89-583

ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E2-89-583

I.V.Amirkhanov, O.M.Juraev*, V.N.Pervushin,
I.V.Puzynin, N.A.Sarikov*, T.A.Strizh

PION AS RELATIVISTIC BOUND STATE IN OSCILLATOR POTENTIAL

Submitted to "Physics Letters B"

* Institute for Nuclear Physics of AS.Uz.SSR,
Tashkent, USSR

1989

The pion is the light quark-antiquark bound-state formed by the potential due to the gluon fields. An exact form of the potential is not yet derived from QCD. Nevertheless, the lattice calculations and investigation of the hadron spectroscopy by means of the quark models indicate that the potential can be represented approximately as a sum of the Coulomb and the rising potentials (which dominate among heavy and light quarks, respectively).

The attempt consistently to describe the bound-states of massless quarks in the instantaneous rising potential has been made in refs. [1,2] where by the combined solutions to the Schwinger - Dyson (SD) and Bethe- Salpeter (BS) equations the authors show the spontaneous chiral symmetry breaking (appearance of the dynamical quark mass) and generation of the Goldstone mode (massless pions) in the energy spectrum of mesons. These results, in ref. [2] , have been exemplified by the oscillator potential. However, the authors of these works could not solve the problem of relativization of the bound-state wave-functions.

The relativistic generalization of the potential model has been proposed in ref.[3], in the framework of the bilocal-field formalism [4,5] for the gauge-field theory quantized in the space of physical variables (defined by the explicit solutions to the Gauss equation to the temporal component of the gauge field) [6]. The relativization of the bound-states is achieved

i) by the Heisenberg - Pauli transformations providing in the Lorentz transformations $(P=(M,0,0,0)\to P'=(P_0,\vec{P}_{10}))$ a change of the potential together with arbitrary time-like vector (the time-axis) η^μ ($\eta^2=1$), $A_0=\eta A \rightarrow A'_0=\eta' A$ ($\eta'^2=1$) and ii) by the choice of the time-axis (η^μ) to be "parallel" to the total-momentum of the bound-state, i.e.,

$$\eta^\mu \sim \frac{P^\mu}{\sqrt{P^2}}, \quad \eta'^\mu \sim \frac{P'^\mu}{\sqrt{P'^2}} . \qquad (I)$$

The condition (I) is in fact equivalent to the Markov - Yukawa principle [7] for the bilocal fields [8].

In our approach, the SD- and BS-equations are derived from the effective action for the bilocal fields [3,5], $M(x_1, x_2)$,

$$S_{eff}[M]=N_c\left\{\frac{1}{2}(M,K^{-1}_{(\eta)}M)-i\,\mathrm{Tr}\,\mathrm{Ln}(-G^{-1}_{m^0}+M)\right\}.$$

Here $N_c = 3$ is the colour number, $G^{-1}_{m^0}=i\partial^\mu\gamma_\mu-m^0$ is the Dirac operator for the quark of bare mass m^0, and (A,B), Tr mean integration over continuous variables and trace over indices, and $K_{(\eta)}(z)$ is the BS-kernel which in an arbitrary Lorentz-frame can be written as

$$K_{(\eta)}(z)_{\alpha_1\beta_1\alpha_2\beta_2}=\hat{\eta}_{\alpha_1\beta_1}V(z^\perp)\delta(z\eta)\hat{\eta}_{\alpha_2\beta_2} \quad (z^\perp_\mu=z_\mu-\eta_\mu(z\eta))$$

where α_i and β_i are the sets of flavour and Lorentz indices, $\hat{\eta}=\eta^\mu\gamma_\mu$ (in the rest frame $\eta=(1,0,0,0)$, $\hat{\eta}=\gamma_0$), $V(z)$ is the potential, for example,

$$V(r)=V_0\ r^2 \qquad (r=|z^\perp|) \qquad (2)$$

with the free parameter V_0 .

The SD-equation has the following form [3]

$$\Sigma_f(\vec{p})=m^0_f+\int\frac{d^4q}{(2\pi)^4}\,\tilde{V}(p^\perp-q^\perp)\,\hat{\eta}\,G_{\Sigma_f}(q)\hat{\eta} \qquad (3)$$

where $G_{\Sigma_f}(q)=\left[\hat{q}-\Sigma_f(q)\right]^{-1}$ is the Green function of the quark with the flavour f , $\hat{q}=q^\mu\gamma_\mu$, $\hat{\eta}=\eta^\mu\gamma_\mu$ (η^μ is defined by (I)), $\Sigma_f(q)$ is the mass-operator, $q^\perp$ is the transversal momentum: $q^\perp_\mu=q_\mu-q^\eta_\mu$, $q^\eta_\mu=\eta_\mu(q\eta)$ and $\tilde{V}(k^\perp)$ is the Fourier-transform of the potential, for (2) it is

$$\tilde{V}(k)=\frac{4}{3}V_0\ \Delta_{k^\perp}\delta^3(k^\perp) \quad \left(\Delta_{k^\perp}=\left(\frac{\partial}{\partial k^\perp}\right)^2\right) \qquad (4)$$

In order to solve this equation it is convenient to use the following representations for Σ and G_Σ:

$$\Sigma_f(p^\perp)=\hat{p}^\perp+S_f(\vec{p})E_f(\vec{p}), \qquad (5)$$

$$G_{\Sigma_f}(p)=-\frac{S_f(p^\perp)\Lambda^\eta_+S_f(p^\perp)}{E_f(p^\perp)-p^\eta-i\varepsilon}-\frac{S_f(p^\perp)\Lambda^\eta_-S_f(p^\perp)}{E_f(p^\perp)+p^\eta-i\varepsilon} \qquad (6)$$

where

$$S_f(p^\perp)=\exp\left\{\frac{1}{2}\bar{p}_i\gamma_i\left[\varphi_f(p^\perp)-\frac{\pi}{2}\right]\right\} \qquad (\bar{p}_i=p_i/|p^\perp|) \qquad (7)$$

$\Lambda^\eta_\pm$ are the projectors,

$$\Lambda^\eta_\pm=\frac{1}{2}(I\pm\hat{\eta}). \qquad (8)$$

Then, equation (5), in the rest frame $\eta=(1,0,0,0)$, takes the form

$$(p^2\varphi'_f(p))'=2p^3\sin\varphi_f(p)-2p^2m^0_f\cos\varphi_f(p)-\sin2\varphi_f(p),$$

$$E_f(p)=m^0_f\sin\varphi_f(p)+p\cos\varphi_f(p)-\frac{1}{2}\left[\varphi'_f(p)\right]^2-\frac{1}{p^2}\cos^2\varphi_f(p) \qquad (9)$$

where the prime denotes differentiation with respect to $p=|p^{\perp}|$.

This equation in the case of a massless quark ($m_f^{\circ}=0$) has been solved numerically in ref.[2] where the spontaneous breaking of chiral symmetry (i.e. generating of the dynamical quark mass) is shown.

We have obtained the numerical solutions to equation (9) in the case of a massive quark ($m_f^{\circ}\neq 0$)[9]. To this end, we have developed the computational scheme which allows one to calculate this equation within a wide range of values of the parameter m_f° . This scheme is based on the application of combination of the continuous analogy of the Newton method[10] and the method of continuation in the parameter [11]. Our results are depicted in Fig.1, they reproduce the solutions of ref.[2] at $m_f^{\circ}=0$. We have found that the effect of the chiral symmetry breaking disappears for $m_f^{\circ}\gtrsim\left(\frac{4}{3}V_0\right)^{1/3}$.

The BS-equation in the momentum space, for the vertex-function $\Gamma_H(p|P)$ has the following form [3]:

$$\Gamma_H(p|P)=-i\int\frac{d^4q}{(2\pi)^4}\tilde{V}(p^{\perp}-q^{\perp})\hat{\eta}\,G_{\Sigma_{f_1}}(q+\frac{P}{2})\Gamma_H(q|P)G_{\Sigma_{f_2}}(q-\frac{P}{2})\hat{\eta} \qquad (10)$$

where P is the 4-momentum of the bound state (H) formed of the quarks with flavours f_1 and f_2 . If we suggest that the vertex-function has no dependence on q^{η} then one can carry out the integration in (10) and arrive at the three-dimensional relativistic covariant equation (of the Salpeter type)[3]

$$[E_T(q^{\perp})\mp\sqrt{P^2}]\Lambda_{\pm}^{\eta}S_{f_1}(q^{\perp})\Psi_H(q^{\perp})S_{f_2}(q^{\perp})\Lambda_{\mp}^{\eta}=$$

$$-\int\frac{d^3p^{\perp}}{(2\pi)^3}\tilde{V}(p^{\perp}-q^{\perp})\Lambda_{\pm}^{\eta}S_{f_1}(q^{\perp})\Psi_H(p^{\perp})S_{f_2}(q^{\perp})\Lambda_{\mp}^{\eta} \qquad (11)$$

here $E_T(q)=E_{f_1}(q)+E_{f_2}(q)$ is the sum of the quark energies (solution to eq. (9)), and $\Psi_H(q^{\perp})$ is the integral

$$\Psi_H(q^{\perp})=i\int\frac{dq^{\eta}}{2\pi}G_{\Sigma_{f_1}}(q+\frac{P}{2})\Gamma_H(q^{\perp}|P)G_{\Sigma_{f_2}}(q-\frac{P}{2})=$$

$$S_{f_1}(q^{\perp})\Lambda_+^{\eta}S_{f_1}(q^{\perp})\Gamma_H(q^{\perp}|P)S_{f_2}(q^{\perp})\Lambda_-^{\eta}S_{f_2}(q^{\perp})\times$$

$$[E_T(q^{\perp})-\sqrt{P^2}-i\varepsilon]^{-1}+$$

$$S_{f_1}(q^{\perp})\Lambda_-^{\eta}S_{f_1}(q^{\perp})\Gamma_H(q^{\perp}|P)S_{f_2}(q^{\perp})\Lambda_+^{\eta}S_{f_2}(q^{\perp})\times$$

$$[E_T(q^{\perp})+\sqrt{P^2}+i\varepsilon]^{-1}.$$

For the pseudoscalar meson (pion) it has the form

$$\Psi_H(q^{\perp})=\gamma_5\left[L_1(q^{\perp})+\frac{\hat{P}}{\sqrt{P^2}}L_2(q^{\perp})\right] \qquad (12)$$

where L_i are functions of $q^{\perp}$. Substitution of (12) into (11) leads to the coupled equations for pion which in the rest frame $\eta=(1,0,0,0)$, $P=(M,0,0,0)$, $q^{\perp}=(0,q_i)$ can be written as

$$M_{\pi}L_{2(1)}(q)=E_T(q)L_{1(2)}(q)+$$

$$\int\frac{d^3q}{(2\pi)^3}\tilde{V}(p-q)\left[C_q^{-(+)}C_p^{-(+)}+S_q^{-(+)}S_p^{-(+)}\bar{p}_i\bar{q}_i\right]L_{1(2)}(p) \qquad (13)$$

where

$$c_p^{-(+)} = \cos\left[\vartheta_{f_1}(p) -(+)\, \vartheta_{f_2}(p)\right] \qquad \left(\vartheta_f(p) = \frac{1}{2}\left[\frac{\pi}{2} - \varphi_f(p)\right]\right)$$
$$s_p^{-(+)} = \sin\left[\vartheta_{f_1}(p) -(+)\, \vartheta_{f_2}(p)\right]$$

and $\varphi_f(p)$ is the solution to equation (9), The solutions to equations (13) must be normalized according to the relation [3]

$$\frac{2N_c}{M_\pi}\int\frac{d^3q^\perp}{(2\pi)^3}\left[L_1(q^\perp)\overset{+}{L}_2(q^\perp) + \overset{+}{L}_1(q^\perp)L_2(q^\perp)\right] = 1\,. \tag{14}$$

For the oscillator potential (4) equations (13) take the following form:

$$M_\pi L_{2(1)}(p) + \left\{-\Delta_p + E_T(p) + \left[\vartheta'_{f_1}(p) -(+)\, \vartheta'_{f_2}(p)\right] + \frac{2}{p^2}\left[s_p^{-(+)}\right]^2\right\} L_{1(2)}(p) = 0. \tag{15}$$

These equations in the chiral limit, $m_f^\circ = 0$, have the solutions[3]

$$L_1(p)\big|_{m_f^\circ=0} = \frac{1}{F}\sin\varphi_f(p),$$
$$L_2(p)\big|_{m_f^\circ=0} = 0, \tag{16}$$

and lead to the Goldstone theorem. The normalization parameter is equal to the pion constant F_π which is given by the expression obtained from the $\pi\to\mu\nu$ decay process[3].

(17)

Defining the quark condensate as

$$\langle\bar{q}_f q_f\rangle = i N_c\, \mathrm{tr}\int\frac{d^4q}{(2\pi)^4}\, G_{\Sigma_f}(q) = -2N_c\int\frac{d^3q^\perp}{(2\pi)^3}\sin\varphi_f(q^\perp)$$

and using the solutions in the neighbourhood of the chiral point

$$L_1 \approx \frac{1}{F_\pi}\sin\varphi_f$$
$$L_2 \approx 2m_f^\circ/(F_\pi M_\pi)$$

(where higher order contributions in m_f° are neglected) we get the well-known low-energy relation

$$2\sum_{f=u,d} m_f^\circ\,\langle\bar{q}_f q_f\rangle \approx -M_\pi^2 F_\pi^2\,. \tag{18}$$

To solve the equations (15) numerically we have used the substitution

$$L_{1(2)}(q) = \frac{1}{q}\, U_{1(2)}(q) \tag{19}$$

which leads to the equations

$$U''_{1(2)} + W_{1(2)}U_{1(2)} + M_\pi U_{2(1)} = 0 \tag{20}$$

where

$$W_1 = \vartheta'_{f_1}(q) + \vartheta'_{f_2}(q) + \frac{2}{q^2}(c_q^+)^2 + E_T(q),$$
$$W_2 = \vartheta'_{f_1}(q) - \vartheta'_{f_2}(q) + \frac{2}{q^2}(s_q^-)^2 + E_T(q).$$

The boundary conditions for equations (20) are

$$U_{1(2)}(q\to 0)\to 0,$$
$$U_{1(2)}(q\to\infty)\to\frac{1}{4}\left[\rho^{-1/4}\exp\left(-\frac{2}{3}\rho^{3/2}\right) +(-)\, \sigma^{-1/4}\exp\left(-\frac{2}{3}\sigma^{3/2}\right)\right] \tag{21}$$

where

$$\rho = 2^{1/3}\left(q - \frac{M_\pi}{2}\right), \quad \sigma = 2^{1/3}\left(q + \frac{M_\pi}{2}\right).$$

We have solved these equations by means of our computational scheme applicated to the SD-equation. In the case of massless quarks ($m^{\circ}_{f_1} = m^{\circ}_{f_2} = 0$) our solutions coincide with those of ref.[2] for the Goldstone pion. The set of solutions obtained in the case of equal masses of quarks ($m^{\circ}_{f_1} = m^{\circ}_{f_2} = m^{\circ}_{f} \neq 0$) is shown in Fig.2 which are represented in terms of $(\frac{4}{3} V_0)^{1/3}$. These solutions are given for three values of the parameter m°_f .

We have estimated the parameter m°_f and $(\frac{4}{3} V_0)^{1/3}$ comparing the solutions for M_{π} and F_{π} (calculated by (17)) with their experimental values M_{π} = 140 MeV and F_{π} = 93 MeV, respectively. The results obtained are

$$m^{\circ}_f = m^{\circ}_u = m^{\circ}_d \simeq 19 \text{ MeV}, \qquad (22)$$

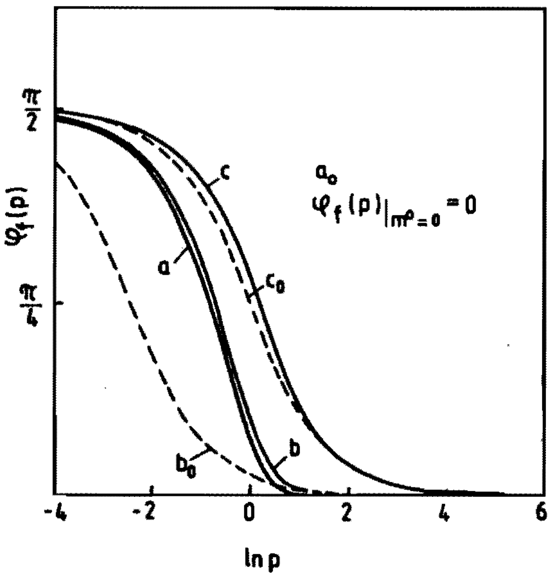
$$\left(\tfrac{4}{3} V_0\right)^{1/3} \simeq 233 \text{ MeV}. \qquad (23)$$

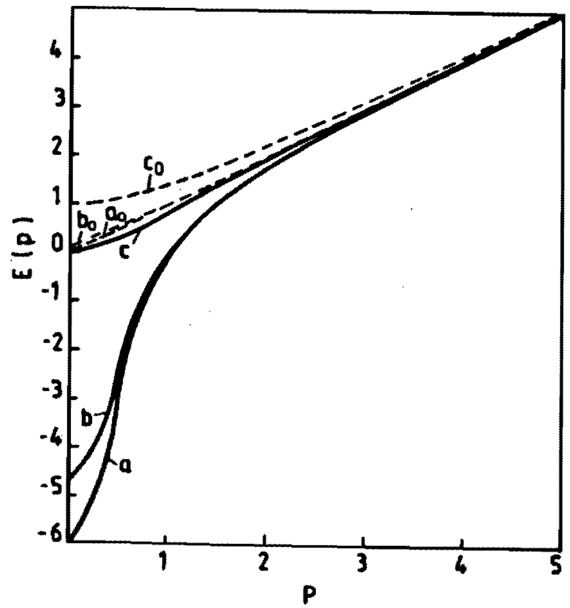
Our result for the bare-quark-mass (22) is two times greater than a rough estimation derived from equation (18) (i.e. ~ 10 MeV), but it is satisfactory with respect to a more careful estimation [12], $m^{\circ}_u + m^{\circ}_d \simeq (27 \pm 8)$ MeV. The estimation for the potential parameter (23) is close to one of ref.[2], 247 MeV, fitted in the massless case.

To summarize, we have obtained solutions to the SD- and BS-equations with the oscillator potential in the case of massive

Fig.1. The numerical solutions ($\varphi_f(p)$, $E_f(p)$) to the SD-equation, in the form (9) obtained for the cases $m^{\circ}_f = 0$, $m^{\circ}_f = 0.085$ and $m^{\circ}_f = 1$ (in units of $(\frac{4}{3} V_0)^{1/3}$ indicated by (a, a_0), (b, b_0) and (c, c_0), respectively. The dashed lines a_0, b_0 and c_0 correspond to the free quark solutions

$$\varphi_f(p) = \arctan \frac{m^{\circ}_f}{p}, \quad E_f(p) = \left[p^2 + m^{\circ 2}_f\right]^{1/2}.$$





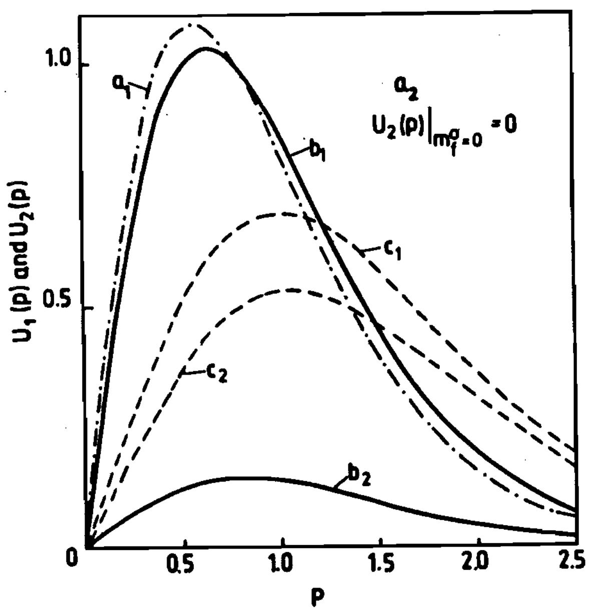


Fig.2. The numerical solutions ($U_1(p)$, $U_2(p)$) to equations (20) for the cases of equal bare-quark-masses ($m_f^o = m_{f_1}^o = m_{f_2}^o$) chosen as $m_f^o = 0$, $m_f^o = 0.085$ and $m_f^o = 1$ (which are indicated by (a_1, a_2), (b_1, b_2) and (c_1, c_2), respectively). The dot -dashed, dashed and solid lines correspond to $m_f^o = 0$, $m_f^o = 0.085$ and $m_f^o = 1$, respectively. In the case of $m_f^o = 0$: $U_2 = 0$, $U_1 \neq 0$.

quarks. The solutions depend on free parameters as the bare-quark masses (m_f^o) and the potential constant (V_0) . In general, the fits of these parameters for pions are in agreement with the available estimations. Our results for the bound-state wave-functions can be tested by describing, for example, the decay process $\pi^o \to 2\gamma$ using the S -matrix for the bound-states proposed in ref.[3]. Of course, for a more exact description of the pions we must take into account the short-distance corrections due to the Coulomb potential as well as consider variants of the rising potentials. These tasks are the subject of our future research.

References

1. S.L.Adler and A.C.Davis, Nucl.Phys. B244 (1984) 469.
2. Le Yaouano, L.Oliver, P.Pene, J.-C.Raunal, Phys.Rev. D29 (1984) 1233; D31 (1985) 137.
3. V.N.Pervushin, W.Kallies, Nguyen Suan Han and N.A.Sakirov, in: Proc. XXI Int.Symp.Ahrenshoop on the Theory of Elem.Particles, PHE-87/13 (Sellin, Oct. 12-16, 1987), p.309;
Yu.L.Kalinovsky et al. Yad.Fiz. 49 (1989) 1709 (in Russian);
V.N.Pervushin, Yu.L.Kalinovsky, W.Kallies and N.A.Sakirov, JINR preprint E2-89-58(1989)(will be published in Forts.Phys. 38 (1990) No 4).
4. J.M.Cornwall, R.Jackiw and E.Tomboulis, Phys.Rev. D10 (1974) 2428;
H.Kleinert, Phys.Lett. B62 (1976) 429;
E.Schrauner, Phys.Rev. D16 (1977) 1877.
5. V.N.Pervushin, H.Reinhardt and D.Ebert, Sov.J.Particles and Nuclear 10 (1979) 444.
D.W.McKay, H.J.Munczek and B.-L.Young, Phys.Rev. D37 (1988)195.
6. V.N.Pervushin, Riv. Nuovo Cim. 8 (1985) 1;
Nguyen Suan Han and V.N.Pervushin, Mod.Phys.Lett. A2 (1987)367;
Nguyen Suan Han, V.N.Pervushin, Forts.Physik 37 (1989) No.8 .
7. M.A.Markov, J.Phys. USSR 3 (1940) 452;
H.Yukawa, Phys.Rev. 77 (1949) 219; 80 (1950) 1047.
8. J. Lukierski and M.Oziewicz, Phys.Lett. B69 (1977) 339.

9. I.V.Amirkhanov et al. JINR Comm. P11-88-506,Dubna 1988 (in Russian).

10. E.P.Zidkov and I.V.Puzynin, Sov. Math. Dokl. 9 (1968) 575.

11. J.M.Ortega and W.C.Rheinboldt, in: Iterative solutions of nonlinear equations in several variables (Acad.Press, N.Y. and L.,1970).

12. S.Narison, E. de Rafael, Phys.Lett. B103 (1981) 57.

Received by Publishing Department
on August 4, 1989.

Амирханов И.В. и др. E2-89-583
Пион как релятивистское связанное состояние
в осцилляторном потенциале

В релятивистской потенциальной модели на примере осцилляторного потенциала получены численные решения уравнения Швингера - Дайсона, в случае массивных голых кварков, и трехмерного уравнения /типа Солпитера/ для массы и волновой функции пиона. Сделана оценка для голых масс кварков и параметра потенциала путем сравнения теоретического и экспериментального значений массы пиона и его лептонной константы /F_π/.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1989

Amirkhanov I.V. et al. E2-89-583
Pion as Relativistic Bound State
in Oscillator Potential

In the relativistic potential model, when the potential is exemplified by the oscillator one, the numerical solutions to both the Schwinger - Dyson equation, in the case of massive bare quarks, and the three-dimensional equation /of the Salpeter type/ for the mass and wave function of pion are obtained. The bare-quark-mass and potential parameter are estimated comparing the calculated and experimental value for the mass of pion and its leptonic decay constant /F_π/.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1989