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G.P.Korchemsky

QUANTUM GEOMETRY OF THE DIRAC FERMIONS

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1. Introduction

It has been realized in the last years [1,2,3,4] that one of the novel properties of $D = 2$ and $D = 3$ dimensional quantum field theories is the transmutation of statistics of elementary particles, i.e., under nontrivial interaction bosons become fermions and vice versa. Recently Polyakov [2] has suggested that in a three-dimensional gauge CP^1 -model with the Chern-Simons form for the kinetic term of the gauge field fermi-boson transmutation occurs. To carry out the proof, Polyakov [2] has supposed that the propagator of the free Dirac electron in $D = 3$ Euclidean space-time may be represented as a bosonic path integral that turns out to coincide with the dressed soliton propagator in the gauge CP^1-model in the limit of low momenta.

Do similar phenomena exist in higher dimensions? There were attempts [5] to answer the question in the case $D = 4$. The present paper is devoted to one aspect of this problem. Namely, only the fermionic part of the problem is analyzed and the final goal of the paper is to obtain the bosonic path integral representation for the correlation functions of Dirac fermions in D-dimensional Euclidean space-time.

It was Feynman [6] who first noted the possibility of the bosonic path integral representation for two-dimensional Dirac fermions. Later the bosonic path integral formalism was used in different problems: for solution of the $D = 2$ Ising model [7]; the spin correlation functions were expressed [8] as a sum over all paths on the sphere S^2; the bosonic path integral representation for the propagator of the three-dimensional Dirac electron was found [2,3,5,9].

In the present paper, the formalism of the bosonic path integrals is developed for interacting Dirac fermions in D-dimensional Euclidean space-time.

2. Dirac fermions in D-dimensional Euclidean space-time

Let us consider, in D-dimensional Euclidean space-time, Dirac fermions with mass M interacting with a nonabelian gauge field $A_\mu = A^a_\mu T^a$ where T^a are some generators of the gauge (or "color") group whose explicit form is not essential for our purposes. We define the effective action and propagator of interacting fermions as follows:

$$W[A] = \log\,\det(\hat{D} + M) = \int d^Dx\,\langle x \mid \mathrm{Tr}\,\log(\hat{D} + M) \mid x\rangle$$

$$S(x, y; A) = \langle x \mid (\hat{D} + M)^{-1} \mid y\rangle$$

where $\hat{D} = \gamma^\mu(\partial_\mu - igA_\mu)$, Tr refers to color indices of the gauge field and spinor indices of Dirac matrices γ^μ in D-dimensional Euclidean space-time, $p_\nu = i\partial_\nu$, $[x_\mu, p_\nu] = -ig_{\mu\nu}$ and $\langle x \mid y\rangle = \delta^D(x - y)$, $\langle p \mid k\rangle = \delta^D(p - k)$, $\langle x \mid p\rangle = (2\pi)^{-D/2}\exp -i(pk)$

Using the identity $A^{-1} = \int_0^\infty dT\,\exp(-TA)$ one gets the expressions:

$$S(x, y; A) = \int_0^\infty dT\,e^{-TM}\,U(x, y; T), \quad W[A] = \int_0^\infty \frac{dT}{T}\,e^{-TM}\int d^Dx\,\mathrm{Tr}\,U(x, x; T) \qquad (1)$$

where the function $U(x, y; T)$ is equal to [10]

$$U(x, y; T) = \langle x| e^{iT(i\hat{D})} |y\rangle$$

Let us treat this function as a matrix element of the evolution operator of a particle with the hamiltonian $H = i\hat{D}$. Then the function $U(x,y;T)$ is an amplitude for a particle to go from point y to point x in the proper time T. Following Feynman [6] one can represent $\langle x \mid e^{iTH} \mid y\rangle$ as a path integral over the phase space of a particle [10,11]:

$$U(x,y;T) = \int_y^x \mathcal{D}x_\mu \, P\exp\left(ig\int_0^T dt\, \dot{x}_\mu(t)A_\mu(x)\right) \mathcal{M}_D[\dot{x}] \tag{2}$$

where integration is performed over all x-paths between the points $x_\mu(0) = y_\mu$ and $x_\mu(T) = x_\mu$ and the functional integral over all unrestricted momentum paths is factorized

$$\mathcal{M}_D[\dot{x}] = \int \mathcal{D}p_\nu \,\exp\left(-i\int_0^T dt\, p(t)\dot{x}(t)\right) P\exp\left(i\int_0^T dt\, \hat{p}(t)\right) \tag{3}$$

$\mathcal{M}_D$ does not depend on a gauge field and the fermion mass M. It accumulates all spinor structure of the evolution operator and is called the spinor functional.

3. Spinor functional

The spinor functional in eq.(3) is ill-defined and it must be properly regularized for large values of momenta. One of the regularization prescriptions proposed in [11] is the insertion of the cut-off factor

$$\exp\left(-\int_0^T dt\, \epsilon(t)\sqrt{p_\mu^2}\right), \ \epsilon \to 0$$

into the right-hand side of (3). To calculate the regularised spinor functional, we split the interval $[0,T]$ into N equal pieces $\tau = \frac{T}{N}$ and define $\mathcal{M}_D[\dot{x}]$ to be given by the following limiting procedure

$$\mathcal{M}_D[\dot{x}] = \lim_{\epsilon(t)\to 0}\lim_{N\to\infty} \mathcal{M}_{reg}(x_N)\cdots\mathcal{M}_{reg}(x_2)\mathcal{M}_{reg}(x_1) \tag{4}$$

where a simple but tiresome calculation yields

$$\begin{aligned}\mathcal{M}_{reg}(x) &= \int d^D k \exp(-i(kx) + i\hat{k}\tau - \epsilon|k|\tau) \\ &= (4\pi)^{\frac{D-1}{2}}\Gamma(\tfrac{D+1}{2})[\epsilon\tau + i(\tau+\hat{x})](x^2 - \tau^2 + 2i\epsilon\tau^2)^{-\frac{D+1}{2}} + \text{h.c.}\end{aligned} \tag{5}$$

and $\tau \to 0$, $x_i = \tau\dot{x}(\tau i) \to 0$ in the limit $N \to \infty$ and $\dot{x}_\mu$ =fixed. Now we perform the limit $\epsilon \to 0$ in the regularized expression (5) and note that as $x^2 \neq \tau^2$: $\mathcal{M}_{reg}(x) \propto \epsilon \to 0$ but for $x^2 = \tau^2$ and $D \neq 4\mathbf{Z}+3$: $\mathcal{M}_{reg}(x) \propto (1+\hat{\dot{x}})\epsilon^{-\frac{D+1}{2}}$. This means that $\mathcal{M}_{reg}(x)$ has a δ-function singularity at $x^2 = \tau^2$ or $\dot{x}^2 = 1$ and after it substitution into eq.(4) one gets the following expression for the spinor functional in the limit $N \to \infty$

$$\mathcal{M}_D[\dot{x}] = \delta(1-\dot{x}_\mu^2)\, I[\dot{x}], \tag{6}$$

where for an arbitrary D-dimensional vector $n_\mu(t)$, $t\in[0,T]$ we denote

$$I[n] = \lim_{N\to\infty}\frac{1+\hat{n}(N\tau)}{2}\cdots\frac{1+\hat{n}(2\tau)}{2}\frac{1+\hat{n}(\tau)}{2} \tag{7}$$

The right-hand side of this equation is an infinite product of matrices of order $2^{[D/2]}$ whose calculation is quite nontrivial. Substituting eqs.(6), (2) into (1) we derive

$$S(x,y;A) = \int_0^\infty dT\, e^{-TM}\int \mathcal{D}\dot{x}_\mu\, \delta\left(x - y - \int_0^T dt\, \dot{x}\right)\delta(1-\dot{x}_\mu^2)\, I[\dot{x}]\, P\exp\left(ig\int_y^x dx_\mu A_\mu(x)\right) \tag{8}$$

and for the effective action analogously. This relation represents the propagator of interacting fermions as a sum over all paths in D-dimensional x-space between points x and y with T being the length of a path.

4. Dimensional extension

To evaluate the infinite product $I[\dot{x}]$ of Dirac matrices in eq.(7), we perform the identical transformation of eq.(8) called the *dimensional extension*

$$\int \mathcal{D}\dot{x}_\mu(\cdots) = \int \mathcal{D}\dot{x}_a\, \delta(\dot{x}_\beta)(\cdots)$$

where $a = 1,\ldots,4d^2-1$, $\beta = D+1,\ldots,4d^2-1$, dots denote the integrand in eq.(8) and $2d = 2^{[D/2]}$ is the dimension of the $su(2d)$ Lie algebra. The expression $\hat{\dot{x}} = \dot{x}_\mu\gamma^\mu$ entering into definition (7) of $I[\dot{x}]$ may be replaced by $\dot{x}_a\Gamma^a$, where traceless, hermitian matrices of order $2d$: $\{\Gamma^a\} = \{\gamma^\mu, i\gamma^{[\mu}\gamma^{\nu]},\ldots, i^n\gamma^{[\mu}\gamma^\nu\cdots\gamma^{\rho]}\}$, $a = 1,\ldots,4d^2-1$ are elements of the $su(2d)$ algebra [12].

It is well-known [12] that in the $su(2d)$ algebra the orthogonal Cartan-Weyl basis consisting of operators $\{H_i,\ E_\alpha\}$ may be chosen, where $i = 1,\ldots,2d-1$, $\alpha = \{e_i - e_j | 1 \le i,j \le 2d\}$ are roots and $\{e_i\}$ is the orthonormal basis in the space $\mathbf{R}^{2d}$. Then an arbitrary element $\dot{x}_a\Gamma^a$ of $su(2d)$ may be decomposed as follows

$$\dot{x}_a\Gamma^a = D(z)(u,H)D^{-1}(z) \tag{9}$$

where $(u,H) \equiv \sum_{i=1}^{2d-1} u_i H_i$ is an element of the Cartan subalgebra $\mathcal{H}$, u is a vector in the space $\mathbf{R}^{2d-1}$ and the unitary matrix $D(z)$ is given by:

$$D(z) = \exp\left(\sum_{\alpha>0}(z_\alpha E_\alpha - \bar{z}_\alpha E_{-\alpha})\right), \quad \bar{z}_\alpha = z^* \tag{10}$$

where the sum runs over all positive roots $\alpha = \{e_i - e_j | 1 \le i < j \le 2d\}$

Eq.(9) relates $4d^2-1$ variables $\dot{x}_a$ to $2d-1$ variables u_i and $d(2d-1)$ complex variables z_α. This fact is expressed as:

$$\dot{x}_a = \dot{x}_a(u, z_\alpha) \tag{11}$$

To obtain this dependence more explicitly, one notes that Γ^a, H_i, E_α are matrices of order $2d$ that act in the space of the a fundamental representation of the $SU(2d)$ group with dimension $2d$ called the minimal fundamental representation [12]. The basis in the representation space consists of the highest weight state $|1\rangle$ and states $|i\rangle$ obtained from $|1\rangle$ under the action of step operators $E_{-\alpha}$ corresponding to the negative roots. All vectors $|i\rangle$ are simultaneously the eigenstates of operators from the Cartan subalgebra: $H|i\rangle = \lambda_i|i\rangle$, $i = 1,2,\ldots,2d$ with vectors $\lambda_i = e_i - \frac{1}{2d}\sum_{j=1}^{2d} e_j$ called weights.

After substitution of the decomposition $H = \sum_{i=1}^{2d} \lambda_i |i\rangle\langle i|$ into eq.(9) we get:

$$\dot{x}_a \Gamma^a = \sum_{i=1}^{2d} (u, \lambda_i) D(z) |i\rangle\langle i| D^{-1}(z) = \sum_{i=1}^{2d} u_i \, |i, z\rangle\langle i, z| \tag{12}$$

where $(u, \lambda_i) = (u, e_i) = u_i$ are coordinates of the vector u that lies in the subspace orthogonal to vector $\sum_{i=1}^{2d} e_i$ and the introduced states

$$|i, z\rangle = D(z)|i\rangle \tag{13}$$

are well-known as the coherent states for the $SU(2d)$ group [13].

Using eq.(12) one finds the relation between variables $\dot{x}_a$ and (u, z_α):

$$\dot{x}_a = \frac{1}{2d} \sum_{i=1}^{2d} u_i \, e_a^{(i)}(z) \tag{14}$$

where the orthogonality condition for Γ^a matrices: $\mathrm{Tr}(\Gamma^a \Gamma^b) = 2d\,\delta^{ab}$ is taken into account and the following notation is introduced:

$$e_a^{(i)}(z) = \langle i, z | \Gamma_a | i, z\rangle \tag{15}$$

where $i = 1, \ldots, 2d$ and $a = 1, \ldots, 4d^2 - 1$. As only u and z_α are known, the vector $\dot{x}_a$ is determined unambiguously from eq.(14). But the reverse statement is wrong. There is the gauge ambiguity in the dependence of u and z_α on vector $\dot{x}_a$ and the corresponding gauge group is the Weyl group [12]. Indeed for an arbitrary root β we may rewrite eq.(9) as

$$\dot{x}_a \Gamma^a = (D(z) S_\beta)(S_\beta^{-1}(u, H) S_\beta)(D(z) S_\beta)^{-1} = D(z_{,\beta})(\sigma_\beta(u), H) D^{-1}(z_{,\beta}) \tag{16}$$

where $S_\beta \equiv \exp\left(\frac{\pi}{2}(E_\beta - E_{-\beta})\right)$ is an element of the Weyl group and it follows from commutation relations of the Cartan-Weyl basis that

$$S_\beta^{-1}(u, H) S_\beta = (\sigma_\beta(u), H), \qquad D(z) S_\beta = D(z_{,\beta}) \exp(i(\phi, H))$$

where variables $z_{,\beta}$ and the $(2d-1)$-dimensional vector ϕ both depend on z_α, u and β. The linear operator $\sigma_\beta(u) = u - 2\beta \dfrac{(u,\beta)}{(\beta,\beta)}$ is called the Weyl reflection. Acting on the vector $u = (u_1, \ldots, u_{2d})$, the operator $\sigma_\beta(u)$, $\beta = e_i - e_j$ permutes coordinates u_i and u_j:

$$\sigma_\alpha(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_{2d}) = (u_1, \ldots, u_j, \ldots, u_i, \ldots, u_{2d}), \quad \alpha = e_i - e_j \tag{17}$$

Thus the Weyl group acts on the components of vector u as the permutation group. Comparing eqs.(9) and (16) one concludes that dependence (14) is invariant under discrete transformations of variables u and z_α

$$\dot{x}_a = \frac{1}{2d} \sum_{i=1}^{2d} u_i \, e_a^{(i)}(z) = \frac{1}{2d} \sum_{i=1}^{2d} (\sigma_\alpha(u), e_i) \, e_a^{(i)}(z_{,\alpha}) \tag{18}$$

This relation is fulfilled for arbitrary values u and α. Therefore assuming $\alpha = e_1 - e_j$ and with eq.(17) one compares coefficients of variables u_i and finds the relations between the functions $e_a^{(j)}(z)$:

$$e_a^{(i)}(z) = e_a^{(1)}(z_{,i}), \quad i \geq 2 \tag{19}$$

where $z_{,i} \equiv z_{,\alpha = e_1 - e_i}$.

Therefore for eq.(18) to have a unique solution, the gauge condition for the Weyl group must be fixed. It follows from eq.(17) that the gauge may be chosen as

$$(u \in C_1) \quad \text{or} \quad (u_1 \geq u_2 \geq \cdots \geq u_{2d} = -u_1 - u_2 - \cdots - u_{2d-1}) \tag{20}$$

and it is in fact the definition of the fundamental Weyl chamber C_1. Indeed, unless the vector u belongs to the boundary of the Weyl chamber, the Weyl reflection $\sigma_\alpha(u)$ sends it from the region (20). To fix the gauge at the boundary of the fundamental Weyl chamber, one has to examine the action of the Weyl group on variables z_α defined in eq.(10) and then choose the gauge by imposing additional constraints on the variables z_α. This program was completed in ref.[14].

Let us express the integration measure over momenta $d^{4d^2-1}\dot{x}$ in terms of the variables u, z_α and $\bar{z}_\alpha$. Since the functions $u = u(\dot{x}_a)$, $z_\alpha = z_\alpha(\dot{x}_a)$ and $\bar{z}_\alpha = \bar{z}_\alpha(\dot{x}_a)$ may be found by solving eq.(9) under gauge condition (20), the general structure of the measure is:

$$d^{4d^2-1}\dot{x} = d\mu(u, z)\, \theta(u \in C_1) \tag{21}$$

where the θ-function takes into account the gauge condition. The explicit form of the measure $d\mu(u, z)$ was derived in ref.[14] but now it is sufficient to establish some properties of $d\mu(u, z)$.

It follows from eq.(18) that vector $\dot{x}_a$ is a gauge invariant quantity and it does not depend on the explicit form of the gauge condition. Hence the measure $d\mu(u, z)$ is unchanged under transformations (18) of the Weyl group:

$$\mathcal{W}: \quad d\mu(u, z) \to d\mu(\sigma_\alpha(u), (z_{,\alpha})) = d\mu(u, z) \tag{22}$$

for an arbitrary root α.

Moreover it follows from eq.(9) that vector $\dot{x}_a$ as well as the integration measures $d\dot{x}_a$ and $d\mu(u, z)$ are invariant under transformations of the Cartan subalgebra

$$\begin{aligned} \mathcal{H}: \quad D(z) &\to D(z) \exp(i(\phi(z), H)) \\ d\mu(u, z) &\to d\mu(u, z) \end{aligned} \tag{23}$$

where $\phi(z)$ is an arbitrary vector in the space $\mathbf{R}^{2d-1}$.

5. One-dimensional Wess-Zumino term

Let us consider one of the terms involved in definition (7) of the function $I[\dot{x}]$

$$\frac{1 + \dot{x}_a \Gamma^a}{2} = \sum_{i=1}^{2d} \frac{1 + u_i}{2} |i, z\rangle\langle i, z| \equiv \sum_{i=1}^{2d} \frac{1 + u_i}{2} P(e^{(1)}(z_{,i})) \tag{24}$$

where eqs.(12) and (19) are used. Then combining relations (11), (14), (21), (7) and (24) one gets the following representation for propagator (8)

$$\begin{aligned} S(x, y; A) \;=\; & \int_0^\infty dT\, e^{-TM} \int \mathcal{D}\mu(u, z)\, \theta(u \in C_1)\, \delta(\dot{x}_a(u, z))\, \delta(2d - u^2) \\ & \times \delta\Big(x - y - \int_0^T dt\, \dot{x}(u, z)\Big) P \exp\left(ig \int_y^x dx_\mu\, A_\mu(x)\right) \\ & \times \prod_\tau \left(\sum_{i=1}^{2d} \frac{1 + u_i(\tau)}{2} P(e^{(1)}(z_{,i}(\tau)))\right) \end{aligned} \tag{25}$$

Let us perform the inverse Weyl transformation:

$$\mathcal{W}^{-1}: \quad (\sigma_i(u), z_{,i}) \to (u, z)$$

in the i-th item of the sum and take into account the gauge invariance (18) and (22) of vector $\dot{x}_a$ and the integration measure to derive, with the use of equality $\sigma_\alpha \sigma_\alpha = 1$, the following relation:

$$\begin{aligned} S(x,y;A) &= \int_0^\infty dT\, e^{-TM} \int \mathcal{D}\mu(u,z) \left(\theta(u \in C_1) + \sum_{j=2}^{2d} \theta(\sigma_j(u) \in C_1) \right) \delta(\dot{x}_a(u,z)) \\ &\times \delta(2d - u^2)\, \delta\left(x - y - \int_0^T dt\, \dot{x}(u,z)\right) P \exp\left(ig \int_y^x dx_\mu A_\mu(x)\right) \\ &\times \prod_\tau \left(\frac{1 + u_1(\tau)}{2} P(e^{(1)}(z(\tau))) \right) \end{aligned} \tag{26}$$

Comparing eqs.(25) and (26) one concludes that it is the gauge invariance of the vector $\dot{x}_a$ that enables us to get rid of the sum of projection operators in the integrand of (25). The final expression (26) for the spinor functional contains only one projection operator onto the state $|1, z\rangle$, defined in eq.(13) and the sum of θ-functions is really the sum over gauge conditions. At $j \geq 2$ vector u lies in the region of the space $\mathbf{R}^{2d-1}$, formed by $(2d-1)$ Weyl chambers C_j obtained from the fundamental Weyl chamber under reflection transformations $\sigma_\alpha, \alpha = e_1 - e_j$. Thus, eq.(26) determines the following region:

$$u \in \Omega = C_1 \cup C_2 \cup \cdots \cup C_{2d} \tag{27}$$

where the Weyl chambers C_j are defined as:

$$C_1 : u_1 \geq u_2 \geq \cdots \geq u_j \geq \cdots \geq u_{2d}, \quad C_j : u_j \geq u_2 \geq \cdots \geq u_1 \geq \cdots \geq u_{2d}, \quad j \geq 2$$

and $u_{2d} = -u_1 - u_2 - \cdots - u_{2d-1}$. Thus, the infinite product matrices occurring in eq.(7) is replaced in eq.(26) by the scalar products:

$$\begin{aligned} &|1, z(T)\rangle\langle 1, z(0)| \lim_{N\to\infty} \prod_{i=1}^{N} \langle 1, z(i\frac{T}{N})|1, z((i-1)\frac{T}{N})\rangle \\ &= |1, z(T)\rangle\langle 1, z(0)| \exp\left(- \int_0^T dt\, \langle 1, z(t)| \frac{d}{dt} |1, z(t)\rangle \right) \\ &\equiv |1, z(T)\rangle\langle 1, z(0)| \exp\left(-\frac{i}{2}\Phi(C) \right) \end{aligned} \tag{28}$$

where eq.(24) is used and C denotes the path $z(t), t \in [0, T]$. We get from eq.(13)

$$\Phi(C) = -2i \int_0^T dt\, \langle 1|D^{-1}(z) \frac{d}{dt} D(z)|1\rangle \tag{29}$$

$$= i \int_0^T dt\, \frac{\dot{\bar{z}} z - \bar{z}\dot{z}}{1 + \bar{z}z} \tag{30}$$

$$= \int \theta(z, \bar{z}) \tag{31}$$

where we denote

$$\langle i|D(z)|1\rangle = N \begin{cases} 1 & , \text{if } i = 1 \\ z_i & , \text{if } i > 1 \end{cases}, \qquad N N^* = (1 + \bar{z}z)^{-1} \tag{32}$$

$z = (z_2, z_3, \ldots, z_{2d})$ is a point of the complex projective space CP^{2d-1} and $d\theta = 2i \frac{\partial F(z,\bar{z})}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j$, $F(z, \bar{z}) = \ln(1 + \bar{z}z)$ is the closed $G = SU(2d)$-invariant 2-form on the Kähler manifold $SU(2d)/U(2d-1) \simeq CP^{2d-1}$ [13,15].

Eq.(31) coincides with the definition of the one-dimensional Wess-Zumino term [1,16] but eq.(29) was rediscovered as the Berry phase [17].

After substitution of eq.(28) into eq.(26) we get for the propagator

$$\begin{aligned} S(x,y;A) &= \int_0^\infty dT\, e^{-TM} \int \mathcal{D}\mu(u,z)\, \theta(u \in \Omega)\, \delta(2d - u^2)\, \delta\left(x - y - \int_0^T dt\, \dot{x}(u,z)\right) \\ &\times \delta(\dot{x}_a(u,z))\, |1, z(T)\rangle\langle 1, z(0)| \exp\left(-\frac{i}{2}\Phi(C)\right) J[u_1]\, P \exp\left(ig \int_y^x dx_\mu A_\mu(x)\right) \end{aligned} \tag{33}$$

where the function $J[u_1]$ is equal to

$$J[u_1] = \prod_\tau \frac{1 + u_1(\tau)}{2} \tag{34}$$

6. Dimensional reduction

Thus, the first part of the problem is completed. The expression (33) for the propagator was obtained where all the spinor structure that appears in the original form (8) and (7) as an infinite product of Dirac matrices is accumulated by the one-dimensional Wess-Zumino term (29)–(31). To this end the space-time dimension was changed from D to $2d$ in the integrand of (8). Now one has to perform the inverse transformation, called the *dimensional reduction*, viz. to explicitly solve $(2d - D)$ restrictions on the variables u_i and $z_j^{(i)}$:

$$\dot{x}_\alpha = \frac{1}{2d} \sum_{j=1}^{2d} u_j e_\alpha^{(j)}(z), \qquad \alpha = D + 1, \ldots, 4d^2 - 1 \tag{35}$$

imposed by δ-functions in eq.(33) and reconstruct the integrand and integration measure in eq.(33) on the space of solutions.

Let us find at first all the restrictions on u_i following from eq.(35). Note that the identity

$$\dot{x}_\mu^2 = \dot{x}_a^2 = (\dot{x}_\mu \gamma^\mu)^2 = (\dot{x}_a \Gamma^a)^2 = \frac{1}{2d} \mathrm{Tr}(\dot{x}_a \Gamma^a)^2 = 1$$

holds where $\mu = 1, 2, \ldots, D$. After substitution into it of decomposition (12) one gets

$$\dot{x}_\mu^2 = u_i^2 = u_j^2 = 1, \quad \forall i, j$$

So, it follows from eq.(35) that

$$u_i = \pm u_j = \pm 1, \quad i \neq j \tag{36}$$

There is an ambiguity in the sign assignment restricted by the only condition: $\sum_{j=1}^{2d} u_j = 0$. The ambiguity disappears, however, when one recalls the definition (27) of the region Ω of vector u.

Combining eqs.(27) and (36) we conclude that components of the vector u being solutions of eq.(35) assume one of the following values:

$$\mathcal{C}_i: \quad u_i = u_2 = \cdots = u_1 = \cdots = u_d = -u_{d+1} = \cdots = -u_j = \cdots = -u_{2d} = 1, \quad i \le d \quad (37)$$

$$\mathcal{C}_j: \quad u_j = u_2 = \cdots = u_i = \cdots = u_d = -u_{d+1} = \cdots = -u_1 = \cdots = -u_{2d} = 1, \quad j > d \quad (38)$$

All these solutions lie on the boundary of the Weyl chambers, that is, on the hyperplanes where, as pointed out before, the residual gauge invariance of $\dot{x}_a$ under transformations of the Weyl group appears and the additional gauge fixing performed in ref.[14] is required.

It follows from eq.(34) that function $J[u_1]$ possesses one of the following values on the space of solutions (37)–(38)

$$J[u_1] = \begin{cases} 1 & , \text{if } u \in \mathcal{C}_i,\ i \le d \\ 0 & , \text{if } u \in \mathcal{C}_j,\ j > d \end{cases}$$

and therefore only the first solution, (37), contributes to eq.(33). Substituting it into eq.(35) and using the property: $\sum_{j=1}^{2d} e_\alpha^{(j)}(z) = 0$ one gets all restrictions on the variables z_i, $i = 2, \ldots, 2d$ in the form:

$$\sum_{j=1}^{d} e_\alpha^{(j)}(z) = 0, \qquad \alpha = D+1, \ldots, 4d^2 - 1 \tag{39}$$

The properties of solutions of this equation are discussed in detail in ref.[14] but now we restrict ourselves to formulation of the main result. It follows from eq.(39) that

$$e_\mu^{(i)}(z) = e_\mu^{(j)}(z), \qquad i,j = 1,2,\ldots,d, \ \mu = 1,\ldots,D \tag{40}$$

Moreover, eq.(39) implies that variables z_i defined in eq.(32) are restricted by $n(D) = 2d - (D-1) \equiv 2^{[D/2]} - D + 1$ additional conditions $\varphi_\alpha(z,\bar{z}) = 0$, $\alpha = 1,2,\ldots,n(D)$. Under the decrease of the space-time dimension from odd to even values the total number $n(D)$ of constraints is increased by unity and the additional constraint is

$$e_{D+1}^{(1)}(z) = 0, \qquad D = even$$

Using eqs.(14), (37) and (40) we get

$$\dot{x}_\mu = \frac{u_1}{2d} \sum_{j=1}^{d} (e_\mu^{(j)}(z) - e_\mu^{(j+d)}(z)) = u_1\, e_\mu^{(1)}(z), \qquad u_1 = 1 \tag{41}$$

and the explicit form (32) of the coherent state $|1,z\rangle$ enables one to express vector $\dot{x}_\mu$ in terms of the variables z_j. It turns out that the number of independent variables z_j (with $n(D)$ constraints taken into account) is much larger than the number of components of vector $\dot{x}_\mu$. The origin of this problem was stressed above. It is the residual gauge invariance of the vector $\dot{x}_\mu$ under the Weyl group gauge transformations of variables z_j. Thus, to eliminate the residual gauge ambiguity, $2(d-1)$ constraints additional to eq.(20) are imposed on the variables $z_j, \bar{z}_j$. Then after simple calculations [14] we derive the final expression for the integration measure on the space of solutions of eq.(35)

$$d^D\dot{x} = d\mu(u,z)\, \delta(\dot{x}_a(u,z)) = \text{const.}\ du_1\, u_1^{d-1}\, d\mu_0(z) \prod_{\alpha=1}^{n(D)} \delta(\varphi_\alpha(z,\bar{z})) \tag{42}$$

where

$$d\mu_0(z) = (2d-1)!\, \frac{d\bar{z}dz}{(2\pi i)^{2d-1}} \frac{1}{(1+\bar{z}z)^{2d}}$$

is G-invariant measure on the manifold $SU(2d)/U(2d-1)$ [13,15].

After substitution of eqs.(41) and (42) into (33) the integral over u_1 is easily computed and we get the following bosonic path representation of the propagator

$$\begin{aligned} S(x,y;A) &= \int_0^\infty dT\, e^{-TM} \int \mathcal{D}\mu_0(z) \prod_{\alpha=1}^{n(D)} \delta(\varphi_\alpha(z,\bar{z}))\, \delta\left(x - y - \int_0^T dt\, e^{(1)}(z)\right) \\ &\times |1, z(T)\rangle\langle 1, z(0)|\, \exp\left(-\frac{i}{2}\Phi(C)\right) P\exp\left(ig\int_y^x dx_\mu\, A_\mu(x)\right) \end{aligned} \tag{43}$$

where the integration contour of the P-exponential is defined as

$$x_\mu(t) = y_\mu + \int_0^t dt\, e_\mu^{(1)}(z), \qquad x_\mu(T) = x_\mu$$

For the effective action we have the analogous relation

$$\begin{aligned} W[A] &= \int_0^\infty \frac{dT}{T} e^{-TM} \int d^Dy \int \mathcal{D}\mu_0(z) \prod_{\alpha=1}^{n(D)} \delta(\varphi_\alpha(z,\bar{z}))\, \delta\left(\int_0^T dt\, e^{(1)}(z)\right) \\ &\times \langle 1, z(0)|1, z(T)\rangle\, \exp\left(-\frac{i}{2}\Phi(C)\right) \mathrm{Tr}\, P\exp\left(ig\oint dx_\mu\, A_\mu(x)\right) \end{aligned} \tag{44}$$

Recall that for even values of the space-time dimension the final representations for the propagator and effective action differ from eqs.(43) and (44) by the factor $\delta(e_{D+1}^{(1)}(z))$.

7. Conclusion

Eqs.(43) and (44) express the propagator and effective action of D-dimensional interacting fermions as sums over all paths on the complex projective space CP^{2d-1}.

Note that for $D = 2,3$ due to the isomorphism $CP^1 \simeq S^2$ the summation in eq.(44) may run over all the paths on the sphere S^2 with function $e_a^{(1)}(z)$ being the tangent field $\dot{x}_a(t)$ for a closed path $C = \{x(t), t \in [0,T] | x(0) = x(T)\}$. Then $\Phi(C)$ is equal to the torsion of the curve C and eq.(44) coincides with the analogous relation proposed in ref.[2]:

$$W[A] = \sum_C e^{-ML(C)} \exp\left(-\frac{i}{2}\Phi(C)\right) \mathrm{Tr}\, P\exp\left(ig\oint_C dx_\mu\, A_\mu(x)\right)$$

For $D = 2$ the additional condition $e_3^{(1)} = 0$ means that curve C lies in the plane and therefore $\Phi(C) = 2\pi N = 2\pi(\nu+1) \pmod 2$ or

$$\exp\left(-\frac{i}{2}\Phi(C)\right) = (-1)^{\nu+1}$$

where N is the number of total rotations of the tangent vector $e_a^{(1)}$ and ν is the number of self-intersections of the path C.

Comparing eqs.(43), (44) and (1) we conclude that all the spinor structure of the original expressions for the propagator and effective action is absorbed by the one-dimensional Wess-Zumino term and it is by no means accidental.

First, there exists a classical mechanics [15] on the space $CP^{2d-1} \simeq SU(2d)/U(2d-1)$ with the action being equal to the spin factor $\Phi(C)$. The Poisson bracket for this mechanics is defined by the closed 2-form $d\theta(z,\bar{z})$ and in terms of the local coordinates z_j it is [18]

$$\{\ ,\ \}_{PB} = i g_{i\bar{j}}(z,\bar{z}) \left(\frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_j} - \frac{\partial}{\partial \bar{z}_j}\frac{\partial}{\partial z_i} \right)$$

where the metric $g_{i\bar{j}}(z,\bar{z})$ is inverse to the Kähler metric $\frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_j}F(z,\bar{z})$. As a result, under the geometrical quantization [9] the commutation relations for the variables $e_a^{(1)}(z)$ reproduce the commutation relations of the $su(2d)$ Lie algebra and functions $e_\mu^{(1)}(z)$ may be thought of as Dirac matrices γ_μ.

Secondly, the expression (44) for the effective action contains the following term

$$\langle 1, z(0)|1, z(T)\rangle \exp\left(-\frac{i}{2}\Phi(C)\right) = |\langle 1, z(0)|1, z(T)\rangle| \exp\left(-\frac{i}{2}\Phi(\bar{C})\right)$$

where $\bar{C}$ is a closed curve on the space CP^{2d-1}. Let us examine gauge invariant properties of the effective action (44) under the action of the Cartan subgroup $\mathcal{H}$. It follows from eqs.(23) and (15) that under the action of $\mathcal{H}$ the integration measure and function $e_a^{(1)}$ are both invariant but the Wess-Zumino term changes as

$$\begin{array}{lccl} \mathcal{H}: & D(z) & \to & D(z)\exp(i(\phi, H)) \\ & |1,z\rangle & \to & |1,z\rangle \exp(i(\phi,\lambda_1)) \\ & \Phi(\bar{C}) & \to & \Phi(\bar{C}) + 2(\phi(1),\lambda_1) - 2(\phi(0),\lambda_1) = \Phi(\bar{C}) + 4\pi k, k \in \mathbf{Z} \end{array}$$

since for closed paths $\bar{C} = \{z(t), t \in [0,1] | z(0) = z(1)\}$ the relation $|1, z(1)\rangle = |1, z(0)\rangle$ implies $\exp(i(\phi(1),\lambda_1)) = \exp(i(\phi(0),\lambda_1))$. Therefore the phase exponential of the action $\exp(-iJ\Phi(\bar{C}))$ is nonmanifestly gauge invariant provided that the quantized condition $2J \in \mathbf{Z}$ is fulfilled. Indeed eq.(44) implies that the spin J of the Dirac fermion is one half, $J = \frac{1}{2}$. Thus, we conclude that the consistency condition of the underlying quantized dynamics leads to the quantized values of the spin of particles. Moreover it was demonstrated [19] that elementary particles with the phase exponential of the action $\exp(-iJ\Phi(\bar{C}))$ possess the Bose statistics for integer J and the Fermi statistics for half-integer J.

Thus, it is the Wess-Zumino term that ensures all necessary properties of Dirac fermions in the bosonic path integral representations (43) and (44) for the effective action and propagator of interacting fermions.

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Квантовая геометрия дираковских фермионов

Для описания дираковских фермионов, взаимодействующих с неабелевым калибровочным полем в D-мерном евклидовом пространстве-времени в работе развивается формализм бозонных интегралов по путям. Получены представления для эффективного действия и корреляционных функций фермионов в виде суммы по путям в комплексном проективном пространстве CP^{2d-1} ($d = 2^{[D/2]-1}$), в которых вся спинорная структура поглощается одномерным членом Весса-Зумино. Именно весс-зуминовский член обеспечивает все необходимые свойства фермионов при квантовании: квантованные значения спина, уравнение Дирака, Ферми-статистику и т.д.

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Quantum Geometry of the Dirac Fermions

The bosonic path integral formalism is developed for Dirac fermions interacting with a nonabelian gauge field in the D-dimensional Euclidean space-time. The representation for the effective action and correlation functions of interacting fermions as sums over all bosonic paths on the complex projective space CP^{2d-1}, ($2d = 2^{[D/2]}$) is derived where all the spinor structure is avsorbed by the one-dimensional Wess-Zumino term. It is the Wess-Zumino term that ensures all necessary properties of Dirac fermions under quantization, i.e., quantized values of the spin, Dirac equation, Fermi statistics.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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