89-616

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дубна



E2-89-616

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# RELATIVE VELOCITY, MOMENTUM AND ENERGY OF POINT PARTICLES IN SPACES WITH GENERAL LINEAR TRANSPORT

1989

## 0. INTRODUCTION

This work is devoted to the application of an apparatus of general linear transports (I-transports) along curves [3] for defining and investigating the concepts of relative velocity, relative momentum and relative energy between two test point particles in manifolds endowed with these transports.

In Sec.1 and Sec.2 we define, without going into details, the relative velocity and relative momentum, respectively, between two test point particles (observed particles) with respect to a third particle (observer). This is done on the basis of a differentiable manifold endowed with a general linear transport (I-transport). In Sec.3 we show how in these manifolds, in which a metric (i.e., a scalar product of the vectors) is defined, one can introduce the concept of relative energy between two particles. Here, as a special case we get the notion of proper (rest) energy of a single particle. Here, we also consider some properties and connections between concepts introduced in this paper. At the end (Sec.4), we give an idea of one way of comparing vectors (or tensors) defined at different points and demonstrate how the general concepts introduced in the present article work in the case of special theory of relativity.

Now we shall introduce some preliminary notions and notation needed for the following.

Let a point particle with a (rest) mass m be moving along the $C^1$-curve (trajectory, world line) $y:[s', s'']\to M$ in the space (differentiable manifold) M in which to any curve $\gamma: J\to M$, $J\subset \boldsymbol{R}$ is assigned a (linear) generalized transport (an I-transport) $I^{\gamma}_{u\to v}$, $u, v\in\gamma(J)$ (for details see Ref. [3]). The velocity of this particle (with respect to the parameter $s\in[s', s'']$) is defined as a tangent vector $\dot{\mathbf{y}}$ to $\mathbf{y}$ with components $\dot{y}^a(s) :=$ $=: dy^a(s)/ds$ at the point $y(s)$, where $y^a(s)$, $a=1,\dots,n=\dim M$ are the coordinates of $\mathbf{y}(s)$ in some local coordinates. Let us note that we do not exclude the case when the mass m may depend upon the position $y(s)$, i.e., m may be a function of s.

By definition (see [5], ch. III, § 3) the vector

$$p = p(s) := \mu(s)\,\dot{y}(s), \quad p^a = p^a(s) = \mu(s)\, dy^a(s)/ds, \qquad (0.1)$$

where $\mu(s)$ has a dimension of mass and is some scalar function on $[s', s'']$ (see below), is called momentum of the given particle. If $m \neq 0$, then $\mu(s) = m$. If $m = 0$ (e.g., when we have a photon), we cannot connect $\mu(s)$ with $m = 0$; in this case, the momentum p is primarily defined and $\mu(s)$ is then uniquely defined by eq.(0.1). (For example, in Sec.4 we shall consider in special relativity the case $m = 0$, see, e.g., eqs.(4.8)-(4.15)). Let us note that in both the cases $\mu(s) \neq 0$ for every s. (The case $m = 0$, $\mu(s) = 0$ describes the vacuum, not a particle).

The parameter u defined by

$$\frac{du}{ds} = \frac{1}{\mu(s)} \quad \text{or} \quad u = \int_{s'}^{s} \frac{dt}{\mu(t)} + \text{const} \tag{0.2}$$

may be called[1] the proper time of the particle. If we put $z(u) := y(s(u))$ we get $\dot z^{\alpha}(u) = dz^{\alpha}(u)/du = \dot y^{\alpha}(s)(ds/du) = \mu(s)\dot y^{\alpha}(s)$ and from (1.1) we see that

$$p = p(u) = \dot z(u), \quad p^{\alpha} = dz^{\alpha}(u)/du = dy^{\alpha}(s(u))/du, \tag{0.3}$$

i.e., when the proper time is used as a parameter, the momentum of a particle coincides with its velocity. It is easy to see that the definition of the momentum is equivalent to the definition of the proper time, i.e., through momentum we can define the proper time and vice versa.

Let two particles 1 and 2 be moving along the $C^1$-curves $x_a : [s'_a, s''_a] \to M$, $a = 1,2$, respectively, and let they be observed from a point particle (observer) with a world line $x : [s', s''] \to M$. Let the $C^1$ - maps $\tau_a : [s', s''] \to [s'_a, s''_a]$, $a = 1,2$ map the parameter $s \in [s', s'']$ uniquely on the parameters $s_a = \tau_a(s) \in [s'_a, s''_a]$, $a = 1,2$. And at the end let there be given two families of curves $\gamma_s : [r'_s, r''_s] \to M$ and $\eta_s : [\rho'_s, \rho''_s] \to M$, such that $\gamma_s(r'_s) = x_1(\tau_1(s))$, $\gamma_s(r''_s) = x_2(\tau_2(s))$, $\eta_s(\rho'_s) = x_1(\tau_1(s))$ and $\eta_s(\rho''_s) = x(s)$, $s \in [s', s'']$. In the next sections we shall define the relative velocity, momentum and energy of particle 2 with respect to particle 1 relatively to the given observer.

## 1. RELATIVE VELOCITY

The velocities of the particles considered at the end of the Introduction are $V_a = \dot x_a(s_a)$, $a = 1,2$ and in any local basis have the components



$$V_a^{\alpha} = V_a^{\alpha}(s_a) = dx_a^{\alpha}(s_a)/ds_a, \quad s_a = \tau_a(s), \quad a = 1,2. \tag{1.1}$$

The I-transport of $V_2$ from $x_2(s_2)$ to $x_1(s_1)$ gives the vector

$$(V_2)_1 := I^{\gamma_s}_{x_2(s_2) \to x_1(s_1)} V_2 \in T_{x_1(s_1)}(M) \tag{1.2}$$

which can be compared with the velocity $V_1$. This leads to the following definition.

The vector

$$\Delta V_{21} := \Delta V_{21}(s, x) := I^{\eta_s}_{x_1(s_1) \to x(s)} ((V_2)_1 - V_1) = I^{\eta_s}_{x_1(s_1) \to x(s)} (I^{\gamma_s}_{x_2(s_2) \to x_1(s_1)} V_2 - V_1) \in T_x(M) \tag{1.3}$$

will be called a (defined by the I-transport) relative velocity of the second particle with respect to the first one (as it is "observed" from an observer with a world line x at the point x(s)).

As is clear, in some sence, this definition is the most natural generalization of the Newtonian concept of relative velocity (defined as difference of the 3-vectors representing the particle velocities).

In the (pseudo-) Euclidean case, when $I^{\gamma}$ is simply the Euclidean parallel transport along $\gamma$ in a (pseudo-) Euclidean space $E_n$, we evidently have

$$((V_2)_1)^{\alpha}|_{E_n} = V_2^{\alpha}, \tag{1.4}$$

$$(\Delta V_{21})^{\alpha}|_{E_n} = V_2^{\alpha} - V_1^{\alpha} = \frac{ds}{ds_2}\frac{dx_2^{\alpha}(\tau_2(s))}{ds} - \frac{ds}{ds_1}\frac{dx_1^{\alpha}(\tau_1(s))}{ds}. \tag{1.5}$$

The relative velocity (1.3) is defined mostly from general theoretical considerations. But for some purposes another concept of "relative velocity" is needed, which is introduced as follows.

Let $M$ be endowed with a covariant derivative $\nabla$, i.e., $M$ be with linear connection, and $D/ds = D/ds|_{\gamma_s} := \nabla_{\dot{\gamma}_s}$ be the covariant derivative along $\gamma_s$.

The deviation vector between the observed particles (as it is "observed" from the observer) is /3/

$$h_{21} = h_{21}(s;x) := I^{\eta_s}_{x_1(\tau_1(s)) \to x(s)} \int_{r'_s}^{r''_s} I^{\gamma_s}_{\gamma_s(r) \to x_1(\tau_1(s))} \dot{\gamma}_s(r)\, dr \in T_{x(s)}(M), \tag{1.6}$$

where $\dot{\gamma}_s(r)$ has components $\dot{\gamma}^{\alpha}_s(r) = \partial \gamma^{\alpha}_s(r)/\partial r$.

As the deviation vector (1.6) has the meaning of a relative coordinate of particle 2 with respect to particle 1 (relative to an observer), we can call the vector

$$V_{21} := Dh_{21}/ds \tag{1.7}$$

a deviation velocity between the particles 2 and 1 because it is a measure of the change of the deviation vector between them with respect to the (proper time of the) observer.

The deviation velocity has a direct physical meaning because it can be measured. In fact, if for example the observer coincides with the first particle, then in one or another way it can measure (e.g., by radiolocation) the position $h_{21}$ of the second particle with respect to himself and then to find the deviation velocity by (1.7) in which parameter $s$ must be interpreted as an observer's "proper time".

The relative and deviation velocities generally do not coincide even in the (pseudo-) Euclidean case ($I^{\gamma}$ = parallel transport along $\gamma$, $\nabla$ = flat connection, i.e. $D/ds = d/ds$) in which

$$V_{21}|_{E_n} = \frac{d(h_{21}|_{E_n})}{ds} = \frac{d(x_2(s_2) - x_1(s_1))}{ds} = \frac{dx_2(\tau_2(s))}{ds} - \frac{dx_1(\tau_1(s))}{ds}. \tag{1.8}$$

Nevertheless, in the Newtonian mechanics, where we have an Euclidean world with an absolute simultaneity ($s_2 \equiv s_1 \equiv s$), due to (1.5) and (1.8) both the definitions coincide.

The difference between relative and deviation velocities as well as their meaning will be deeper examined elsewhere (see also Sec.4).

In manifold endowed with a linear connection and an I-transport one can introduce in an analogous way the concepts for relative and deviation accelerations between the observed particles: to do this it is enough in the above definitions to replace the velocities $V_a$ with the accelerations $A_a := \frac{D}{ds_a}\Big|_{x_a} V_a$, $a = 1,2$ and $h_{21}$ with $V_{21}$.

## 2. RELATIVE MOMENTUM

The momenta of the observed particles are

$$p_a := p_a(s_a) := \mu_a(s_a) V_a, \quad a = 1, 2, \tag{2.1}$$

where $\mu_1(s_1)$ and $\mu_2(s_2)$ are scalars (see Sec.0) and the velocities $V_a = \dot{x}_a(s_a)$, $a = 1,2$ are with components (1.1).

Let

$$(p_2)_1 := I^{\gamma_s}_{x_2(s_2) \to x_1(s_1)} p_2 \in T_{x_1(s_1)}(M). \tag{2.2}$$

This is the I-transport vector $p_2$ from $x_2(s_2)$ to $x_1(s_1)$ which can be compared with the vector $p_1$. Due to (2.1) and (1.2) the vector (2.2) may be written also as

$$(p_2)_1 = \mu_2(s_2)(V_2)_1. \tag{2.3}$$

We define the relative momentum of the second particle with respect to the first one, as it is "seen" from the observer, at the point $x(s)$ as

$$\begin{aligned} \Delta p_{21} &:= I^{\eta_s}_{x_1(s_1) \to x(s)}((p_2)_1 - p_1) = \\ &= I^{\eta_s}_{x_1(s_1) \to x(s)}(I^{\gamma_s}_{x_2(s_2) \to x_1(s_1)} p_2 - p_1) = \\ &= I^{\eta_s}_{x_1(s_1) \to x(s)}(\mu_2(s_2)\, I^{\gamma_s}_{x_2(s_2) \to x_1(s_1)} V_2 - \mu_1(s_1) V_1). \end{aligned} \tag{2.4}$$

From (2.4) it is not difficult to derive (see also /3/) that

$$\Delta p_{21} = \mu_2(s_2)\, \Delta V_{21} + (\mu_2(s_2)/\mu_1(s_1) - 1)\, I^{\eta_s}_{x_1(s_1) \to x(s)} p_1, \tag{2.5}$$

where the relative velocity $\Delta V_{21}$ of the second particle with respect to the first one is defined by (1.3).

It is also easy to see that in the (pseudo-) Euclidean case ($I^{\cdots}_{\cdots}$-parallel transport; see [3]) the relative momentum takes its well-known Newtonian expression

$$\Delta p_{21}|_{E_n} = p_2 - p_1 . \tag{2.6}$$

By analogy with (2.4) one can define the relative momentum $\Delta p_{12}$ of the first particle with respect to the second one:

if $\eta^*_s : [\rho^{*\prime}_s, \rho^{*\prime\prime}_s] \to M$, $\eta^*_s(\rho^{*\prime}_s) = x_2(s_2)$, $\eta^*_s(\rho^{*\prime\prime}_s) = x(s)$

and we use the same curves $\gamma_s$, then

$$\Delta p_{12} = I^{\eta^*_s}_{x_2(s_2) \to x(s)} (I^{\gamma_s}_{x_1(s_1) \to x_2(s_2)} p_1 - p_2) . \tag{2.7}$$

Hereof, we find that (see [3])

$$\Delta p_{12} = -(I^{\eta^*_s}_{x_2(s_2) \to x(s)} \circ I^{\gamma_s}_{x_1(s_1) \to x_2(s_2)} \circ I^{\eta_s}_{x(s) \to x_1(s_1)}) \Delta p_{21} . \tag{2.8}$$

In the special cases when $\eta^*_s$ is a composition of $\gamma_s$ and $\eta_s$ [3] or when $I^{\gamma}_{\cdots}$ does not depend on $\gamma$, eq.(2.8) reduces to

$$\Delta p_{12} = -\Delta p_{21} . \tag{2.9}$$

## 3. RELATIVE ENERGY

One has to be careful when defining the concept of "relative energy" of particles 1 and 2, because when it is applied to a single particle only, one gets a quantity which has a sense of energy of this particle (see below (3.8) and (4.7)) and hence, due to the established opinion [6], it must be positive.

Let the manifold M be endowed with a metric and $g_{\alpha\beta}(y)$ be the covariant components of a metrical tensor at $y \in M$.

We shall denote the scalar product of the vectors by a dot: if $A = A^\alpha \partial_\alpha \in T_y(M)$ and $B = B^\alpha \partial_\alpha \in T_y(M)$, then $A \cdot B := A^\alpha B^\beta g_{\alpha\beta}(y)$. The square of the vector A will be written as $(A)^2 := A^\alpha A^\beta g_{\alpha\beta}(y) = A \cdot A$ for it has to be distinguished from the second component $A^2$ of A (for $\dim(M) \geq 2$). The metric is not supposed to be positive, so generally $(A)^2$ can take any real values.



The relative energy of particle 2 with respect to particle 1 is defined as a scalar

$$E_{21} := E_{21}(s) = \epsilon((V_1(s_1))^2)(I^{\gamma_s}_{x_2(s_2) \to x_1(s_1)} p_2(s_2)) \cdot V_1(s_1) = \epsilon((V_1(s_1))^2)\, p_{21} \cdot V_1(s_1) , \tag{3.1}$$

where $\epsilon(\lambda) = -1$ for $\lambda < 0$ and $\epsilon(\lambda) = +1$ for $\lambda \geq 0$ (for some purposes one may put $\epsilon(0) = -1$ instead of $\epsilon(0) = +1$; our general results are independent of this choice).

If there is $s_0 \in [s', s'']$ such that $x_1(\tau_1(s_0)) = x_2(\tau_2(s_0))$, i.e., for $s = s_0$ the world lines of the observed particles intersect with each other, then from (3.1) we get

$$E_{21}(s_0) = \epsilon((V_1(\tau_1(s_0)))^2)\, p_2(\tau_2(s_0)) \cdot V_1(\tau_1(s_0)) . \tag{3.2}$$

In the case of the space-time of general relativity this expression coincides with the definition of a relative energy given in [5], ch.III, § 5, eq.(23). The deficiency of the cited definition is that it defines the relative energy only for the "moment" $s = s_0$ because of which one cannot study the evolution of the relative energy in time. Evidently, our definition (3.1) is free from this deficiency.

Substituting (2.4) into (3.1) we get:

$$E_{21}(s) = \epsilon((V_1(s_1))^2)\, \mu_2(s_2) V_{21} \cdot V_1(s_1) = \epsilon((V_1(s_1))^2)\, \mu_2(s_2) (I^{\gamma_s}_{x_2(s_2) \to x_1(s_1)} V_2(s_2)) \cdot V_1(s_1) . \tag{3.3}$$

Due to (3.1) and (3.3) the relative energy of particle 1 with respect to particle 2 is

$$E_{12}(s) = \epsilon((V_2(s_2))^2)(I^{\gamma_s}_{x_1(s_1) \to x_2(s_2)} p_1(s_1)) \cdot V_2(s_2) = \epsilon((V_2(s_2))^2)\, \mu_1(s_1) (I^{\gamma_s}_{x_1(s_1) \to x_2(s_2)} V_1(s_1)) \cdot V_2(s_2) . \tag{3.4}$$

If we use the most general I-transports [3], then in the general case the relative energies $E_{21}(s)$ and $E_{21}(s)$ are not connected with each other. However, let us consider the class of not so general I-transports having the following property (in the notation of Sec.0):

$$E_i \cdot E_j = (I^{\gamma}_{u\to v} E_i) \cdot (I^{\gamma}_{u\to v} E_j) , \tag{3.5}$$

where $\{E_i\}$, $i, j = 1, \dots, n$ is any local (coordinate or not) basis in $T_u(M)$. This condition is equivalent to

$$A \cdot B = (I^{\gamma}_{u\to v} A) \cdot (I^{\gamma}_{u\to v} B) , \quad A, B \in T_u(M) . \tag{3.5'}$$

(One can say that the metric and I-transport are consistent if the last equality is valid for every $\gamma$, u, v, A and B).

As examples of I-transports obeying eq.(3.5) or (3.5') equivalent to it we shall mention, e.g., the parallel transport and Fermi-Walker transport [5]. A complete description of all I-transports satisfying (3.5) is given in [2].

It is not difficult to check (see (2.5) and (3.1)) that for I-transport satisfying (3.5') the relative energy and relative momentum are related by

$$E_{21} = \epsilon((V_1(s_1))^2)(\Delta p_{21} \cdot I^{\eta_s}_{x_1(s_1)\to x(s)} V_1 + p_1 \cdot V_1) . \tag{3.6}$$

For I-transports obeying (3.5') from (3.3) we get:

$$E_{21}(s) = \epsilon((V_1(s_1))^2)\mu_2(s_2)(I^{\gamma_s}_{x_2(s_2)\to x_1(s_1)} V_2(s_2)) \cdot V_1(s_1) =$$

$$= \epsilon((V_1(s_1))^2)\mu_2(s_2)[(I^{\gamma_s}_{x_1(s_1)\to x_2(s_2)} \circ I^{\gamma_s}_{x_2(s_2)\to x_1(s_1)}) V_2(s_2)] \times$$

$$\times (I^{\gamma_s}_{x_1(s_1)\to x_2(s_2)} V_1(s_1)) =$$

$$= \epsilon((V_1(s_1))^2)\mu_2(s_2) V_2(s_2) \cdot (I^{\gamma_s}_{x_1(s_1)\to x_2(s_2)} V_1(s_1)) .$$

Multiplying this equality by $\epsilon((V_2(s_2))^2)\mu_1(s_1) \neq 0$ and using (3.4) (and the symmetry of the metric $g_{\alpha\beta} = g_{\beta\alpha}$ or $A \cdot B = B \cdot A$), we find

$$\epsilon((V_2(s_2))^2)\mu_1(s_1) E_{21}(s) = \epsilon((V_1(s_1))^2)\mu_2(s_2) E_{12}(s) . \tag{3.7}$$



Let us remind that this relation takes place only if the condition (3.5) is satisfied.

Applying the definition (3.1) to the first particle itself, for which one has to put $x_2 = x_1$ and $s_2 = s_1$ (or $r_2 = r_1$), we find the energy of this particle (with respect to itself) to be

$$E_{11} = \epsilon((V_1(s_1))^2)(I^{\gamma_s}_{x_1(s_1)\to x_1(s_1)} p_1(s_1)) \cdot V_1(s_1) =$$

$$= \epsilon((V_1(s_1))^2) p_1(s_1) \cdot V_1(s_1) = \epsilon((V_1(s_1))^2)\mu_1(s_1)(V_1(s_1))^2 =$$

$$= \mu_1(s_1)|(V_1(s_1))^2| = |(p_1(s_1))^2| / \mu_1(s_1) , \tag{3.8}$$

where $|\lambda| := \epsilon(\lambda)\lambda$ is the absolute value of $\lambda \in \boldsymbol{R}$.

One can call $E_{11}$ a proper energy (or an intrinsic, or rest energy) of the particle (for the case of special relativity see Sec.4). If $m_1 > 0$, then due to $\mu_1(s_1) = m_1$ and (3.8), we have $E_{11} \geq 0$ and if besides this $(V_1(s_1))^2 \neq 0$, then $E_{11} > 0$ which corresponds to the usual case of nonzero mass material particle.

If $m_1 \neq 0$, then $\mu_1(s_1) = m_1$ and (3.8) shows that $E_{11}$ is proportional to $m_1$; hence, if we suppose that $E_{11}$ is a continuous function of $m_1$ at the point $m_1 = 0$, we get

$$\lim_{m_1\to 0} E_{11} = 0 , \tag{3.9}$$

which, due to (3.8) (and $\mu_1(s_1) \neq 0$ for any $m_1$), is equivalent to

$$(V_1(s_1))^2 = 0 \quad \text{for} \quad m_1 = 0 , \tag{3.10}$$

or

$$E_{11} = 0 \quad \text{for} \quad m_1 = 0 . \tag{3.10'}$$

Remark. If we accept $\lim_{m_2\to 0} E_{21} = 0$, i.e., the continuous dependence of $E_{21}$ on $m_2$, then (3.3) and the fact that the I-transport is an isomorphism leads to $V_2(s_2) = 0$, which contradicts the assumption that we are dealing with a material particle (and not with the vacuum). Moreover, the equality $\lim_{m_2\to 0} E_{21} = 0$

is unacceptable from a physical viewpoint, for instance, it means that the photon, which is a zero-mass particle, has zero energy with respect to any other particle, something which is not true.

Let us note that without further assumptions one cannot derive that $(V_1(s_1))^2 = 0$ implies $m_1 = 0$.

The relation (3.10) must be considered as a direct analog and generalization of the well known fact that in special and general relativity zero-mass particles are moving with the velocity of light (in vacuum).

The energies $E_{21}$ (or $E_{12}$) and $E_{11}$ can be connected with the components of $\Delta p_{21}$ (or $\Delta p_{12}$), $p_{21}$ (or $p_{12}$) and $p_1$ in the following way.

Let $(V_1)^2 \neq 0$. Then, along $x$ we can define a basis $\{\lambda_a\}$, such that $\lambda_1 := V_1/\sqrt{|(V_1)^2|}$ and $\lambda_1 \cdot \lambda_a = 0$ for $a \neq 1$ ( if $\dim M = n > 1$), so $(\lambda_1)^2 = \epsilon((V_1)^2)$. (The concrete choice of $\lambda_a$ for $a \neq 1$ is insignificant for us). The component $A^1$ of any vector $A = A^a \lambda_a$ in $\{\lambda_a\}$ is given by

$$A^1 = A \cdot \lambda_1/(\lambda_1)^2 = \epsilon((V_1)^2)(A \cdot V_1)/\sqrt{|(V_1)^2|}, \qquad (3.11)$$

because of $A \cdot \lambda_1 = A^a \lambda_a \cdot \lambda_1 = A^1 \lambda_1 \cdot \lambda_1$.

Applying (3.11) to the defined by (2.1) vector $p_1$ and using (3.8), we get

$$p_1^1 = E_{11}/\sqrt{|(V_1)^2|}, \quad p_1^a = 0, \quad a \neq 1. \qquad (3.12)$$

In a similar way applying (3.11) to $p_{21}$ (see (2.2)) and taking into account (3.1), we find

$$(p_2)_1^1 = E_{21}/\sqrt{|(V_1)^2|}, \qquad (3.13)$$

According to (2.4) the relative momentum of particle 2 with respect to particle 1 as it is "seen" from particle 1 is

$$\Delta\pi_{21} := (p_2)_1 - p_1 (= \Delta p_{21}|_{x = x_1}). \qquad (3.14)$$

So, in the basis $\{\lambda_a\}$ we have

$$\Delta\pi_{21}^1 = (p_2)_1^1 - p_1^1 = (E_{21} - E_{11})/\sqrt{|(V_1)^2|}. \qquad (3.15)$$

Let us suppose now eq.(3.5) to be satisfied and $(V_2)^2 \neq 0$. Then, defining a basis $\{\lambda_{a'}\}$ along $x_1$ so that

$$\lambda_{1'} := I^{\gamma_s}_{x_2(s_2) \to x_1(s_1)} V_2/\sqrt{|(V_2)^2|}$$

and $\lambda_{1'} \cdot \lambda_{a'} = 0$ for $a \neq 1$ and using (3.5´), we find the first component of $p_1$ in $\{\lambda_{a'}\}$ to be

$$p_1^{1'} = p_1 \cdot \lambda_{1'}/(\lambda_{1'})^2 =$$

$$= p_1 \cdot (I^{\gamma_s}_{x_2(s_2) \to x_1(s_1)} V_2) \cdot \epsilon((V_2)^2)/\sqrt{|(V_2)^2|} =$$

$$= \epsilon((V_2)^2)(I^{\gamma_s}_{x_1(s_1) \to x_2(s_2)} p_1) \cdot V_2/\sqrt{|(V_2)^2|}.$$

Thus, because of (3.4), we get

$$p_1^{1'} = E_{12}/\sqrt{|(V_2)^2|}. \qquad (3.16)$$

If $(V_1)^2 \neq 0$ and (3.5´) is satisfied, then along $x$ there is a basis $\{\ell_a\}$ such that

$$\ell_1 := I^{\eta_s}_{x_1(s_1) \to x(s)} V_1 |(V_1)^2|^{-1/2} = I^{\eta_s}_{x_1(s_1) \to x(s)} \lambda_1$$

and $\ell_1 \cdot \ell_a = 0$ for $a \neq 1$ in which the first component of $\Delta p_{21}$ (see (2.4)) is

$$\Delta p_{21}^1 = \Delta p_{21} \cdot \ell_1/(\ell_1)^2 = \Delta\pi_{21} \cdot \lambda_1/(\lambda_1)^2 =$$

$$= \Delta\pi_{21}^1 = (E_{21} - E_{11})/\sqrt{|(V_1)^2|}. \qquad (3.17)$$

At last, if $(V_1)^2 = 0$, then (see (3.8)) $E_{11} = 0$ and the invariant $(p_2)_1 \cdot V_1 = \Delta\pi_{21} \cdot V_1 = E_{21}$ cannot be connected with a single component of $(p_2)_1$ (or $\Delta\pi_{21}$) in some basis; so the relative energy $E_{21}$ is spread over (all) the components of $(p_2)_1$ (or $\Delta\pi_{21}$).

For I-transports obeying (3.5´) we want also to mention the invariant (see (3.88), (3.1) and (2.4))

$$(\Delta p_{21})^2 = (\Delta\pi_{21})^2 = ((p_2)_1)^2 + (p_1)^2 - ((p_2)_1 \cdot p_1 + p_1 \cdot (p_2)_1) =$$

$$= (p_1)^2 + (p_2)^2 - ((p_2)_1 \cdot p_1 + (p_1)_2 \cdot p_2) =$$

$$= \epsilon((V_1)^2)\mu_1 E_{11} + \epsilon((V_2)^2)\mu_2 E_{22} -$$

$$-\epsilon(\epsilon((V_1)^2)\mu_1 E_{21}+\epsilon((V_2)^2)\mu_2 E_{12}), \qquad (3.18)$$

the two terms in the parenthesis being equal due to (3.7).

## 4. DISCUSSION

From a mathematical viewpoint the basis of Sec.1 and Sec.2 is a method for comparing vectors (or more generally tensors) defined at different points by means of I-transports (generalized linear transports) along some (smooth) curve connecting these points. The general scheme of that method is as follows. Let $A_a \in T_{z_a}(M)$, $a = 1,2$, $\gamma:[s_1, s_2]\to M$, $\gamma(s_a) = z_a$, $a = 1,2$ and

$$(A_2)_1 := I^{\gamma}_{z_2\to z_1} A_2,\ (A_1)_2 := I^{\gamma}_{z_1\to z_2} A_1 \qquad (4.1)$$

for some I-transport $I^{\gamma}$ along $\gamma$ [3]. Now instead of the vectors $A_1$ and $A_2$, one can compare the vectors $A_1 \in T_{z_1}(M)$ and $(A_2)_1 \in T_{z_1}(M)$. For instance, we have met several times the difference

$$\Delta A_{21} := \Delta A_{21}(z_1, z_2, \gamma) := (A_2)_1 - A_1 . \qquad (4.2)$$

In an utterly equivalent way one can compare the vectors $A_2 \in T_{z_2}(M)$ and $(A_1)_2 \in T_{z_2}(M)$. For example, if we put $\Delta A_{12} = (A_1)_2 - A_2$, then due to the group property of the I-transport [3], we have

$$\Delta A_{12} = I^{\gamma}_{z_1\to z_2} \Delta A_{21},\ \Delta A_{21} = I^{\gamma}_{z_2\to z_1} \Delta A_{12}, \qquad (4.3)$$

As an illustration and as a reason for the introduction of the concepts defined in this work we shall reveal their meaning in case of the special theory of relativity. (As a standard reference about special relativity see, e.g., [4]).

Let us have a standard (4-dimensional, flat, with signature (+- - -)) Minkowski space-time in which a parallel transport will be used as a concrete realization of the general I-transport, so the components of the vectors (or tensors) remain unchanged under its action. Let two point particles 1 and 2 with proper masses $m_1 \neq 0$ and $m_2 \neq 0$ be moving with constant 3-velocities $\vec{v}_1$ and $\vec{v}_2$ with respect to a fixed inertial frame of reference along the world lines $x_a(s_a) = (ct, t\vec{v}_a) + y_a$, $a = 1,2$ where c is the velocity of light in vacuum, t is the time in this frame, $s_a := t\sqrt{1-\vec{v}^2/c^2}$, $a=1,2$ are the corresponding proper times and $y_1$ and $y_2$ are fixed points.

Due to (1.1) the 4-velocities [4] of the particles are

$$V_a = (c, \vec{v}_a)/\sqrt{1 - \vec{v}_a^2/c^2}, \quad a = 1, 2. \qquad (4.4)$$

and consequently

$$(V_a)^2 = (c^2 - \vec{v}_a^2)/(\sqrt{1 - \vec{v}_a^2/c^2})^2 = c^2,\ \epsilon((V_a)^2) = -1, a=1,2. \qquad (4.5)$$

Thus, using (2.2)-(2.4), (3.14), (3.1), (3.4) and (3.8), we get

$$p_a = m_a(c, \vec{v}_a)/\sqrt{1-\vec{v}_a^2/c^2},\quad a = 1, 2 \quad (\mu_1 = m_1, \mu_2 = m_2), \qquad (4.6a)$$

$$(p_1)_2 = p_1,\ (p_2)_1 = p_2,\ \Delta p_{21} = \Delta\pi_{21} = p_2 - p_1, \qquad (4.6b)$$

$$E_{21} = m_2 c^2(1 - \vec{v}_1\cdot\vec{v}_2/c^2)[(1 - \vec{v}_1^2/c^2)(1 - \vec{v}_2^2/c^2)]^{-1/2}, \qquad (4.7a)$$

$$E_{12} = m_1 c^2(1 - \vec{v}_1\cdot\vec{v}_2/c^2)[(1 - \vec{v}_1^2/c^2)(1 - \vec{v}_2^2/c^2)]^{-1/2}, \qquad (4.7b)$$

$$E_{11} = m_1 c^2,\quad E_{22} = m_2 c^2. \qquad (4.7c)$$

Evidently, $E_{11}$ and $E_{22}$ are the proper energies (rest energies) of the particles and if, e.g. $\vec{v}_1 = 0$, then $E_{21} = m_2 c^2/\sqrt{1 - \vec{v}_2^2/c^2} = E_2$ is simply the energy of the second particle with respect to the given frame [4].

If $m_1 \neq 0$ and $m_2 = 0$, then above we have to change $x_2$ and $s_2$ to $x_2(s_2) = (ct, ct\vec{n}_2) + y_2$ and $s_2 = t$, respectively, where $\vec{n}_2$ is a unit 3-vector ($\vec{n}_2^2 = 1$) showing the direction of the propagation of the second particle, i.e., $\vec{v}_2 = c\vec{n}_2$ and

$$V_2 = c(1, \vec{n}_2),\ \vec{n}_2^2 = 1,\ (V_2)^2 = 0,\ \epsilon((V_2)^2) = +1. \qquad (4.8)$$

If $E_2$ is the energy of the second particle with respect to the used frame, then (see [4]) its 4-momentum is

$$p_2 = (E_2/c, \vec{p}_2) = (E_2/c, (E_2/c)\vec{n}_2) = (E_2/c)(1, \vec{n}_2) \qquad (4.9)$$

and, due to (1.1), we have

$$\mu_2 = \mu_2(s_2) = E_2/c^2. \qquad (4.10)$$

In this case (4.6b) hold and (4.7) take the form

$$E_{21} = E_2 (1 - \vec{v}_1 \cdot \vec{n}_2 / c) / \sqrt{1 - \vec{v}_1^{\,2}/c^2}\,, \tag{4.11a}$$

$$E_{12} = m_1 c^2 (1 - \vec{v}_1 \cdot \vec{n}_2 / c) / \sqrt{1 - \vec{v}_1^{\,2}/c^2}\,, \tag{4.11b}$$

$$E_{11} = m_1 c^2, \; E_{22} = 0, \tag{4.11c}$$

the last equality being in accordance with (3.10´).

Note that $E_{21} = E_2$ for $\vec{v}_1 = 0$, whence it follows that (4.11a) expresses the usual Doppler effect /4,5/ in terms of energies. In fact, if we have a moving with 3-velocity $\vec{v}_1 = \vec{v}$ source of zero mass particles (e.g. photons) with 3-velocities $\vec{v}_2 = c\vec{n}$ and energy (with respect to the source) $E_{21} = E_0$ which are detected from an observer at rest to the given frame, then, the observer will find the energy of these particles to be $E = E_2$ which according to (4.11a) is

$$E = E_0 \frac{\sqrt{1 - v^2/c^2}}{1 - \vec{v} \cdot \vec{n}/c} = E \frac{\sqrt{1 - v^2/c^2}}{1 - (v/c)\cos\theta}\,, \tag{4.12}$$

where $v := \sqrt{\vec{v}^{\,2}}$ and $\theta$ is the angle (in the given frame) between the direction of emission and the direction in which the source moves.

The corresponding formulae for $m_1 = 0$ and $m_2 \neq 0$ are obtained from the last considered case by the change $1 \to 2$, $2 \to 1$ of all subscripts.

And at the end, when $m_1 = m_2 = 0$, we have $x_a(s_a) = (ct, ct\vec{n}_a) + y_a$, $s_a = t$, $a = 1,2$ and (of. (4.8)-(4.10):

$$\vec{v}_a = c\vec{n}_a, \; \vec{n}_a^{\,2} = 1, \; V_a = c(1, \vec{n}_a), \; (V_a)^2 = 0, \; \epsilon((V_a)^2) = +1, \; a = 1,2, \tag{4.13}$$

$$p_a = (E/c)(1, \vec{n}_a), \; \mu_a = \mu_a(t) = E_a/c^2, \; a = 1,2. \tag{4.14}$$

and equations (4.6) hold, whence we find:

$$E_{21} = E_2 (1 - \vec{n}_1 \cdot \vec{n}_2), \quad E_{12} = E_1 (1 - \vec{n}_1 \cdot \vec{n}_2), \tag{4.15a}$$

$$E_{11} = E_{22} = 0. \tag{4.15b}$$

So if $\vec{n}_1 = \vec{n}_2$, then $E_{21} = E_{12} = 0$ and vice versa.



One can easily check that in all the considered above cases equation (3.7) is satisfied.

At the end, let us look of the concepts of relative and deviation velocities in the case of special relativity.

Let K be arbitrary fixed inertial frame in which the arbitrary moving particle 2 has a 4-radius vector $x_2(s_2)|_K = (ct, \vec{x}_2(t))$, $s_2 = t\sqrt{1 - \vec{v}_2^{\,2}/c^2}$, $\vec{v}_2 := d\vec{x}_2(t)/dt$, t being the time in K. Let the inertial frame K´be attached to the particle 1 whose world line in K is $x_1(s_1)|_K = (ct, t\vec{v}_1)$, $\vec{v}_1 = \overline{\text{const}}$, $s_1 = t\sqrt{1 - \vec{v}_1^{\,2}/c^2}$. The observer´s world line is fully arbitrary.

We have in the frame K

$$V_a|_K = \frac{dx_a(s_a)|_K}{ds_a} = \frac{(c, \vec{v}_a)}{\sqrt{1 - \vec{v}_a^{\,2}/c^2}}\,, \quad a = 1, 2, \quad \vec{v}_2 = \frac{d\vec{x}_2}{dt}\,, \tag{4.16a}$$

and in the frame K´

$$V_1|_{K'} = (c, \vec{0}), \quad V_2|_{K'} = (c, \vec{v}'_2) / \sqrt{1 - \vec{v}'^{\,2}_2/c^2}\,, \tag{4.16b}$$

where $\vec{v}'_2$ is the 3-velocity of the particle 2 with respect to the frame K´(i.e., to the particle 1) in a sense of special relativity whose explicit form when K and K´have one and the same orientation is /4, 5/

$$\vec{v}'_2 = \{(1 - \vec{v}_1^{\,2}/c^2)^{1/2} \vec{v}_2 + $$
$$+ [(1 - (1 - \vec{v}_1^{\,2}/c^2)^{1/2})(\vec{v}_1 \cdot \vec{v}_2 / \vec{v}_1^{\,2}) - 1] \vec{v}_1 \} (1 - \vec{v}_1 \cdot \vec{v}_2/c^2)^{-1}.$$

So, as we are dealing with a pseudo-Euclidean case, due to (1.5) the relative velocity is $\Delta V_{21} = V_2 - V_1$ and

$$\Delta V_{21}|_K = (1 - \vec{v}_2^{\,2}/c^2)^{-1/2} (c, \vec{v}_2) - (1 - \vec{v}_1^{\,2}/c^2)^{-1/2} (c, \vec{v}_1), \tag{4.17a}$$

$$\Delta V_{21}|_{K'} = (1 - \vec{v}'^{\,2}_2/c^2)^{-1/2} (c, \vec{v}'_2) - (c, \vec{0}). \tag{4.17b}$$

Moreover, in the pseudo-Euclidean case $h_{21} = x_2(s) - x_1(s)$, so that

$$h_{21}|_K = (0, \vec{x}_2(t) - t\vec{v}_1), \tag{4.18a}$$

$$h_{21}|_{K'} = (0, \vec{x}'_2), \tag{4.18b}$$

where $\vec{x}'_2$ is obtained from $\vec{x}_2(t)$ through a Lorentz transformation when we pass from K to K′ /4/.

Due to (1.8) the deviation velocity is

$$V_{21} = dh_{21}/ds = (ds_1/ds)(dh_{21}/ds_1) = (dt/ds)(dh_{21}/dt),$$

where $s_1 = t'$ is the time in K′ /4/, hereof we get

$$V_{21}|_K = \frac{dt}{ds}(0, \vec{v}_2 - \vec{v}_1), \tag{4.19a}$$

$$V_{21}|_{K'} = \frac{ds_1}{ds}(0, \vec{v}'_2). \tag{4.19b}$$

So, if the observer coincides with the first particle, then $s_1 \equiv s$ and $V_{21}|_{K'} = (0, \vec{v}'_2)$ which shows that in fact the deviation velocity is a straightforward generalization of the "relative velocity" in a sense of special relativity.

The author thanks Professor N.A.Chernikov for the useful discussion of the problems put in this work.

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Received by Publishing Department
on August 23, 1989.



Илиев Б.З. E2-89-616

Относительная скорость, импульс и энергия точечных частиц в пространствах с общим линейным переносом

На основе понятия об общих линейных переносах вдоль кривых определяются относительная скорость и импульс двух точечных частиц в любых пространствах /многообразиях/, в которых задан какой-нибудь перенос /например, параллельный, Ферми-Уолкера и т.д./, а если в этих пространствах задана и метрика, то определяется и относительная энергия между этими частицами. Введены некоторые взаимосвязи между этими величинами и обращено внимание на случай безмассовых частиц. В качестве примера подробно рассмотрен случай специальной теории относительности.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

**Сообщение Объединенного института ядерных исследований. Дубна 1989**

Iliev B.Z. E2-89-616

Relative Velocity, Momentum and Energy of Point Particles in Spaces with General Linear Transport

On the basis of the concept of general linear transport along curves, the relative velocity, momentum and energy of two point particles in any spaces (manifolds) are defined in which some transport (e.g., parallel, Fermi-Walker, etc.), and in the last case the metric, is given. Some connections between these quantities are derived and attention is paid to the case of massless particles. As an example the case of special relativity is investigated in detail.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

**Communication of the Joint Institute for Nuclear Research. Dubna 1989**