+ = T_{g^{-1}}$. Now we verify easily that $Q(a^*, a) = Q(T_g a^*, T_g a)$. It follows both from the unitarity of T_g and invariance of the measure $d\mu(g_1 g g_2) = d\mu(g)$. It remains for us to prove projectional properties of Q. After simple calculations we get

$$\int d^N(b^*, b)\, Q(a^*, b)\, \psi(b^*) = \mu_G^{-1} \int d\mu(g)\, \psi(T_g^+ a^*), \quad (22)$$

where N is a total number of degrees of freedom. To derive equality (22), we have used definition (21) and the change of integration variables $b_i^* \to b_i^* - T_g^+ a_i^*$ has been done. If $\psi(T_g a^*) = \psi(a^*)$, i.e., it is a physical state, Q is the kernel of the unit operator in the physical subspace, as it follows from (22). Also obviously there is a projector on physical states in the right-hand side of Eq.(22). The derivation of PI without gauge fixation in the Lagrange form will be suggested elsewhere.

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Received by Publishing Department
on September 28, 1989.