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TOPOLOGICAL SUSCEPTIBILITY, DISLOCATIONS AND UNIVERSALITY IN SU(2) LATTICE GAUGE THEORY WITH MIXED ACTION

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1. Introduction

The lattice approximation of quantum chromodynamics does not provide only a scheme for calculating non-perturbative quantities like hadron masses, the string tension, the critical temperature of the deconfinement transition etc., but also a convenient tool to explore its vacuum structure starting from first principles. It is of particular interest to study the role of those gauge field excitations which carry a topological charge

$$Q_t \equiv \int d^4x \, q_t(x) \equiv -\frac{1}{16\pi^2} \int d^4x \, \mathrm{tr}\{G_{\mu\nu} {}^*G_{\mu\nu}\} \in \mathbf{Z}, \qquad (1.1)$$

$$ {}^*G_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} G_{\rho\sigma} .$$

As an indicator of the topologically non-trivial vacuum structure usually serves the topological susceptibility

$$\chi_t \equiv \int_V d^4x \, \langle q_t(x) q_t(0) \rangle = \frac{1}{V} \langle Q_t^2 \rangle . \qquad (1.2)$$

Via Ward-Takahashi identities and $1/N_c$ arguments within the quenched approximation it has been related to the mass of the η'-meson [1]

$$\chi_t^{quenched} = \frac{f_\pi^2}{2N_f} (m_{\eta'}^2 + m_\eta^2 - 2m_K^2) \approx (180 \text{ MeV})^4 , \qquad (1.3)$$

N_f denoting the number of light flavours. Eq. (1.3) resolves the so-called $U_A(1)$ problem.
First attempts directly to check relation (1.3) in SU(2) and SU(3) lattice gauge theories have been undertaken with naive lattice discretizations of $q_t(x)$ [2], e.g.

$$a^4\, q_{naive}(n) = -\frac{1}{2^9\pi^2} \sum_{\substack{\mu\nu\rho\sigma= \\ \mp 1,\ldots,\mp 4}} \tilde{\epsilon}_{\mu\nu\rho\sigma}\, \mathrm{tr} U_{n\mu\nu} U_{n\rho\sigma} \qquad (1.4)$$

where $U_{n\mu\nu}$ denotes the Wilson loop around an elementary lattice plaquette of size a at site n in the $\mu\nu$-plane, and $\tilde{\epsilon}_{1234}=-\tilde{\epsilon}_{2134}=-\tilde{\epsilon}_{-1234}=\ldots=1$. $\sum q_{naive}(n)$ cannot be written in terms of a four-divergence like Q_t. Therefore it retains its geometrical nature only in the continuum limit or on sufficiently smoothed lattice fields. The first Monte Carlo (MC) χ_t estimates [2] with expression (1.4) failed by two orders of magnitude. Only very recently it has been shown that the reason for the discrepancy can be traced back to the original neglect of the proper renormalization of the operator (1.4) [3].

In the mean time several authors invented topological charge definitions having a real geometric meaning for lattice gauge fields [4,5,6]. They derive the second Chern number related to Q_t from a reconstruction of the coordinate bundle by interpolating the lattice fields. The most attractive algorithm became the one of Phillips and Stone [6]. It is very fast in the SU(2) case [7,8]. The main problem with the geometric algorithm is that certain continuity conditions for the lattice fields should be satisfied in order to define the charge unambiguously. The conditions quoted in papers [5,6] are mostly violated for MC equilibrium configurations at accessible bare couplings. Therefore, different algorithms or different ways to interpolate the lattice fields can yield different Q_t numbers event by event. Moreover, Pugh and Teper have recently shown [9] the Wilson action to produce dislocations due to which χ_t measured with the Phillips-Stone algorithm is expected to diverge in the continuum limit. This observation has been confirmed in Ref. [10]. But then the beautiful scaling behaviour produced for SU(2) with the Wilson action in a high statistics MC run [8] calls for explanation.

There is another recipe to find a topological charge. Starting from MC equilibrium configurations one can iteratively minimize the action to freeze out ultraviolet fluctuations and determine a "background" topological charge [11,12,13]. The "cooling" procedure is aimed to kill dislocations disturbing the Q_t measurements just in the equilibrium. Cooling has nice properties. Doing it sufficiently carefully it seems to conserve for a

while fluctuations responsible for confinement [14] and might be interpreted as a kind of MC renormalization group transformation [10].

More obviously, cooling can be used to explore the semi-classical situation and to establish possible classical background fields like instantons, monopoles etc. [12,15,16]. However, it deserves further study in as far topologically relevant excitations are lost during cooling. Numerical χ_t estimates using the cooling method are in reasonable agreement with the one based on Eq.(1.4) with the renormalization of q_{naive} taken into account as well as with the fermionic method invented by Smit and Vink [17].

For SU(3) all the methods produce similar results. But in the SU(2) case there is a clear disagreement between the geometric methods and the other ones quoted before.

If this disagreement at the available β-values is due to dislocations, then the disagreement should disappear for lattice actions which suppress them from the early beginning. Such actions are provided e.g. by Migdal-Kadanoff renormalization group transformations in the space of couplings belonging to different group representations of the plaquette term in the action. Bitar et al. [18] have shown that for SU(2) the Migdal-Kadanoff iterations lead to points lying on a universal line in the plane of couplings β_f, β_a corresponding to the fundamental and adjoint representations, respectively. Choosing the couplings along this line with fixed $\beta_a/\beta_f<0$ should correspond to physics nearer to the continuum limit.

In our investigation presented here we used just this action. It has been approbated in recent calculations of glueball masses in medium-sized volumes [19][2]. Here we compute the topological susceptibility with the Phillips-Stone algorithm in MC-equilibrium and during cooling. The results are compared with the corresponding ones of the Wilson action. As far as we are going to present them in units of the lattice spacing of both theories, we have to find relations between them, which at small β's necessarily include non-perturbative effects. Such relations can be found via $1/N_c$ arguments [22,23]. They allow to express the results in terms of a unique effective coupling $\beta_{eff}=4/g^2$, according to which the lattice spacing has to scale.

[2] A first topology study with the mixed fundamental-adjoint action was carried out a couple of years ago by Bhanot and Seiberg [20]. They used the algorithm of Ref.[4] and found the results in a reasonable agreement with universality.

We will show that in the range of couplings $2.2 \lesssim \beta_{eff} \lesssim 2.4$ the χ_t-values produced with the mixed action are very sensitive with respect to the non-perturbative relation used. In particular, it turns out that the method proposed by Makeenko and Polikarpov leads to universality. At the same time we demonstrate the mixed theory clearly to suppress dangerous fluctuations with small plaquette loop values $W_p = \frac{1}{2} \mathrm{tr}\, U_{n\mu\nu} \simeq -1$. During cooling the suppression of dislocations for the mixed action becomes even more pronounced. Nevertheless, the corresponding χ_t values for both the actions are of the same magnitude and definitely smaller than the equilibrium results. We are led to the conclusion that the disagreement between results of the geometric charge method and those of cooling cannot be simply explained by the presence of dislocations in the former one.
The outline of our paper is as follows. In section 2 we will specify our choice of the lattice theory. Section 3 is devoted to a brief discussion of the topological charge algorithm and the cooling method we employed. In section 4 we present plaquette value distributions and demonstrate the suppression of dislocations. The main results of the MC simulations can be found in section 5. Finally we summarize our findings and formulate questions deserving further studies.

2. Specifying the Lattice Action

The mixed fundamental-adjoint lattice action we are going to use is defined as follows

$$S = \beta_f \sum_{n,\mu<\nu} \left(1 - \frac{1}{2} \mathrm{tr}_{fund} U_{n\mu\nu}\right) + \beta_a \sum_{n,\mu<\nu} \left(1 - \frac{1}{3} \mathrm{tr}_{adj} U_{n\mu\nu}\right), \qquad (2.1)$$

$$U_{n\mu\nu} = U_{n\mu} U_{n+\mu,\nu} U^{+}_{n+\nu,\mu} U^{+}_{n\nu}$$

where the contribution of the adjoint representation of SU(2) can be expressed by the fundamental one

$$\mathrm{tr}_{adj} U_{n\mu\nu} = (\mathrm{tr}_{fund} U_{n\mu\nu})^2 - 1 \ .$$

Let the theory be quantized in the standard way by means of the functional integral

$$Z = \int \prod_{n,\mu} dU_{n\mu} \ \exp{-S(U_{n\mu\nu})} \ .$$

$dU_{n\mu}$ denotes the Haar measure for the link variable μ at the lattice site n. Our final aim is to calculate the topological susceptibility for the action (2.1) with such a choice of couplings β_f, β_a which allows to suppress lattice artifacts in the longer distance physics. Migdal-Kadanoff renormalization group transformations have been shown to lead to points on a stable trajectory in the (β_f,β_a)-plane with [18]

$$\beta_a / \beta_f = -0.24 \qquad (2.2)$$

In the following we want to use this mixed improved action and will compare the results with those of the Wilson theory $(\beta_a = 0)$ thought to be defined at the same lattice scale a.
In the classical continuum limit the couplings are related by

$$\frac{4}{g^2} = \beta_f + \frac{8}{3} \beta_a \qquad (2.3)$$

In order to fix a common lattice scale of the quantized theory one usually considers lines in the (β_f,β_a)-plane along which Wilson loops and the string tension, respectively, have the same values. These lines of constant physics are parameterized by an effective coupling

$$\beta_{eff} = \beta_{eff}(\beta_f,\beta_a) = \beta_{eff}(\tilde\beta_f, 0) \equiv \tilde\beta_f = \mathrm{const.}$$

of the corresponding Wilson theory. I.e. the Wilson theory at β_{eff} is viewed as an effective theory for the mixed one at (β_f,β_a). Then β_{eff} is related to the lattice scale by the two-loop renormalization group formula (for SU(2))

$$a\Lambda_L = \left[\frac{6\pi^2 \beta_{eff}(\beta_f,\beta_a)}{11} \right]^{51/121} \exp\left[-\frac{3\pi^2}{11} \beta_{eff}(\beta_f,\beta_a) \right] \qquad (2.4)$$

with a unique parameter Λ_L. Perturbation theory provides [21]

$$\beta_{eff} = \beta_f + \frac{8}{3}\beta_a - \frac{10}{3}\frac{\beta_a}{\beta_f+8\beta_a/3} . \qquad (2.5)$$

However this is obviously a bad approximation for negative β_a such that $\beta_f+8\beta_a/3$ becomes small.
The relation can be improved non-perturbatively within the framework of an $1/N_c$-expansion. In Ref. [22] the relation

$$\beta_{eff} = \beta_f + \frac{8}{3}\beta_a\left[\frac{5}{3}\omega(\beta_{eff})-\frac{2}{3}\right] \qquad (2.6)$$

has been proved with

$$\omega(\beta_{eff}) = \langle \frac{1}{2}\mathrm{tr}U_{n\mu\nu}\rangle_{\beta_{eff}}$$

to be determined in the Wilson theory.
Using this relation the lines of constant string tension determined numerically were quite satisfactorily reproduced. The authors of Ref. [19] have recently used this relation for a computation of the string tension and glueball mass ratios at finite volumes. Their results reasonably confirm universality.
By taking into account also self-correlations of plaquettes one can arrive with the relation [23]

$$\beta_{eff} = \beta_f + \frac{4}{3}\beta_a\left[2\omega(\beta_{eff}) + \frac{\rho'(\beta_{eff})}{\omega'(\beta_{eff})}\right] , \qquad (2.7)$$

where $$\rho(\beta_{eff}) = \langle (\frac{1}{2}\mathrm{tr}U_{n\mu\nu})^2\rangle - \left[\omega(\beta_{eff})\right]^2$$

is understood in the Wilson theory as well. ω' and ρ' denote the derivatives with respect to β_{eff}. By means of (2.7) lines of constant physics have been even better reproduced. We used both formulae (2.6) and (2.7) for our simulations.
As an input for the practical determination of $\beta_{eff}(\beta_f,\beta_a)$ we used ω-

and ρ-data of Ref. [24]. In order to find the derivatives ω' and ρ' we did a polynomial fit which turned out to be in good agreement with some own data for these quantities directly produced on a 4^4 lattice.
All combinations of couplings used throughout this paper are collected in Tab. 1. Monte Carlo simulations were done on a 4^4 lattice with periodic boundary conditions at β_{eff}=2.2 and on a 6^4 lattice at β_{eff}=2.36. These values according to expression (2.4) correspond to the same physical lattice size.
Such a small volume, in which the one-instanton contribution is expected to dominate the vacuum-to-vacuum transition amplitude [25], is well suited for a study of the role of short-range fluctuations. For physical quantities, of course, we have to recover strong finite-size effects.

3. The Topological Charge and the Cooling Algorithm

For determining the topological charge we used both the combinatoric algorithm of Phillips and Stone [6] and the naive charge definition (1.4). Since the first method is defined on a simplicial lattice, we have to slice our hypercubic lattice into a simplicial one. Its diagonal link variables are generated from link variables of the original hypercubic lattice by means of some interpolation. For slicing and interpolating we adopted the procedure of Ref. [7] in a slightly modified way. Our computer code is approximately as fast as that of Ref. [7]. Simultaneously with the computation of the geometric topological charge we have monitored continuity conditions invented in Ref. [6] (cf. definition (2.7) of that Ref.)

$$d\left(1, \prod_{l\in\mathcal{C}} U_l\right) < \frac{\pi}{2} \qquad (3.1)$$

here l runs over closed loops $\mathcal{C}$ around a particular simplex and d denotes the geodesic distance in SU(2). On a lattice of size L^4, the number of such conditions is 256 L^4. By obeying them a certain smoothness of the corresponding lattice configuration is guaranteed, and hence the independence of the topological charge value of a local ordering of the vertices of the simplicial lattice. The reader should keep in mind that these conditions are sufficient but not necessary ones.
To check whether the computed charge depends on the way of interpolation to

Table 1

Lattice sizes and couplings used throughout this paper.

$Q^{(2)}_t$ \ $Q^{(1)}_t$	-3	-2	-1	0	1	2	3	$N_{i.}$
$N_{.j}$	2	20	44	114	51	10	3	
3					2	1		3
2				1	1	2	2	6
1		2	3	24	32	5	1	67
0	1	3	17	68	13	2		104
-1	1	6	16	17	3			43
-2		7	8	4				19
-3		2						2

a)

latt. size	action	β_f	β_a	β_{eff} [Eq.(2.7)]	β_{eff} [Eq.(2.6)]
4^4	Wilson	2.20	0	2.20	2.20
	mixed	2.84	-0.68	2.20	
		2.68	-0.64	2.13	2.20
6^4	Wilson	2.36	0	2.36	2.36
	mixed	3.18	-0.76	2.36	
		3.08	-0.74	2.32	2.36

a simplicial lattice and possibly from a local ordering of the corners we calculated the charges $Q^{(1)}_t$ and $Q^{(2)}_t$ in two different interpolation schemes for the same lattice configuration. We take the equality between them as a necessary condition to be satisfied for a well-defined topology. Tables 2a and 2b show the number of events N_{ij} with $i \equiv Q^{(1)}_t$, $j \equiv Q^{(2)}_t$ for Wilson and mixed improved action, respectively, both measured in MC equilibrium at β_{eff}=2.2 according to relation (2.7). In approximately half of the cases we found $Q^{(1)}_t \neq Q^{(2)}_t$. Nevertheless, the charges are correlated. For a more detailed discussion of the correlation matrix N_{ij} we refer to the next paragraph.

By cooling it is possible to smooth the original equilibrium configuration. Dislocations are expected to be removed [13]. We used a Langevin-type relaxation procedure which has been invented in Ref. [16]. The link variables "move" in accordance with the equation of motion

$$\frac{dU_{n\mu}}{d\tau} = - \frac{\delta S}{\delta U_{n\mu}} \tag{3.2}$$

in the "computer time" τ. The discretization into time steps $\Delta\tau$ leads to the iteration procedure

$$(U_{n\mu})_{m+1} = (U_{n\mu})_m \,(1+\Delta\tau\ \eta_{n\mu})_m \det{}^{-1/2}(1+\Delta\tau\ \eta_{n\mu})_m \tag{3.3}$$

with

$$\eta_{n\mu} = -\frac{1}{8} \sum_{\substack{\nu=\pm 1 \\ |\nu|\neq\mu}}^{\pm 4} (\beta_f + \frac{4}{3}\beta_a \mathrm{tr} U_{n\mu\nu})\ (\tilde{U}_{n\mu\nu} - \tilde{U}^{+}_{n\mu\nu})$$

and

$$\tilde{U}_{n\mu\nu} = U^{+}_{n\mu} U_{n\mu\nu} U_{n\mu} \ .$$

The links can be simultaneously updated after all new ones have been computed. Adopting this prescription the result will not depend on the sequence the links are exposed to changes during a cooling iteration.

A sample of typical cooling histories is shown in Table 3. The action values $\underline{S}$ in units of the classical one-instanton action

Tables 2a,b

Correlation tables showing frequencies N_{ij} of MC equilibrium configurations with topological charges $Q^{(1)} \equiv i$ and $Q^{(2)} \equiv j$.

a) mixed improved action at $\beta_{eff} = 2.2$ [Eq. (2.7)], number of configurations investigated $N_{conf} = 244$.

b) Wilson action at $\beta_{eff} = 2.2$, $N_{conf} = 200$.

The lattice size is 4^4 .

$Q_t^{(2)}$ \ $Q_t^{(1)}$	-3	-2	-1	0	1	2	3	4	5	$N_{.j}$
$N_{i.}$	1	15	41	95	34	12	2			
5						1				1
4										
3					1					1
2		1		3	4	2				10
1			3	18	13	3	1			38
0	1	4	12	57	12	4	1			91
-1		4	22	14	4	2				46
-2		6	4	3						13
-3										

b)

$$S_o = 2\pi^2(\beta_f + 8\beta_a/3) = 8\pi^2/g^2$$

are shown together with $Q_t^{(1)}$, $Q_t^{(2)}$ and the naive topological charge Q_{naive}. Lowering $\Delta\tau$ in the range $\Delta\tau \le 0.05$ we convinced ourselves that the same histories with the same $Q_t^{(i)}$ - values occured only within a changed (computer) time scale.

Already after a few cooling steps with $\Delta\tau = 0.05$ for both mixed and Wilson actions we arrive in a region where $Q_t^{(1)}$ and $Q_t^{(2)}$ typically agree. As one can see also from Table 3 the naive charge at the same time usually tends to a plateau value which exactly corresponds to the number $Q_t^{(1)} = Q_t^{(2)}$. (Of course, the naive charge fails to provide an integer on the so small lattices we consider.) Mostly the right geometric charge is well-established earlier than the naive one. Comparing the mixed with the Wilson case we find that the mixed one lowers the rate the action is minimized step by step.

After having investigated in very detail samples of 25 cooling histories, for each action case we decided to measure the topological charges (additionally to the MC equilibrium case) only once in that region, where $Q_t^{(1)} = Q_t^{(2)}$ is mostly achieved. We take just 15 cooling steps ("half-way cooling"). Finally we compute the charges at those steps, where $d^2S/d\tau^2$ changes its sign. The latter case happens definitely at later stages of the relaxation process and can mostly be related to the occurrence of approximate solutions of the lattice equations of motion $\delta S/\delta U_{n\mu} = 0$. We call it "down-to-plateau cooling".

The half-way cooling stage is characterized by the following observations. Indeed, only O(10%) of configurations are left, where the values $Q_t^{(1)}$ and $Q_t^{(2)}$ fail to agree. Moreover, typically 0.04% of all continuity conditions (3.1) are yet violated for both the actions, compared with 1% in the equilibrium. After 15 iterations with $\Delta\tau = 0.05$ the average total action has been lowered from $\langle S/S_o \rangle = 33.1 \mp 0.3$ (53.5 ∓ 0.3) down to $\langle S/S_o \rangle = 3.5 \mp 0.2$ (6.8 ∓ 0.2) for the Wilson (mixed) action. So, we stay more or less yet in a range, where Campostrini et al. [14] have observed a constant string tension during cooling. From this point of view the "half-way cooling" stage is physically very interesting. The lattice fields seem to contain yet all those fluctuations being responsible for confinement.

Table 3

Typical cooling histories for the mixed action [$\beta_{eff} = 2.20$, acc. to Eq. (2.7)] with relaxation time step $\Delta\tau = .05$.
For the geometric charge $Q_t^{(i)}$ the measurements corresponding to both versions of interpolation are indicated. Only one value is shown if they agree. The lattice size is 4^4 .

I_{cool}	S/S_0	$Q_t^{(i)}$	Q_{naive}	S/S_0	$Q_t^{(i)}$	Q_{naive}	S/S_0	$Q_t^{(i)}$	Q_{naive}
0	52.2	-2/-1	-.07	56.0	0/-1	.04	53.1	+2/+1	.20
5	21.5	-2/-1	-.29	22.3	0/-1	-.01	21.4	0	.23
10	11.3	-2	-.44	11.8	0/-1	-.01	10.8	0	.20
15	7.5	-2	-.53	7.9	0/-1	.02	6.7	0	.14
20	5.7	-2	-.59	5.8	0	.06	4.5	0	.10
25	4.6	-2	-.63	4.7	+1	.12	3.2	0	.06
30	3.9	-2	-.67	3.9	+1	.17	2.3	0	.04
35	3.4	-2	-.70	3.4	+1	.22	1.7	0	.02
40	3.1	-2/-1	-.71	3.1	+1	.26	1.3	0	.02
45	2.9	-1	-.72	2.8	+1	.31	1.0	0	.01
50	2.6	-1	-.71	2.6	+1	.35			
60	2.3	-1	-.67	2.3	+1	.43			
70	2.0	-1	-.62	2.1	+1	.50			
80	1.7	-1	-.59	2.0	+1	.54			
0	55.1	-2	-.17	54.0	0/+1	.09	53.1	-1	.01
5	24.6	-1	-.21	22.0	+1	.06	21.5	-1	-.18
10	13.9	-1	-.22	11.2	+1/ 0	.03	10.4	-1	-.19
15	9.5	-1	-.22	7.3	0	.02	6.5	-1	-.17
20	7.2	-1	-.21	5.3	0	.03	4.5	-1	-.14
25	5.8	-1	-.20	4.3	0	.06	3.5	-1	-.13
30	4.8	-1/ 0	-.19	3.6	0	.09	2.8	-1	-.12
35	4.2	-1/ 0	-.18	3.2	+1	.13	2.3	-1/ 0	-.11
40	3.7	-2/-1	-.17	2.8	+1	.18	2.0	0	-.11
45	3.3	-1	-.16	2.6	+1	.23	1.7	0	-.09
50	2.9	-1/ 0	-.14	2.4	+1	.28	1.5	0	-.07
60	2.3	0	-.10	2.0	+1	.38	1.0	0	-.03
70	1.8	0	-.05	1.7	+1	.43			
80	1.3	0	-.02	1.4	+1	.45			

4. The Role of Dislocations

Dislocations i.e. fluctuations living at the scale size of one lattice spacing and carrying a non-trivial topological charge can be dangerous for the existence of the continuum limit of the topological susceptibility (1.2). This may happen, indeed, if such fluctuations possess an action value

$$S_{disl} < 4\,\frac{3\pi^2}{11}\;\beta_{eff} \approx 10.8\;\beta_{eff}. \qquad (4.1)$$

Their density $\sim \exp(-S_{disl})/a^4$ taken from semi-classical arguments will diverge in the limit $\beta_{eff} \longrightarrow \infty$ and consequently χ_t as well. Lüscher has applied this reasoning in order to explain the strong scaling violations for the topological susceptibility observed in the 2D non-linear O(3) σ-model [26].
Recently for the Wilson theory Pugh and Teper discovered a dislocation in the $Q_t = 1$ sector (with respect to the Phillips-Stone charge) with an action value $S_{disl} = 9.6\,\beta_f$ [9]. Göckeler et al. were able to lower this number by systematically minimizing the action with the Phillips-Stone charge held fixed at $Q_t = 1$ [10]. They found even $S^{min}_{disl} = 6.8\,\beta_f$. For our further argumentation it is important that these dislocations always carry one plaquette with $\mathrm{tr}\, U_{\mu\nu}/2 \approx -1$ at their centre and after a small variation they run immediately into the $Q_t = 0$ sector with very high probability.
One can also ask how the mixed action (2.1) behaves for these configurations. One finds [10]

$$S^{min}_{disl} = 6.8\,\beta_f + 11.7\,\beta_a \quad . \qquad (4.2)$$

Using far in the continuum limit the classical relation $\beta_{eff} = \beta_f + 8\beta_a/3$ we easily convince ourselves that (4.1) holds for $\beta_a > -0.23\beta_f$ ($\beta_f > 0$). Therefore, the improved mixed action corresponding to Eq. (2.2) should allow to determine χ_t with the geometric charge algorithm in the limit $\beta_{eff} \rightarrow \infty$, unless there are yet other dislocations with a smaller action. This does not mean that (4.1) cannot be satisfied at such small β_{eff} we use in our practical calculations. In fact we find

$$S^{min}_{disl} = \begin{cases} 5.1\,\beta_{eff} \\ 5.4\,\beta_{eff} \end{cases} \quad \text{for} \quad \beta_{eff} = \begin{cases} 2.2 \\ 2.36 \end{cases} \text{acc. to Eq. (2.7).} \qquad (4.3)$$

There is a simple way to check how many dislocations of the kind described above can maximally appear in the Monte Carlo equilibrium or after cooling. One can count the number of those plaquettes with $W_p = \frac{1}{2}\,\mathrm{tr}U_{n\mu\nu} \approx -1$. So we have computed the distribution of plaquette values $P(W_p)$ in the MC equilibrium. It is presented in Fig. 1 for the Wilson and mixed improved action at $\beta_{eff} = 2.2$ acc. to Eq. (2.7) . First of all, the comparison shows that the mixed action really stronger suppresses plaquettes with negative W_p-values.
In particular we have

$$R \equiv P(W_p \le -0.9)_{mixed} \,/\, P(W_p \le -0.9)_{Wilson} \approx 1/4 \quad .$$

The suppression becomes stronger at larger β_{eff} . We get $R \approx 1/6$ at $\beta_{eff} = 2.36$. A similar somewhat less pronounced behaviour can be observed if Eq. (2.6) is employed. So the mixed improved action works in the right way. Moreover, the small absolute value of $P(W_p \le -0.9)$, e.g. for the Wilson action at $\beta_{eff} = 2.2$ it is 0.03% , tells us that roughly speaking only every second 4^4 lattice configuration carries one plaquette with $W_p \le -0.9$. Certainly only a small number among such plaquettes represents dislocations with $Q_t \neq 0$. Consequently the possibility to find a dislocation of the kind described in Refs. [9,10] is practically very small in spite of their small action values (4.3) .
Now let us return to the correlation Tables 2a and 2b , respectively. Are the dislocations with $W_p \approx -1$ the main reason for the ambiguity in assigning the right topological charge to the equilibrium lattice fields ? The answer is definitely not. For the Wilson action there are only approximately as much violations of $Q_t^{(1)} = Q_t^{(2)}$ as single plaquettes with $W_p \le -0.9$ are available. It is highly unprobable that all these plaquettes belong to real dislocations carrying a topological charge. For the mixed action the number of the dangerous plaquettes is even lowered by the factor R quoted above, whereas the number of cases with $Q_t^{(1)} \neq Q_t^{(2)}$ is of the same order of magnitude as for the Wilson action. So there must be other short-range fluctuations (see also Ref. [27]) common to both actions due to which the topological charge is not well-defined event by event. How they look like and whether they can spoil the continuum limit deserves further study.

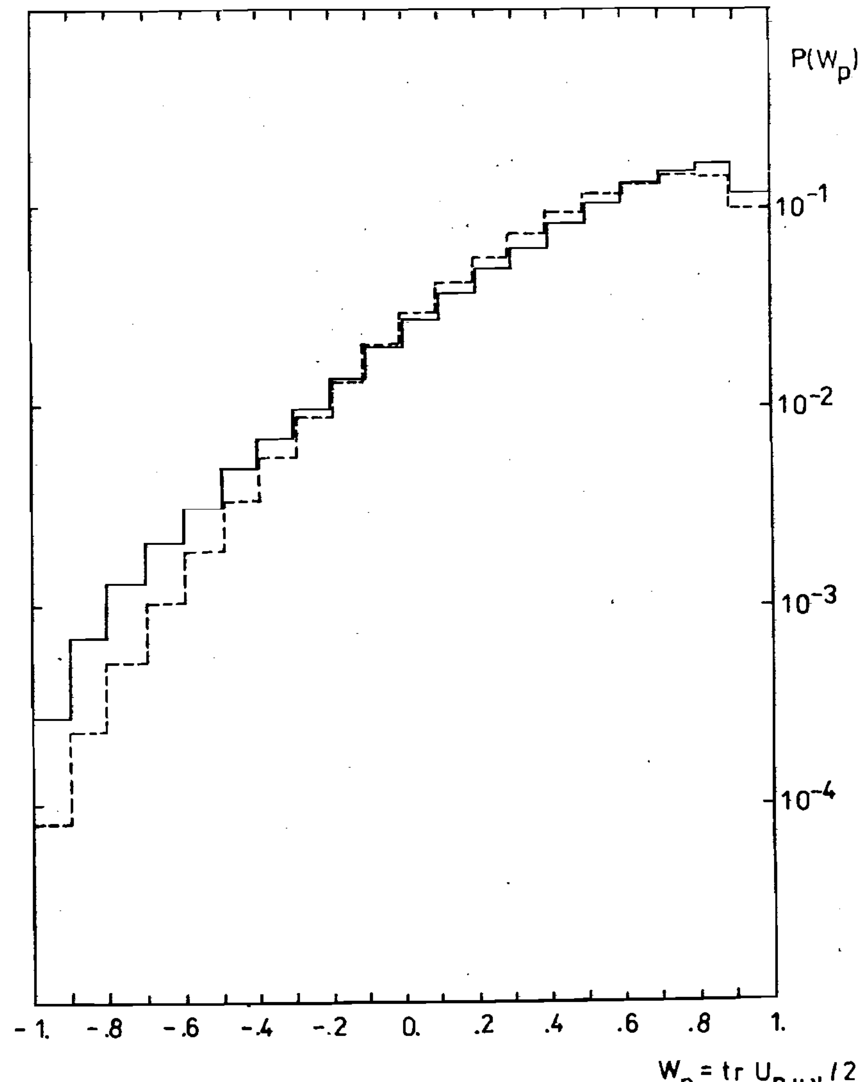

Fig. 1 Plaquette distributions $P(W_p)$ in the MC equilibrium at $\beta_{eff}=2.2$ acc. to Eq. (2.7) for Wilson action (straight lines) and for mixed improved action (dashed lines), lattice size 4^4 .

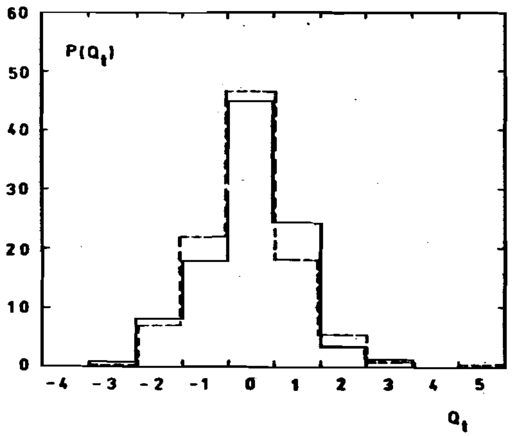

Fig. 2 Distributions of topological charge Q_t in the MC equilibrium at β_{eff} acc. to Eq. (2.7) for Wilson action (dashed lines) and for mixed action (straight lines), lattice size 4^4. For drawing these distributions both charge versions $Q_t^{(1)}$ and $Q_t^{(2)}$ have been taken into account. The statistics is 244 and 200 configurations for the mixed and Wilson action, respectivenly.

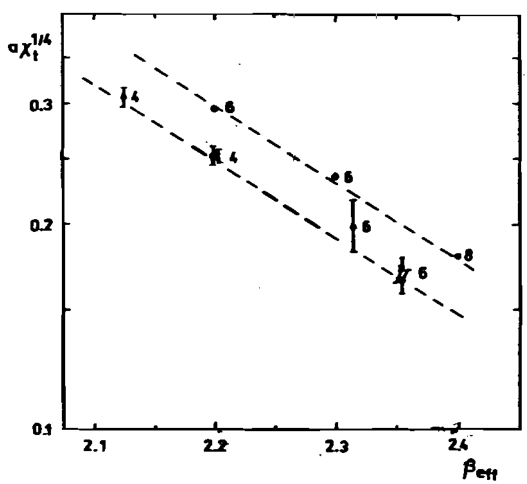

Fig. 3 Topological susceptibility in MC equilibrium as a function of β_{eff} acc. to prescription (2.7). Open circles correspond to Wilson action, crosses and triangles to mixed action. Dots show data points of Ref. [8]. The integers indicate the linear lattice size.

The correlation matrix N_{ij} allows to quantify in as far the topology is statistically ill- or well-defined for MC equilibrium gauge fields. Let us calculate the quantity

$$\rho^2 = \sum_i \sum_j \frac{\left(N_{ij} - \frac{N_{i.} N_{.j}}{N} \right)^2}{N_{i.} N_{.j}} \qquad (4.4)$$

with $N_{i.} = \sum_j N_{ij}$, $N_{.j} = \sum_i N_{ij}$ and $N = \sum_{ij} N_{ij}$. Let r_n be the number of the different values $Q_t^{(n)}$ found during the N measurements. Then in the limit $N \rightarrow \infty$ $N\rho^2$ satisfies a χ^2-distribution with $f = (r_1-1)(r_2-1)$ degrees of freedom if $Q_t^{(1)}$ and $Q_t^{(2)}$ are statistically independent. From Tables 2a and 2b we obtain for $f = 36$ $N\rho^2 = 107$ and 205 , respectively. The confidence level CL for $Q_t^{(1)}$ and $Q_t^{(2)}$ to be really statistically independent turns out for the Wilson action $CL < 0.01\%$. For the mixed action it is even some orders of magnitude smaller. Thus there is a strong correlation in both cases, and it seems to be slightly better in the mixed action case. We are led to the conclusion that the topology is well-defined in a statistical sense at least for small lattice sizes.

During cooling the correlation between $Q_t^{(1)}$ and $Q_t^{(2)}$ becomes still better. This was mentioned already before. At the same time after half-way-cooling the average minimal plaquette becomes $\langle \min_P W_P \rangle = -0.10 \mp 0.09$ in the Wilson case and even $\langle \min_P W_P \rangle = +0.30 \mp 0.03$ in the mixed one, respectively. We have convinced ourselves that in fact all plaquettes with $W_P \leq -0.9$ were removed. Nevertheless, in $O(10\%)$ of all cases $Q_t^{(1)}$ and $Q_t^{(2)}$ disagree. This once more points out to the existence of other fluctuations rather than those of Refs. [9,10] which prevent the unambiguous assignment of the topological charge.

5. Estimates of the Topological Susceptibility

Now let us discuss the results of our numerical simulations. We applied the Metropolis method and separated the topological charge measurements by 50 sweeps. This number is sufficient to remove all correlations between consecutive measurements. Our statistics is based on O(200) configurations for $\beta_{eff} = 2.2$ (4^4 lattice) and O(100) for $\beta_{eff} = 2.36$ (6^4 lattice).

The equilibrium topological charge distributions at $\beta_{eff} = 2.2$ can be easily read off Tables 2a,b and are plotted together in Fig. 2. There is no significant difference between both theories considered. The resulting topological susceptibilities $a^4\chi_t$ for all couplings including both the prescriptions (2.6) and (2.7) are shown in Table 4. The statistical errors (in parentheses) are rather pessimistic ones estimated mostly averaging over consecutive bunches of measurements with length O(60).

Our MC equilibrium data show that the χ_t values are rather sensitive with respect to the non-perturbative relation $\beta_{eff} = \beta_{eff}(\beta_f, \beta_a)$. The method of Makeenko and Polikarpov [23] (Eq. (2.7)) leads to an impressive agreement between mixed and Wilson action data at $\beta_{eff} = 2.2$ as well as at 2.36, i.e. to universality, whereas according to the prescription (2.6) of Grossman and Samuel [22] the mixed action results lie significantly above the Wilson ones. Anyway, there is no suppression of the topological susceptibility for the mixed action although the latter suppresses dislocations as argued in section 4 ! Thus the topological susceptibility estimated with the Phillips-Stone algorithm in the equilibrium at realistic β_{eff} is certainly not disturbed or overestimated due to dislocations described in Refs. [9,10].

In Fig. 3 all our equilibrium data are shown in terms of β_{eff} determined from Eq.(2.7). The crosses correspond to points $(\beta_f, \beta_a) = (2.68, -0.64)$ and $(3.08, -0.74)$ originally aimed to belong to $\beta_{eff} = 2.2$ and 2.36 with respect to Eq. (2.6). Physically these couplings produce a somewhat larger volume on a 4^4 and 6^4 lattice, respectively. Into the same Figure we have plugged some of the high statistics data of Ref. [8] obtained with the Wilson action and corresponding to the large volume limit. We see that our data are definitely smaller due to finite size effects. Nevertheless, as far as we have produced the same physical lattice sizes at $\beta_{eff} = 2.2$ and 2.36 , these data nicely fit to the scaling behaviour (2.4) .

Let us finally comment on our data found by cooling (see also Tab. 4). After "half-way-cooling" with 15 iterations and $\Delta\tau = 0.05$ we arrived at configurations where the topology was well-defined for approximately 90% of them. It is difficult really to compare the χ_t results at this stage. Nevertheless, the numbers quoted in Table 4 indicate that the mixed action

Table 4

Overview of numerical results for the topological susceptibility obtained with the Phillips-Stone algorithm in equilibrium and during cooling. N_{conf} denotes the number of configurations investigated. Statistical errors are given in parentheses. For the corresponding (β_f, β_a) - values we refer to Table 1.

lattice size	action	β_{eff} [Eq.(2.7)]	MC equilibrium $a^4\chi_t$	MC equilibrium N_{conf}	cooling $a^4\chi_t$ half-way	cooling $a^4\chi_t$ plateau	cooling N_{conf}
4^4	Wilson	2.20	0.00410(53)	200	0.00158(30)	0.00078(23)	100
	mixed	2.20	0.00412(24)	244	0.00239(23)	0.00110(12)	227
	"	2.13	0.00886(80)	300	0.00289(44)	0.00160(20)	100
6^4	Wilson	2.36	0.00074(12)	100			
	mixed	2.36	0.00089(18)	100			
	"	2.32	0.00160(57)	100			

allows to enlarge the cooling estimate for χ_t from below. Such a tendency is seen as well at the "action plateaus". In this respect we would like to argue that the use of the mixed improved action acc. to Eq. (2.2) may improve the situation and make the disagreement between the equilibrium-geometric method and the cooling algorithm to become smaller.

6. Conclusions

In this paper we have investigated the topological vacuum structure of the quantized SU(2) lattice gauge theory from a more technical point of view. Our main question was the following. Can the difference between the topological susceptibility calculated with geometric charge algorithms in the MC equilibrium on the one hand and with the cooling prescription on the other hand be explained by the existence of dislocations, which in the Wilson case are expected to produce a divergent result at large β for the geometric methods? To attack this question we employed the mixed action with parameters corresponding to a Migdal-Kadanoff renormalization group improvement and made the comparison with the Wilson action. We considered a small lattice size and reobserved clear finite size effects.

Our main result is that the topological susceptibility at $\beta_{eff} \approx 2.2 \div 2.4$ does not become smaller for the mixed improved action, although the latter clearly suppresses those excitations having small plaquette loop values $\mathrm{tr}\, U_{n\mu\nu}/2 \approx -1$. This holds for equilibrium configurations as well as for cooled ones. So we have a strong indication that the disagreement mentioned before cannot be simply explained by the dislocations described in Refs. [9,10]. How other short-range fluctuations causing ambiguities in defining the Phillips-Stone charge and being common to both actions influence the results especially in the continuum limit deserves further studies.
We have seen that the topological susceptibility at small β_{eff} is sensitive with respect to the non-perturbative lattice scale fixing. It distinguishes the $1/N_c$ method of Makeenko and Polikarpov [23], which allows to establish universality at small lattice sizes. It seems to be worthwhile to reconsider the glueball and string tension data of Ref. [19] in view of this result.
By measuring the Phillips-Stone charge with two different kinds of interpolations of lattice fields we were also able to demonstrate in as far the topology is well-defined. We have seen that in the equilibrium both charges are statistically strongly correlated and that the correlation became slightly better for the mixed action. During cooling the geometric charge was found to become properly determined. This allowed us to define a cooling stage, where one can study the vacuum structure more thoroughly. Nevertheless, even on a small lattice for configurations having not any plaquette with $\mathrm{tr}\, U_{n\mu\nu}/2 < -0.9$ we have observed a few cases, where the topology was ambiguously defined and where a small number of violated continuity conditions (3.1) was left.
What remains to be done? First of all the study presented here should be extended to larger lattice sizes and β_{eff}. A comparison with other actions e.g. non-local ones with six-link loop contributions was propagated in Refs. [10,11]. Of course, universality should be checked yet by expressing the susceptibility in physical units like the string tension or the 0^{++} glueball-mass.

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References

[1] E. Witten, Nucl. Phys. B156 (1979) 269;
G. Veneziano, Nucl. Phys. B159 (1979) 213

[2] P. DiVecchia, K. Fabricius, G. C. Rossi and G. Veneziano, Nucl. Phys. B192 (1981) 392; Phys. Lett. 108B (1982) 323;
N.V. Makhaldiani and M. Müller-Preussker, Pis'ma v ZhETF 37 (1983) 440;
K. Fabricius and G. C. Rossi, Phys. Lett. 127B (1983) 229;
K. G. Ishikawa, G. Schierholz, H. Schneider and M. Teper, Phys. Lett. 128B (1983) 309

[3] M. Campostrini, A. Di Giacomo and H. Panagopoulos, Phys. Lett. 212B (1988) 206;
M. Campostrini, A. Di Giacomo, H. Panagopoulos and E. Vicari, Pisa University preprint IFUP-TH 2/89 (1989)

[4] P. Woit, Phys. Rev. Lett. 51 (1983) 638; Nucl. Phys. B262 (1985) 284;
Y. Arian and P. Woit, Nucl. Phys. B268 (1986) 521

[5] M. Lüscher, Comm. Math. Phys. 85 (1982) 39;
P. van Baal, Comm. Math. Phys 85 (1982) 529

[6] A. Phillips and D. Stone, Comm. Math. Phys. 103 (1986) 599

[7] A. S. Kronfeld, M. L. Laursen, G. Schierholz and U.-J. Wiese, Nucl. Phys. B292 (1987) 330;
A. S. Kronfeld et al., Preprint DESY 88-134 (1988)

[8] M. Kremer et al., Nucl. Phys. B305 (1988) 109

[9] D.J.R. Pugh and M. Teper, Oxford University preprint OUTP-89-10 P

[10] M. Göckeler et al., Preprint DESY 89-065 (1989)

[11] M. Teper, Phys. Lett. 162B (1985) 357; 171B (1986) 81; 86

[12] E.-M. Ilgenfritz, M. L. Laursen, M. Müller-Preussker, G. Schierholz, and H. Schiller, Nucl. Phys. B268 (1986) 693

[13] J. Hoek, Comp. Phys. Comm. 39 (1986) 21; Phys. Lett. 166B (1986) 199;
J. Hoek, M. Teper and J. Waterhouse, Phys. Lett. 180B (1986) 112;
Nucl. Phys. B288 (1987) 589

[14] M. Campostrini et al., Pisa University preprint IFUP-TH 18/89 (1989)

[15] M. I. Polikarpov and A. I. Veselov, Nucl. Phys. B297 (1988) 34;
M. L. Laursen and M. Müller-Preussker, Nucl. Phys. B313 (1989) 1;
M. L. Laursen and G. Schierholz, Z. Phys. C38 (1988) 501

[16] S. Grunewald, E.-M. Ilgenfritz and M. Müller-Preussker, Z. Phys. C33 (1987) 561

[17] J. Smit and J. C. Vink, Nucl. Phys. B284 (1987) 234; B298 (1988) 557;
J. C. Vink, Nucl. Phys. B (Proc. Suppl.) 4 (1988) 519;
Phys. Lett. 212B (1988) 483
[18] K. M. Bitar, S. Gottlieb and C. K. Zachos, Phys. Rev. D26 (1982) 2853
[19] A. Billoire, K. Decker and R. Henzi, Phys. Lett. 211B (1988) 124
[20] G. Bhanot and N. Seiberg, Phys. Rev. D29 (1984) 2420
[21] A. Di Giacomo and G. Paffuti, Nucl. Phys. B205 (1982) 313
[22] B. Grossman and S. Samuel, Phys. Lett. 120B (1983) 383
[23] Yu. M. Makeenko and M. I. Polikarpov, Nucl. Phys. B205 (1982) 386;
Yu. M. Makeenko, M. I. Polikarpov and A. V. Zhelonkin,
Phys. Lett. 126B (1983) 82
[24] B. Berg and J. Stehr, Z. Phys C9 (1981) 333
[25] M. Lüscher, Nucl. Phys. B205 [FS5] (1982) 483
[26] M. Lüscher, Nucl. Phys. B200 [FS4] (1982) 61
[27] D. J. R. Pugh and M. Teper, Phys. Lett. 218B (1989) 326

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Топологическая восприимчивость, дислокации
и универсальность в калибровочной теории
на решетке со смешанным действием

В пределе маленького объема вычисляется топологическая восприимчивость в теории Янга-Миллса со смешанным улучшенным действием с помощью метода Монте-Карло. Результаты сравниваются со стандартным Вильсоновским действием. Применяется геометрический алгоритм топологического заряда Филлипса и Стоуна для равновесных и для тех конфигураций, полученных во время "охлаждения". Результаты оказываются чувствительными относительно метода определения решеточного шага. Универсальность получается при применении метода введенного Макеенко и Поликарповым. При этом в случае смешанного действия дислокации сильно подавливаются. Поэтому, они не являются опасными для определения топологической вакуумной структуры при малых константах β.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Bornyakov V.G. et al. E2-89-606

Topological Susceptibility, Dislocations and Universality
in SU(2) Lattice Gauge Theory with Mixed Action

In a small volume limit we present Monte Carlo measurements of the topological susceptibility for an appropriately chosen mixed fundamental adjoint action and compare the results with the case of the standard Wilson action. We apply the geometric algorithm of Phillips and Stone directly to equilibrium as well as to background configurations at different stages of cooling. We show the results to be sensitive to the way of determining the corresponding lattice spacing. A method proposed in this respect by Makeenko and Polikarpov allows to establish universality for the topological susceptibility in equilibrium. Since plaquette distributions show a strong suppression of plaquettes with $trU_{n\mu\nu}/2 \approx -1$ for the mixed action case, we conclude that dislocations seem not to be dangerous for topological charge measurements with the geometric algorithm at least at small β-values in the scaling region.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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