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дубна

E2-89-500

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QUANTUM GEOMETRY OF THE DIRAC FERMIONS
Dimensional Reduction of the Spinor Functional

Submitted to "Nuclear Physics B"

1989

1. Introduction

In the previous papers [1,2] the formalism of the bosonic path integrals was developed for interacting Dirac fermions in D-dimensional Euclidean space-time. Representations (I.1.14) and (I.1.15)[1] for the effective action and propagator of interacting D-dimensional Dirac fermions as sums over all paths in the x-space were obtained

$$S(x,y;A) = \int_0^\infty dT e^{-TM} \int_y^x \mathcal{D}x_\mu P\exp\left(ig\int_y^x dx_\mu A_\mu(x)\right)\mathcal{M}_D[\dot{x}] \tag{1.1}$$

and

$$W[A] = \int_0^\infty \frac{dT}{T} e^{-TM} \int \mathcal{D}x_\mu\, \delta(x(0)-x(T))\,\mathrm{Tr}\, P\exp\left(ig\oint dx_\mu A_\mu(x)\right)\mathrm{Tr}\,\mathcal{M}_D[\dot{x}] \tag{1.2}$$

and for the spinor functional $\mathcal{M}_D[\dot{x}]$ for arbitrary D relation was found

$$\mathcal{M}_D[\dot{x}] = \int \mathcal{D}\dot{x}_\alpha\, \mathcal{M}_{2d}[\dot{x}], \quad \alpha = D+1,\cdots,4d^2-1 \tag{1.3}$$

The expression for the spinor functional $\mathcal{M}_{2d}[\dot{x}]$ was obtained

$$\mathcal{M}_{2d}[\dot{x}] = \int_\Omega \mathcal{D}\mu(y)\,\mathcal{D}\mu(z)\,|1,z(T)\rangle\langle 1,z(0)|\exp\left(-\frac{i}{2d}\int_0^T dt\sum_{i=1}^{2d} y_i e_a^{(i)}(z)\dot{x}_a + i\int_0^T dt\, y_1 - \frac{i}{2}\Phi(C)\right) \tag{1.4}$$

where all the spinor structure that appears in the original form (I.2.8) and (I.2.9) as an infinite product of Dirac matrices is accumulated by the one-dimensional Wess-Zumino term (II.2.62). To this end the space-time dimension was changed from D to $2d$ and the dimensionally extended spinor functional $\mathcal{M}_{2d}[\dot{x}]$ was calculated.

2. Dimensional reduction of the spinor functional

To evaluate the propagator and effective action of fermions, one has to perform the inverse transformation, called *dimensional reduction*, on the spinor functional $\mathcal{M}_{2d}[\dot{x}]$ with the use of relation (1.3). We substitute (1.4) into eq.(1.3) and perform integration over variables $\dot{x}_\alpha(t)$, $(\alpha = D+1,\ldots,4d^2-1)$

$$\begin{aligned}\mathcal{M}_D[\dot{x}] &= \int_\Omega \mathcal{D}\mu(y)\mathcal{D}\mu(z)\,\delta\left(\frac{1}{2d}\sum_{i=1}^{2d} y_i e_\alpha^{(i)}(z)\right)|1,z(T)\rangle\langle 1,z(0)| \\ &\quad\times\exp\left(-\frac{i}{2d}\int_0^T dt\sum_{j=1}^{2d} y_j e_a^{(j)}(z)\dot{x}_a(t) + i\int_0^T dt\, y_1 - \frac{i}{2}\Phi(C)\right)\end{aligned} \tag{2.1}$$

to obtain δ-functions that impose restrictions on the components of vector k_α or on the variables y_i and $z_j^{(i)}$:

$$k_\alpha = \frac{1}{2d}\sum_{j=1}^{2d} y_j e_\alpha^{(j)}(z) = 0, \quad \alpha = D+1,\ldots,4d^2-1 \tag{2.2}$$

[1] Henceforth eqs.(I.X.Y) and (II.X.Y) should be understood as equation (X.Y) of refs.[1] and [2], respectively.

where functions $e_a^{(j)}(z)$ are related to $z_j^{(i)}$ by equality (II.2.30):

$$e_a^{(j)}(z) = \langle j, z|\Gamma_a|j, z\rangle = \bar{u}^{(j)}(z)\,\Gamma_a\, u^{(j)}(z) \tag{2.3}$$

To simplify eq.(2.1) one has to solve equations (2.2) and reconstruct the integrand and integration measure on the space of solutions.

We note that hereafter the space-time dimension is implied to have only odd values

$$D = 2\nu + 1, \quad \nu \in \mathbf{Z}$$

since the spinor functional for even values of D is easily calculable with the use of eq.(II.2.2).

2.1. Solution of the reduction equation

Let us find at first all the restrictions on y_i following from eq.(2.2). Note that the identity:

$$(k_a\Gamma^a)^2 = (k_\mu\gamma^\mu)^2 = k_\mu^2 \cdot \mathbf{1} = k_a^2 \cdot \mathbf{1} = \frac{1}{2d}\mathrm{Tr}\,(k_a\Gamma^a)^2$$

holds where $\mu = 1, 2, \ldots, D$. After substitution into it of decomposition (II.2.26) in the form $k_a\Gamma^a = \sum_{j=1}^{2d} y_j P_j(z)$ one gets:

$$k_\mu^2 = \sum_{j=1}^{2d} y_j^2 P_j(z) = \frac{1}{2d}\sum_{j=1}^{2d} y_j^2 \cdot \mathbf{1}$$

or

$$k_\mu^2 = y_i^2 = y_j^2, \quad \forall i, j$$

So, it follows from eq.(2.2) that

$$y_i = \pm\, y_j, \quad i \neq j \tag{2.4}$$

There is an ambiguity in the sign assignment restricted by the only condition:

$$\sum_{j=1}^{2d} y_j = 0$$

The ambiguity disappears, however, when one recalls the definition (II.2.42) of the region Ω of vector $y = (y_1, \ldots, y_{2d})$:

$$y \in \Omega, \quad \Omega = C_1 \cup C_2 \cup \cdots \cup C_{2d} \tag{2.5}$$

and Weyl chambers C_i are defined in eq.(II.2.43).

Combining eqs.(2.4) and (2.5) we conclude that components of the vector y being solutions of eq.(2.2) assume one of the following values:

$$\begin{array}{ll} C_1: & y_1 = y_2 = \cdots = y_i = \cdots = y_d = -y_{d+1} = \cdots = -y_j = \cdots = -y_{2d} \\ C_i: & y_i = y_2 = \cdots = y_1 = \cdots = y_d = -y_{d+1} = \cdots = -y_j = \cdots = -y_{2d}, \quad i \leq d \\ C_j: & y_j = y_2 = \cdots = y_i = \cdots = y_d = -y_{d+1} = \cdots = -y_1 = \cdots = -y_{2d}, \quad j > d \end{array} \tag{2.6}$$

and the first two expressions are identical. So, in the region Ω eq.(2.2) has $(d+1)$ solutions for vector y. All these solutions lie at the boundary of the Weyl chambers, that is, on the hyperplanes where, as shown in sect.2.2.2 of ref.[2], the residual gauge invariance of $\mathcal{M}_D[\dot{x}]$ under transformations of the Weyl group appears and the gauge fixing additional to eq.(II.2.20) is required.

Let us substitute solutions (2.6) into the original eq.(2.2). For $y \in C_i$, $(i \leq d)$ we have

$$\sum_{j=1}^{d} e_\alpha^{(j)}(z) - \sum_{j=d+1}^{2d} e_\alpha^{(j)}(z) = 0, \quad \alpha = D+1, \ldots, 4d^2-1$$

and using the property (II.2.31a): $\sum_{j=1}^{2d} e_\alpha^{(j)}(z) = 0$ one gets

$$e_\alpha^{(1)}(z) + \sum_{j=2}^{d} e_\alpha^{(j)}(z) = 0, \quad y \in C_i, \quad i \leq d \tag{2.7}$$

For $y \in C_j$, $j \geq d+1$ eq.(2.7) is replaced by the analogous relation:

$$e_\alpha^{(1)}(z) + \sum_{\substack{k=d+1 \\ k \neq j}}^{2d} e_\alpha^{(k)}(z) = 0, \quad y \in C_j, \quad j \geq d+1 \tag{2.8}$$

Each of the functions $e_\alpha^{(i)}(z)$ depends for a fixed index i only on variables $z_j^{(i)}$, $j = 1, 2, \ldots, 2d$. At the same time eqs.(2.7) and (2.8) differ in upper indices of functions $e_\alpha^{(i)}(z)$, $(i \geq 2)$. Therefore the solutions of these equations may be obtained from one another by a mere redefinition of the variables $z_j^{(i)}$, $i \geq 2$. For this reason only eq.(2.7) is solved in the next section.

2.2. The choise of ansatz

Using eq.(2.3) we rewrite eq.(2.7) in terms of harmonic coordinates as:

$$\sum_{j=1}^{d} \bar{u}^{(j)}(z)\, \Gamma_\alpha\, u^{(j)}(z) = 0, \quad \alpha = D+1, \ldots, 4d^2-1 \tag{2.9}$$

where Γ_α denote all possible matrices of order $2d$ except for the unit matrix and Dirac matrices γ^μ, $\mu = 1, 2, \ldots, D$. The general form of Γ_α is

$$\Gamma_\alpha = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \tag{2.10a}$$

where C means a traceless matrix of order d, A and B are a matrices of order d restricted by the conditions:

$$A^\dagger = B, \quad A \neq 1, \quad A \neq -i\hat{\gamma}^\mu \tag{2.10b}$$

$\mu = 1, 2, \ldots, D-2$ and $\hat{\gamma}^\mu$ are Dirac matrices in the $(D-2)$-dimensional Euclidean space.

Let us find the solutions of eq.(2.9) in the form:

$$u^{(i)}(z) = \begin{pmatrix} \psi^{(i)} \\ \chi^{(i)} \end{pmatrix} \tag{2.11a}$$

where

$$\psi_\alpha^{(i)} = N_i z_\alpha^{(i)}, \quad \chi_\alpha^{(i)} = N_i z_{\alpha+d}^{(i)}, \quad \alpha = 1, 2, \ldots, d \tag{2.11b}$$

$z_i^{(i)} = 1$ and $N_i \bar{N}_i = \left(1 + \bar{z}^{(i)} z^{(i)}\right)^{-1}$, $\bar{z}^{(i)} z^{(i)} \equiv \sum_{j \neq i} \bar{z}_j^{(i)} z_j^{(i)}$. Substituting this ansatz into eq.(II.2.51) one finds the restrictions on functions $\psi^{(i)}$ and $\chi^{(i)}$ following from the properties of harmonics:

$$\sum_{i=1}^{2d} \quad \bar{\psi}_\alpha^{(i)} \psi_\beta^{(i)} = \delta_{\alpha\beta} \tag{2.12a}$$
$$\sum_{i=1}^{2d} \quad \bar{\chi}_\alpha^{(i)} \chi_\beta^{(i)} = \delta_{\alpha\beta} \tag{2.12b}$$
$$\sum_{i=1}^{2d} \quad \bar{\psi}_\alpha^{(i)} \chi_\beta^{(i)} = 0 \tag{2.12c}$$
$$\sum_{\alpha=1}^{d} \quad \left(\bar{\psi}_\alpha^{(i)} \psi_\alpha^{(j)} + \bar{\chi}_\alpha^{(i)} \chi_\alpha^{(j)}\right) = \delta^{ij} \tag{2.12d}$$

where $\alpha, \beta, i, j = 1, 2, \ldots, d$.

Eqs.(2.9) and (2.10) imply that $\psi^{(i)}$ and $\chi^{(i)}$ satisfy the system of equations:

$$\sum_{i=1}^{d} \quad \bar{\psi}^{(i)} C \psi^{(i)} = 0 \tag{2.13a}$$
$$\sum_{i=1}^{d} \quad \bar{\chi}^{(i)} C \chi^{(i)} = 0 \tag{2.13b}$$
$$\sum_{i=1}^{d} \quad \left(\bar{\chi}^{(i)} A \psi^{(i)} + \bar{\psi}^{(i)} B \chi^{(i)}\right) = 0 \tag{2.13c}$$

Let us examine the properties of eqs.(2.12) and (2.13).

2.3. The properties of the system of equations

We decompose the traceless matrix C over the basis of matrices of order d defined in (A.4) and obtain from eqs.(2.13a,b):

$$\sum_{i=1}^{d} \bar{\psi}_\alpha^{(i)} \psi_\beta^{(i)} = C_1 \, \delta_{\alpha\beta}, \qquad \sum_{i=1}^{d} \bar{\chi}_\alpha^{(i)} \chi_\beta^{(i)} = C_2 \, \delta_{\alpha\beta}$$

where C_1 and C_2 are some positive definite functions of variables $z_j^{(i)}$. It follows from eq.(2.12d) that

$$C_1 + C_2 = 1, \quad 0 < C_1 \leq 1, \quad 0 \leq C_2 < 1 \tag{2.14}$$

Now we take eq.(2.13c) and in it set subsequently $A = B = \hat{\Gamma}_a$ and then $A = -B = i\hat{\Gamma}_a$, where $\hat{\Gamma}_a$ are all possible matrices of order d except for the unit matrix and Dirac matrices $\hat{\gamma}_\mu$. The two resulting equations imply that $\sum_{i=1}^{d} \bar{\chi}^{(i)} \hat{\Gamma}^a \psi^{(i)} = 0$ or

$$\sum_{i=1}^{d} \psi_\alpha^{(i)} \bar{\chi}_\beta^{(i)} = C_I \, \delta_{\alpha\beta} - i C_\mu \, (\hat{\gamma}^\mu)_{\alpha\beta} \tag{2.15}$$

where $\alpha, \beta = 1, 2, \ldots, d$ and C_I, C_μ are some unknown functions. C_I and C_μ may possess only real values. To prove it, one chooses two allowed values of matrices in eq.(2.13c):

$$A = -B = i\mathbf{1} : \sum_{i=1}^{d} \quad \left(\bar{\chi}^{(i)} \psi^{(i)} - \bar{\psi}^{(i)} \chi^{(i)}\right) = 0$$
$$A = B = \hat{\gamma}^\mu : \sum_{i=1}^{d} \quad \left(\bar{\chi}^{(i)} \hat{\gamma}^\mu \psi^{(i)} + \bar{\psi}^{(i)} \hat{\gamma}^\mu \chi^{(i)}\right) = 0$$

and after substitution of eq.(2.15)

$$C_I^* = C_I, \quad C_\mu^* = C_\mu$$

Thus the original system (2.13) may be rewritten as

$$\sum_{i=1}^{d} \quad \bar{\psi}_\alpha^{(i)} \psi_\beta^{(i)'} = C_1 \, \delta_{\alpha\beta} \tag{2.16a}$$

$$\sum_{i=1}^{d} \quad \bar{\chi}_\alpha^{(i)} \chi_\beta^{(i)} = C_2 \, \delta_{\alpha\beta} \tag{2.16b}$$

$$\sum_{i=1}^{d} \quad \psi_\alpha^{(i)} \bar{\chi}_\beta^{(i)} = C_I \, \delta_{\alpha\beta} - iC_\mu \, (\hat{\gamma}^\mu)_{\alpha\beta} \tag{2.16c}$$

where all functions C_i are real and obey eq.(2.14). The left-hand side of eqs.(2.16) contains the sum of different harmonics (the sum over the harmonic index). Let us transform eq.(2.16) to the form where the sum of harmonics is replaced by the sum over components of a fixed harmonic (with a fixed index of the harmonic).

The function $\psi_\alpha^{(i)}$ may represented as

$$\psi_\alpha^{(i)} = \sqrt{C_1} \, \langle \alpha | G | i \rangle$$

and it follows from eq.(2.16a) that G is a unitary operator $G\, G^\dagger = \mathbf{1}$. Hence, $G^\dagger G = \mathbf{1}$ or

$$\sum_{\alpha=1}^{d} \bar{\psi}_\alpha^{(i)} \psi_\alpha^{(j)} = C_1 \, \delta^{ij} \tag{2.17a}$$

In an analogous manner one has from eq.(2.16b)

$$\sum_{\alpha=1}^{d} \bar{\chi}_\alpha^{(i)} \chi_\alpha^{(j)} = C_2 \, \delta^{ij} \tag{2.17b}$$

Now we multiply the two last relations by $\bar{\chi}_\beta^{(j)}$ and $\psi_\beta^{(j)}$, respectively, and sum over repeated indices with the use of eq.(2.16):

$$C_2 \, \psi_\alpha^{(i)} = C_I \, \chi_\alpha^{(i)} - iC_\mu \, (\hat{\gamma}^\mu)_{\alpha\beta} \, \chi_\beta^{(i)} \tag{2.18a}$$

and

$$C_1 \, \chi_\alpha^{(i)} = C_I \, \psi_\alpha^{(i)} + iC_\mu \, (\hat{\gamma}^\mu)_{\alpha\beta} \, \psi_\beta^{(i)} \tag{2.18b}$$

The quantities C_1, C_2, C_I, C_μ are new additional variables in the system (2.16). The functions C_1 and C_2 may be determined from eqs.(2.17) but the functions C_I and C_μ are found from eq.(2.18a):

$$\begin{aligned} 2C_I \delta^{ij} &= \bar{\chi}^{(i)} \psi^{(j)} + \bar{\psi}^{(i)} \chi^{(j)} \\ 2C_\mu \delta^{ij} &= i(\bar{\chi}^{(i)} \hat{\gamma}^\mu \psi^{(j)} - \bar{\psi}^{(i)} \hat{\gamma}^\mu \chi^{(j)}) \end{aligned} \tag{2.19}$$

Recalling definition (2.11) of $\psi^{(i)}$ and $\chi^{(i)}$ we conclude that C_1, C_2, C_I, C_μ as functions of the coordinates of harmonics do not depend on the index of harmonics:

$$C_i = C_i(z^{(1)}) = C_i(z^{(2)}) = \cdots = C_i(z^{(2d)}), \quad i = 1, 2, I \text{ and } \mu \tag{2.20}$$

It turns out that there are some relations between the functions C_i. The simplest one follows from eq.(2.18a):

$$C_2{}^2 \left(\bar{\psi}_\alpha^{(i)} \psi_\alpha^{(i)} \right) = (C_I{}^2 + C_\mu{}^2) \left(\bar{\chi}_\alpha^{(i)} \chi_\alpha^{(i)} \right)$$

or with the use of eqs.(2.17)

$$C_1 C_2 = C_I{}^2 + C_\mu{}^2$$

Together with eq.(2.20) the last relation is one of the constraints on the coordinates $z_j^{(i)}$ of harmonics. The total number of the constraints is found from eq.(2.18a). At a fixed index i of harmonics and $\alpha = 1, 2, \ldots, d$ we have $2d$ real equations (2.18a) for $(D-1)$ variables C_I, C_μ (C_1, C_2 are given by eqs.(2.17)). After their elimination

$$n(D) = 2d - (D-1) \equiv 2^{[D/2]} - D + 1 \tag{2.21}$$

equations remain that are constraints on variables $z_j^{(i)}$. For different values of the space-time dimension the number of constraints is:

$$n(D) = 0 \text{ for } D \leq 5, \quad n(D=7) = 2, \quad n(D=9) = 8 \ldots \tag{2.22}$$

Their explicit form may be obtained from eq.(2.18a). For instance, at $D = 7$ we have two constraints: one of them

$$z_1^{(i)} z_6^{(i)} - z_2^{(i)} z_5^{(i)} + z_3^{(i)} z_8^{(i)} - z_4^{(i)} z_7^{(i)} = 0, \quad (z_i^{(i)} = 1) \tag{2.23}$$

and its complex conjugate.

2.4. Properties of the solutions

Let us assume in this section that all the solutions of eq.(2.7) are known and one tries to calculate the integrand of eq.(2.1) on the space of solutions.

All the dependence of the integrand in eq.(2.1) on the variables y_i and $z_j^{(i)}$ is contained in the components of the vector k_μ:

$$k_\mu = \frac{1}{2d} \sum_{j=1}^{2d} y_j e_\mu^{(j)}(z) = \frac{y_1}{2d} \left(\sum_{j=1}^{d} e_\mu^{(j)}(z) - \sum_{j=d+1}^{2d} e_\mu^{(j)}(z) \right) = \frac{y_1}{d} \sum_{j=1}^{d} e_\mu^{(j)}(z)$$

where eqs.(2.6) and (II.2.31a) are applied. With eqs.(2.3) and (2.11) and the explicit form (A.6) of the Dirac matrices γ_μ we get for the function $e_\mu^{(i)}$:

$$e_\mu^{(i)}(z) = (e_\alpha^{(i)}, e_{D-1}^{(i)}, e_D^{(i)}) = \left(i(\bar{\chi}^{(i)} \hat{\gamma}^\alpha \psi^{(i)} - \bar{\psi}^{(i)} \hat{\gamma}^\alpha \chi^{(i)}), (\bar{\chi}^{(i)} \psi^{(i)} + \bar{\psi}^{(i)} \chi^{(i)}), (\bar{\psi}^{(i)} \psi^{(i)} - \bar{\chi}^{(i)} \chi^{(i)}) \right)$$

where $\alpha = 1, 2, \ldots, D-2$. The components of $e_\mu^{(i)}$ are easily identified with the right-hand side of eqs.(2.17) and (2.19) and

$$e_\mu^{(i)}(z) = (2C_\alpha, 2C_I, C_1 - C_2)$$

The dependence of $e_\mu^{(i)}$ on the index i of harmonics disappears in this relation and therefore:

$$e_\mu^{(i)}(z) = e_\mu^{(j)}(z), \quad i, j = 1, 2, \ldots, d \tag{2.24}$$

and for the vector k_μ we get

$$k_\mu = y_1 e_\mu^{(1)}(z) = y_1 (2C_\alpha, 2C_I, C_1 - C_2) \tag{2.25}$$

Thus on the space of solutions the components of k_μ are proportional to the variables C_i that appear in eq.(2.16) and all one needs now is to calculate C_i in terms of the variables $z_j^{(i)}$. The expressions for C_i are given by eqs.(2.17) and (2.19). According to eq.(2.20) C_i do not depend on the index of harmonic and they may be chosen to depend only on the components $z_j^{(1)}$ of the harmonic $u^{(1)}$.

As a result, one concludes from eqs.(2.25) and (2.20) that on the space of solutions of eq.(2.7) the vector k_μ and the integrand in eq.(2.1) are both functions only of the variables $z_j^{(1)}$:

$$k_\mu = k_\mu(y_1, z_j^{(1)}), \quad y_1 \geq 0 \tag{2.26a}$$

where the gauge condition (II.2.43) and eq.(2.6) are used.

Eq.(2.26a) means that for the calculation of the integrand in (2.1) it is not necessary to solve eq.(2.2). It is sufficient to use some properties of the system (2.13). Nevertheless, a comment is in order. We have noticed in sect.2.3 that eqs.(2.18) implies that variables $z_j^{(1)}$ are restricted by $n(D)$ additional conditions. Therefore determining vector k_μ with the use of eq.(2.26a) one has to keep in mind $n(D)$ constraints

$$\varphi_\alpha(z_j^{(1)}, \bar{z}_j^{(1)}) = 0, \quad \alpha = 1, 2, \ldots, n(D) \tag{2.26b}$$

following from eqs.(2.18).

All the above considerations were related to eq.(2.7) that was one of the solutions of eq.(2.2). Let us turn to eq.(2.8). It was pointed out at the end of sect.2.1 that the solutions of eqs.(2.7) and (2.8) are related to one another by the mere redefinition of the coordinates $z_j^{(i)}$, $i \geq 2$ of harmonics. However the solution (2.26) does not depend on the variables $z_j^{(i)}$, $i \geq 2$ and the above transformation does not change its form. So on the space of solutions of eq.(2.8) we have finally the relations:

$$\begin{gathered} k_\mu = k_\mu(y_1, z_j^{(1)}), \quad y_1 \leq 0 \\ \varphi_\alpha(z_j^{(1)}, \bar{z}_j^{(1)}) = 0, \quad \alpha = 1, 2, \ldots, n(D) \end{gathered} \tag{2.27}$$

differing from (2.26) only by the replacement of the gauge condition on y_1.

2.5. Calculation of the integration measure

The integration measure in eq.(2.1) may be written as

$$d^D k = d^{4d^2-1} k \, \delta(k_\alpha) = d\mu(y) \, d\mu(z) \, \delta\left(\frac{1}{2d} \sum_{i=1}^{2d} y_i e_\alpha^{(i)}(z)\right), \quad \alpha = D+1, \ldots, 4d^2 - 1 \tag{2.28}$$

where the measures $d\mu(y)$ and $d\mu(z)$ have been obtained earlier in eqs.(II.2.70) and (II.2.77). If one tries to eliminate some integration variables with the use of the δ-functions, then with a formal substitution of solutions (2.6) of eq.(2.2) into eq.(2.28) the equality $d\mu(y) = 0$ will be obtained. We will demonstrate in this section that this strange result is a consequence of the gauge invariance of the spinor functional under residual transformations (II.2.21) of the Weyl group.

Consider metric (II.2.66) on the space of solutions (2.6), (2.26) and (2.27):

$$ds^2 = dk_\mu dk_\mu = dy_1^2 + y_1^2 \, de_\mu^{(1)}(z) \, de_\mu^{(1)}(z) \tag{2.29}$$

where eq.(2.25) is used. Since all $e_\mu^{(1)}(z)$ depend only on variables $z_j^{(1)}$, the above relation implies for the integration measure

$$d^D k = dy_1 \, y_1^{D-1} \, d\mu_D(z^{(1)}) \tag{2.30}$$

where the power of y_1 is fixed by dimensional counting. To find the integration measure $d\mu_D(z^{(1)})$, one has to resolve the following problem: the number of independent variables $z_j^{(1)}$ (with $n(D)$ constraints taken into account) is much larger than the number of components of vector k_μ. A problem like that did not appear when one calculated the measure $d\mu(y,z)$ in eq.(II.2.68). The origin of this problem was stressed above. It is the residual gauge invariance of the metric under gauge transformations of variables $z_j^{(i)}$.

2.5.1. Residual gauge invariance

It is proved in sect.2.4 of ref.[2] that the spinor functional $\mathcal{M}_{2d}[\dot{x}]$ is invariant under gauge transformations of the Weyl group and gauge condition (II.2.20) has been fixed demanding for vector y to lie in the fundamental Weyl chamber $y \in \mathcal{C}_1$. Solution (2.6) implies that vector y lies at the boundary of the Weyl chambers where, as pointed out in sect.2.2.2 of ref.[2] , the residual gauge invariance exists. Indeed, some components of the vector y in eq.(2.6) are equal to each other and, as a result, vector y is invariant under the following transformations of the Weyl group:

$$\mathcal{W}: \quad y \to \sigma_\alpha(y) = y, \quad y \in \mathcal{C}_i, \quad \alpha = e_1 - e_i, \quad d \geq i \geq 2 \tag{2.31a}$$

The action of the Weyl group on the harmonics is obtained from eqs.(II.2.60) and (II.2.50):

$$\mathcal{W}: \quad u_j^{(1)}(z) \to u_j^{(1)}(z_{,\alpha}) = u_j^{(i)}(z), \quad \alpha = e_1 - e_i \tag{2.31b}$$

that is the Weyl group acting on $u_j^{(1)}$ replaces an upper index of a harmonic.

We conclude from eqs.(2.31) that the sets of variables $(y_1, z_j^{(1)})$ and $(y_1, z_j^{(i)})$ are gauge equivalent, that is, the solutions (2.26) and (2.27) and metric (2.29) are invariant under discrete transformations (2.31). The validity of this important property may be easily checked with the use of eq.(2.25). Namely, vector k_μ being a function (2.25) of variables C_i (that do not depend on the index of a harmonic according to eq.(2.20)) is invariant under transformations (2.31).

Thus, to eliminate the residual gauge ambiguity, we have to impose constraints additional to eq.(2.21) on variables $z_j^{(1)}$, $\bar{z}_j^{(1)}$. Their number

$$n_1(D) = 2(d-1)$$

is twice as large as the number of different transformations (2.31). As a result, from $2(2d-1)$ variables $z_j^{(1)}$, $\bar{z}_j^{(1)}$, $j \geq 2$ only

$$2(2d-1) - n(D) - n_1(D) = D - 1$$

variables are really independent and this is in accordance with the number of integration variables in eq.(2.30). To derive the explicit form of $n_1(D)$ constraints and to evaluate the integration measure in eq.(2.30), consider the special case $D = 5$.

2.5.2. Special case: $D = 5$

At $D = 5$ (or $d = 2$) we denote the coordinates of a harmonic $u^{(1)}$ by

$$u^{(1)} = N \begin{pmatrix} 1 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \psi^{(1)} \\ \chi^{(1)} \end{pmatrix} \tag{2.32}$$

where $N\bar{N} = (1 + \bar{z}_1 z_1 + \bar{z}_2 z_2 + \bar{z}_3 z_3)^{-1}$. After substitution of this expression into system (2.13) one succeeds in solving the equations to derive an expression for a harmonic $u^{(2)}$:

$$u^{(2)} = Ne^{i\varphi} \begin{pmatrix} -\bar{z}_1 \\ 1 \\ -\bar{z}_3 \\ \bar{z}_2 \end{pmatrix} \tag{2.33}$$

where φ is an arbitrary function. Its appearance is connected with the fact that system (2.13) is invariant for general D under the following transformation of harmonics:

$$u^{(i)}(z) \to G_{ij} u^{(j)}(z)$$

where G is a unitary matrix of order d. At $D = 5$ there are, in addition, two harmonics $u^{(3)}$ and $u^{(4)}$. We denote

$$u^{(3)} = N' \begin{pmatrix} 1 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix} \tag{2.34}$$

and notice that according to the orthogonality condition of harmonics (II.2.51): $\bar{u}^{(1)} u^{(3)} = \bar{u}^{(2)} u^{(3)} = 0$ the variables t_i satisfy the equations:

$$1 + \bar{z}_1 t_1 + \bar{z}_2 t_2 + \bar{z}_3 t_3 = 0, \qquad z_1 - t_1 + z_3 t_2 - z_2 t_3 = 0 \tag{2.35}$$

The solutions of these equations imply, for instance, that t_2 and t_3 are functions of variables z_i, $\bar{z}_i$ and t_1. The expression for a harmonic $u^{(4)}$ is found with the use of relations (II.2.52) and (2.35):

$$u_i^{(4)} = \varepsilon^{ijkl} \bar{u}_j^{(1)} \bar{u}_k^{(2)} \bar{u}_l^{(3)} = -N' e^{-i\varphi} \begin{pmatrix} -\bar{t}_1 \\ 1 \\ -\bar{t}_3 \\ \bar{t}_2 \end{pmatrix}$$

The harmonics $u^{(1)}$ and $u^{(2)}$ are related to one another by the gauge transformation (2.31b) of the Weyl group. With the use of eqs.(2.31),(2.32) and (2.33) we find the transformation law of variables z_i under this transformation:

$$\mathcal{W}: \quad N_{,\alpha} = -Ne^{i\varphi} \bar{z}_1 \quad N_{,\alpha} z_{1,\alpha} = Ne^{i\varphi} \quad N_{,\alpha} z_{2,\alpha} = -Ne^{i\varphi} \bar{z}_3 \quad N_{,\alpha} z_{3,\alpha} = Ne^{i\varphi} \bar{z}_2$$

or

$$\mathcal{W}: \quad z_1 \to z_{1,\alpha} = -\frac{1}{\bar{z}_1} \quad z_2 \to z_{2,\alpha} = \frac{\bar{z}_3}{\bar{z}_1} \quad z_3 \to z_{3,\alpha} = -\frac{\bar{z}_2}{\bar{z}_1} \tag{2.36}$$

There is an analogous relation between harmonics $u^{(3)}$ and $u^{(4)}$. To eliminate the residual gauge ambiguity (2.36) one has to impose $n_1(D = 5) = 2$ additional gauge conditions on variables z_i, $\bar{z}_i$. It follows from eq.(2.36) that gauge conditions may be chosen as

$$z_1 = \text{const.}, \qquad \bar{z}_1 = \text{const.} \tag{2.37}$$

Let us express vector k_μ in terms of variables z_i with the use of eq.(2.25):

$$k_\mu = y_1 \left(v + \bar{v}, i(v - \bar{v}), -i(w - \bar{w}), w + \bar{w}, (1 - 4v\bar{v} - 4w\bar{w})^{1/2} \right) \tag{2.38}$$

where $\bar{v} = v^*$, $\bar{w} = w^*$ and

$$v = \frac{z_1\bar{z}_2 - z_3}{1 + \bar{z}_1 z_1 + \bar{z}_2 z_2 + \bar{z}_3 z_3}, \quad w = \frac{z_1\bar{z}_3 + z_2}{1 + \bar{z}_1 z_1 + \bar{z}_2 z_2 + \bar{z}_3 z_3}$$

It may be easily verified that v, w and k_μ are invariant under gauge transformations (2.36). After substitution of eq.(2.38) into relation (2.29) the metric is given by:

$$ds^2 = dy_1^2 + \frac{4y_1^2}{(1+\bar{z}z)^2}\{d\bar{z}_1 dz_1(\bar{z}_2 z_2 + \bar{z}_3 z_3) + (d\bar{z}_2 dz_2 + d\bar{z}_3 dz_3)(1 + \bar{z}_1 z_1) + (-dz_1[d\bar{z}_2(z_1\bar{z}_2 - z_3) + d\bar{z}_3(z_1\bar{z}_3 + z_2)] + \text{c.c.})\}$$

and $\bar{z}z \equiv \bar{z}_i z_i$. It is invariant under (2.36). Now we fix gauge (2.37) and derive

$$ds^2 = dy_1^2 + 4y_1^2\frac{1+\bar{z}_1 z_1}{(1+\bar{z}z)^2}(d\bar{z}_2\, dz_2 + d\bar{z}_3\, dz_3) \tag{2.39}$$

Hence, the normalized integration measure in gauge (2.37) is

$$d^5k = 16\pi^2 dy_1\, d\bar{z}_2\, dz_2\, d\bar{z}_3\, dz_3\, y_1^4 \frac{(1+\bar{z}_1 z_1)^2}{(1+\bar{z}z)^4}(2\pi i)^{-2}, \quad y_1 \geq 0 \tag{2.40}$$

Let $f(k)$ be an arbitrary even function and consider an integral (for odd fuction the integral vanishes)

$$Z = \int d^5k\, f(k) = 16\pi^2 \int_0^\infty dy_1\, y_1^4 \int \frac{d\bar{z}_2\, dz_2\, d\bar{z}_3\, dz_3}{(2\pi i)^2(1+\bar{z}z)^4}(1+\bar{z}_1 z_1)^2 f(y_1 e^{(1)}(z))$$

where eqs.(2.25) and (2.40) are used. In this relation variables $\bar{z}_1$, z_1 play a special role: they fix gauge (2.37) and therefore they are arbitrary constant parameters. At the same time Z is gauge invariant under transformations of the Weyl group (2.36) and does not depend on the gauge condition (2.37), that is, on the variables $\bar{z}_1$, z_1. To prove this statement, the integration variables $\bar{z}_2, z_2, \bar{z}_3, z_3$ are replaced in Z by the variables $\bar{t}_2, t_2, \bar{t}_3, t_3$ defined in eq.(2.34) and with eqs.(2.24),(2.25) and (II.2.31a) taken into account:

$$k_\mu = y_1 e_\mu^{(1)}(z) = \frac{1}{2}y_1(e_\mu^{(1)}(z) + e_\mu^{(2)}(z)) = -\frac{1}{2}y_1(e_\mu^{(3)}(t) + e_\mu^{(4)}(t)) = -y_1 e_\mu^{(3)}(t) = -y_1 e_\mu^{(1)}(t)$$

The integral Z is replaced by the expression

$$Z = 16\pi^2 \int_0^\infty dy_1\, y_1^4 \int \frac{d\bar{t}_2\, dt_2\, d\bar{t}_3\, dt_3}{(2\pi i)^2(1+\bar{t}t)^4}(1+\bar{t}_1 t_1)^2 f(-y_1 e^{(1)}(t))$$

that differs from the original expression only by the replacement of $\bar{z}_1$, z_1 by arbitrary parameters $\bar{t}_1$, t_1. Hence, Z does not depend on $\bar{z}_1$, z_1. To eliminate the dependence of Z on the gauge condition, we apply a trick analogous to the Faddeev-Popov one [3]. Namely, let us multiply the right-hand side of the last equation by unity:

$$\frac{1}{2\pi i}\int \frac{d\bar{z}_1 dz_1}{(1+\bar{z}_1 z_1)^2} = 1$$

and identify the integration variables $\bar{z}_1$, z_1 in this relation with the parameters $\bar{z}_1$, z_1 in integral Z:

$$Z = 16\pi^2 \int_0^\infty dy_1\, y_1^4 \int \frac{d\bar{z}_1 dz_1 d\bar{z}_2 dz_2 d\bar{z}_3 dz_3}{(2\pi i)^3 (1+\bar{z}z)^4} f(y_1 e^{(1)}(z))$$

Thus, at $D=5$ the integration measure (2.30) on the space of solutions of eq.(2.2) is:

$$d^5k = 16\pi^2 dy_1\, y_1^4 \frac{d\bar{z}_1 dz_1 d\bar{z}_2 dz_2 d\bar{z}_3 dz_3}{(1+\bar{z}_1 z_1 + \bar{z}_2 z_2 + \bar{z}_3 z_3)^4}(2\pi i)^{-3} = \frac{8}{3}\pi^2 dy_1\, y_1^4\, d\mu_0(z) \qquad (2.41)$$

and it is surprising that the measure $d\mu_0(z)$ coincides with G-invariant measure (II.2.56) on the manifold $SU(4)/U(3) \simeq CP^3$.

Let us compare expressions (II.2.77) and (2.41). We notice that at $D=5$ on the space of solutions of eq.(2.2) only one factor remains in eq.(II.2.77) corresponding to the integration measure over coordinates of a harmonic $u^{(1)}$. It is natural to expect that an analogous effect takes place for arbitrary D. However, for $D>5$ there are $n(D)$ additional constraints on variables $z_j^{(1)}$ and the integration measure is expected to be the product of the G-invariant measure on the manifold $SU(2d)/U(2d-1)$ and corresponding δ-functions:

$$d^D k = \text{const}\, dy_1\, y_1^{D-1}\, d\mu_0(z^{(1)}) \prod_{\alpha=1}^{n(D)} \delta\left(\varphi_\alpha(z^{(1)}, \bar{z}^{(1)})\right) \qquad (2.42)$$

We will prove this relation in the next section.

2.5.3. Integration measure for arbitrary D

It was the explicit form (2.33) of solutions of eq.(2.2) that enabled us to derive eq.(2.41) and fix gauge (2.37) at $D=5$. At arbitrary D the solutions of eq.(2.2) are unknown. Nevertheless, the measure $d^D k$ can be found in that case.

For arbitrary D the additional $n_1(D) = 2(d-1)$ gauge conditions for variables $z_j^{(1)}$, $\bar{z}_j^{(1)}$ are chosen in the form:

$$z_\beta^{(1)} = \text{const.}, \qquad \bar{z}_\beta^{(1)} = \text{const.}, \qquad \beta = 2, \ldots, d \qquad (2.43a)$$

or

$$(1+\bar{z}^{(1)}z^{(1)})^{1/2}\psi_\beta^{(1)}(z) = \text{const.}, \qquad (1+\bar{z}^{(1)}z^{(1)})^{1/2}\bar{\psi}_\beta^{(1)}(z) = \text{const.}, \qquad \beta = 2, \ldots, d \qquad (2.43b)$$

At $D=5$ eqs.(2.37) and (2.43) are identical. Now we have to prove that conditions (2.43) really fix gauge ambiguity (2.31). Under residual gauge transformations (2.31) we have:

$$\mathcal{W}: \quad \psi_\beta^{(1)}(z) \to \psi_\beta^{(1)}(z_{,\alpha}) = \psi_\beta^{(j)}(z), \qquad \alpha = e_1 - e_j$$

The gauge (2.43) implies that

$$(1+\bar{z}^{(1)}z^{(1)})^{1/2}\psi_\beta^{(1)}(z) = (1+\bar{z}_{,\alpha}^{(1)}z_{,\alpha}^{(1)})^{1/2}\psi_\beta^{(1)}(z_{,\alpha}) = (1+\bar{z}^{(j)}z^{(j)})^{1/2}\psi_\beta^{(j)}(z), \qquad \alpha = e_1 - e_j$$

but this relation contradicts the orthogonality condition (II.2.51) of harmonics:

$$\sum_{\beta=1}^{d} \bar{\psi}_\beta^{(1)}(z)\psi_\beta^{(j)}(z) = \delta^{1j} C_1, \qquad C_1 > 0$$

Thus, gauge (2.43) unambiguously fixes gauge transformations in eqs.(2.31). Then expression (2.29) for the metric looks like

$$ds^2 = dy_1^2 + 4y_1^2(dC_I^2 + dC_\alpha^2 + dC_1^2) \tag{2.44}$$

where

$$\begin{aligned} dC_1 = d(\bar{\psi}^{(1)}(z)\psi^{(1)}(z)) &= d\left([\bar{\psi}^{(1)}\psi^{(1)}(1+\bar{z}^{(1)}z^{(1)})](1+\bar{z}^{(1)}z^{(1)})^{-1}\right) \\ &= \bar{\psi}^{(1)}\psi^{(1)}(1+\bar{z}^{(1)}z^{(1)})d(1+\bar{z}^{(1)}z^{(1)})^{-1} \end{aligned}$$

Using eq.(2.18b) one gets

$$d\left((1+\bar{z}^{(1)}z^{(1)})^{1/2}C_1\chi_\alpha^{(1)}\right) = (dC_I\,\delta_{\alpha\beta} + i\,dC_\mu\,(\hat{\gamma}^\mu)_{\alpha\beta})\,(1+\bar{z}^{(1)}z^{(1)})^{1/2}\psi_\beta^{(1)}$$

or

$$dC_I^2 + dC_\mu^2 = (1+\bar{z}^{(1)}z^{(1)})(\bar{\psi}^{(1)}\psi^{(1)})d\left((1+\bar{z}^{(1)}z^{(1)})^{-1/2}\bar{\chi}_\alpha^{(1)}\right)d\left((1+\bar{z}^{(1)}z^{(1)})^{-1/2}\chi_\alpha^{(1)}\right)$$

Hence, after a simple calculation metric (2.44) is given by:

$$ds^2 = dy_1^2 + 4y_1^2\frac{1+\bar{z}_i^{(1)}z_i^{(1)}}{(1+\bar{z}^{(1)}z^{(1)})^2}\,d\bar{z}_\alpha^{(1)}dz_\alpha^{(1)} \tag{2.45}$$

where $i = 2,\ldots,d$ and $\alpha = d+1,\ldots,2d$.

At $D = 5$ eqs.(2.39) and (2.45) coincide. To find the measure using eq.(2.45), one has to resolve $n(D)$ constraints on variables $z_j^{(1)}$ and single out $(D-1)$ independent integration variables. The general form of constraints follows from eqs.(2.17a),(2.19), (2.18b):

$$\begin{aligned} \varphi_\alpha(z^{(1)},\bar{z}^{(1)}) &= 2(\bar{\psi}^{(1)}\psi^{(1)})\chi_\alpha^{(1)} - (\bar{\chi}^{(1)}\psi^{(1)} + \bar{\psi}^{(1)}\chi^{(1)})\psi_\alpha^{(1)} \\ &\quad -(\bar{\psi}^{(1)}\hat{\gamma}^\mu\chi^{(1)} - \bar{\chi}^{(1)}\hat{\gamma}^\mu\psi^{(1)})(\hat{\gamma}^\mu)_{\alpha\beta}\psi_\beta^{(1)} = 0 \qquad (2.46a) \\ \varphi_{\alpha+n(D)/2}(z^{(1)},\bar{z}^{(1)}) &= \left(\varphi_\alpha(z^{(1)},\bar{z}^{(1)})\right)^* \end{aligned}$$

and $\alpha = 1,\ldots,n(D)/2$. This is a system of $n(D)$ real linear equations for variables $\chi_\alpha^{(1)}$, $\bar{\chi}_\alpha^{(1)}$. At $D = 7$ the last relation is replaced by eq.(2.23):

$$\varphi_1(z^{(1)},\bar{z}^{(1)}) = \frac{z_6^{(1)} - z_2^{(1)}z_5^{(1)} + z_3^{(1)}z_8^{(1)} - z_4^{(1)}z_7^{(1)}}{1+\bar{z}^{(1)}z^{(1)}} = 0 \tag{2.46b}$$

and

$$\varphi_2(z^{(1)},\bar{z}^{(1)}) = \left(\varphi_1(z^{(1)},\bar{z}^{(1)})\right)^*$$

With eqs.(2.45) and (2.46) the general expression for the integration measure is:

$$d^Dk = \text{const}\,dy_1\,y_1^{D-1}g^{1/2}(D)\left(\frac{(1+\bar{z}_i^{(1)}z_i^{(1)})}{(1+\bar{z}^{(1)}z^{(1)})^2}\right)^{d-n(D)/2}\prod_{\alpha=D+1}^{2d}\frac{d\bar{z}_\alpha^{(1)}dz_\alpha^{(1)}}{2\pi i}\prod_{a=1}^{n(D)}\delta(\varphi_a(z^{(1)},\bar{z}^{(1)})) \tag{2.47}$$

where

$$g(D) = \det\left|\frac{\partial}{\partial z_\beta^{(1)}}\varphi_a(z^{(1)},\bar{z}^{(1)})\frac{\partial}{\partial \bar{z}_\beta^{(1)}}\varphi_b(z^{(1)},\bar{z}^{(1)}) + (a \leftrightarrow b)\right|$$

and $i = 2,\ldots,d$; $\beta = d+1,\ldots,2d$; $a,b = 1,\ldots,n(D)$.

We carry out the calculation of $g(D)$ for two values of the space-time dimension: for $D = 5$ there are no constraints at all and

$$g(D=5) = 1;$$

for $D = 7$ using eq.(2.46b) we have

$$g(D=7) = \left(\frac{(1+\bar{z}_i^{(1)}z_i^{(1)})}{(1+\bar{z}^{(1)}z^{(1)})^2}\right)^2$$

Therefore for arbitrary D we assume that

$$g(D) = \text{const.}\left(\frac{(1+\bar{z}_i^{(1)}z_i^{(1)})}{(1+\bar{z}^{(1)}z^{(1)})^2}\right)^{n(D)}$$

The arguments of δ-functions in eq.(2.47) are the constraints invariant under transformations of the Weyl group (2.31). In this equation the variables $\bar{z}_i^{(1)}$, $z_i^{(1)}$, $(i = 2,\ldots,d)$ are arbitrary constant parameters, and any integral of the form $\int d^Dk\, f(k)$ rewritten in terms of variables y_1 and $z^{(1)}$ is gauge invariant under the transformations (2.31) and therefore it does not depend on the variables $\bar{z}_i^{(1)}$, $z_i^{(1)}$, $(i \le d)$. The proof of this statement is analogous to the one of eq.(2.41). Hence, multiplying the right-hand side of (2.47) by unity

$$(d-1)!\int\prod_{j=2}^{d}\frac{d\bar{z}_j^{(1)}dz_j^{(1)}}{2\pi i}\frac{1}{(1+\bar{z}_i^{(1)}z_i^{(1)})^d} = 1$$

we derive the final expression for the integration measure on the space of solutions of eq.(2.2)

$$d^Dk = \text{const}\, dy_1\, y_1^{D-1}\frac{d\bar{z}^{(1)}dz^{(1)}}{(2\pi i)^{2d-1}}\frac{1}{(1+\bar{z}^{(1)}z^{(1)})^{2d}}\prod_{\alpha=1}^{n(D)}\delta(\varphi_\alpha(z^{(1)},\bar{z}^{(1)}))$$

suggested in eq.(2.42).

Now we are completely prepared to calculate the spinor functional defined by eq.(2.1).

3. Conclusions

We find the final expression for the spinor functional (2.1), for odd values of the space-time dimension, combining equations (2.28),(2.42) and (2.25) for the integration measure and momentum k_μ, respectively, on the space of solutions of the dimensional reduction equation (2.2):

$$\begin{aligned}\mathcal{M}_D[\dot{x}] &= \int \mathcal{D}\mu(y_1)\mathcal{D}\mu_0(z)\prod_{\alpha=1}^{n(D)}\delta(\varphi_\alpha(z,\bar{z}))\,|1,z(T)\rangle\langle 1,z(0)| \\ &\times \exp\left(-i\int_0^T dt\, y_1 e_\mu^{(1)}(z)\dot{x}_\mu + i\int_0^T dt\, y_1 - \frac{i}{2}\Phi(C)\right) \qquad (3.1)\end{aligned}$$

where $\mathcal{D}\mu_0(z)$ is the G-invariant measure on the manifold $SU(2d)/U(2d-1)$ and integration measure $\mathcal{D}\mu(y_1)$ is defined as

$$\int \mathcal{D}\mu(y_1) = \lim_{N\to\infty} \prod_{i=1}^{N} \int_{-\infty}^{\infty} dy_1(i\tau)\,(y_1(i\tau))^{D-1}\,, \qquad \tau = \frac{T}{N}$$

The integration region in this relation:

$$-\infty < y_1 < \infty$$

is the union of regions (2.26a) and (2.27). After integration of eq.(3.1) over y_1 one gets

$$\mathcal{M}_D[\dot{x}] = \int \mathcal{D}\mu_0(z) \prod_{\alpha=1}^{n(D)} \delta(\varphi_\alpha(z,\bar{z}))\,|1, z(T)\rangle\langle 1, z(0)|\,\delta^{(D-1)}\left(1 - e_\mu^{(1)}(z)\dot{x}_\mu\right) \exp\left(-\frac{i}{2}\Phi(C)\right) \quad (3.2)$$

Let us compare this equation with the initial relation for the spinor functional (I.2.8a) derived in sect.2.3 of ref.[1]. We note that all the transformations of the spinor functional described in the previous sections are reduced to the following replacements of variables in eq.(I.2.8a)

$$\begin{aligned} e_\mu &\rightarrow e_\mu^{(1)}(z) \\ I[e] &\rightarrow |1, z(T)\rangle\langle 1, z(0)| \exp\left(-\frac{i}{2}\Phi(C)\right) \\ \mathcal{D}e_\mu \delta(1 - e_\mu^2) &\rightarrow \int \mathcal{D}\mu_0(z) \prod_{\alpha=1}^{n(D)} \delta(\varphi_\alpha(z,\bar{z})) \end{aligned} \quad (3.3)$$

The dimension D of space-time in eq.(3.3) has to have only odd values.

Eq.(3.2) expresses the spinor functional as a sum over all paths on the complex projective space CP^{2d-1}. To calculate this sum it is useful to return back to eq.(I.2.8b) where all the integrations were made and perform in it transformations of the variables $\dot{x}_\mu$ and $I[\dot{x}]$ analogous to eq.(3.3). To this end we recall the relations (1.1) and (1.2) for the propagator and effective action and consider the following expression involved in them:

$$\begin{aligned} & \int_y^x \mathcal{D}x_\mu\, \mathcal{M}_D[\dot{x}]\, P\exp\left(ig\int_y^x dx_\mu A_\mu(x)\right) \\ = & \int \mathcal{D}\dot{x}_\mu\, \delta\left(x - y - \int_0^T dt\,\dot{x}\right) \mathcal{M}_D[\dot{x}]\, P\exp\left(ig\int_0^T dt\,\dot{x}_\mu A_\mu(x(t))\right) \\ = & \int \mathcal{D}\dot{x}_\mu\, \delta\left(x - y - \int_0^T dt\,\dot{x}\right) \delta^{(\frac{D-1}{2})}(1 - \dot{x}^2)\, I[\dot{x}]\, P\exp\left(ig\int_0^T dt\,\dot{x}_\mu A_\mu(x(t))\right) \end{aligned} \quad (3.4)$$

where eq.(I.2.8b) is used and the P-exponential depends on the function $x_\mu(t)$ expressed in terms of integration variables

$$x_\mu(t) = y_\mu + \int_0^t d\tau\,\dot{x}_\mu(\tau), \qquad x_\mu(T) = x_\mu$$

The δ-function in eq.(3.4) vanishes as $\dot{x}^2 \neq 1$ but its action on $(1 - \dot{x}^2)$ differs from zero. To deal with this function we assume following ref.[4] that only terms with the maximum singularity at $\dot{x}^2 = 1$ contribute to the path integral (3.4). This means (as remarked in sect.2.4 of ref.[1]) that the integrand of eq.(3.4) may be calculated under the additional condition $\dot{x}^2 = 1$. After decomposition of the vector $\dot{x}_\mu$ onto radial and angular parts the functional integral over the radial variables is easily calculated with the use of the δ-function but the remaining integral over the anglular variables is reduced after transformation (3.3) to the following relation

$$\int_y^x \mathcal{D}x_\mu \mathcal{M}_D[\dot{x}]\, P\exp\left(ig\int_y^x dx_\mu A_\mu(x)\right) = \int \mathcal{D}\mu_0(z)\prod_{\alpha=1}^{n(D)}\delta(\varphi_\alpha(z,\bar{z}))\delta\left(x-y-\int_0^T dt\, e^{(1)}(z)\right)$$
$$\times\exp\left(-\frac{i}{2}\Phi(C)\right)|1,z(T)\rangle\langle 1,z(0)|P\exp\left(ig\int_0^T dt\, e^{(1)}_\mu(z)A_\mu\left(y+\int_0^t d\tau\, e^{(1)}(z)\right)\right) \qquad (3.5)$$

Hence, for odd values of the space-time dimension we derive the following bosonic path integral representation for the propagator of interacting fermions

$$\begin{aligned} S(x,y;A) &= \int_0^\infty dT\exp(-TM)\int \mathcal{D}\mu_0(z)\prod_{\alpha=1}^{n(D)}\delta(\varphi_\alpha(z,\bar{z}))\,\delta\left(x-y-\int_0^T dt e^{(1)}(z)\right) \\ &\quad\times|1,z(T)\rangle\langle 1,z(0)|\exp\left(-\frac{i}{2}\Phi(C)\right)P\exp\left(ig\int_y^x dx_\mu A_\mu(x)\right) \end{aligned} \qquad (3.6)$$

where the integration contour of the P-exponential is defined as

$$x_\mu(t) = y_\mu + \int_0^t d\tau e^{(1)}(z), \qquad x_\mu(T) = x_\mu$$

For the effective action we have the relation (D = odd)

$$\begin{aligned} W[A] &= \int_0^\infty \frac{dT}{T}\exp(-TM)\int d^Dy\int \mathcal{D}\mu_0(z)\prod_{\alpha=1}^{n(D)}\delta(\varphi_\alpha(z,\bar{z}))\,\delta\left(\int_0^T dt e^{(1)}(z)\right) \\ &\quad\times\langle 1,z(0)|1,z(T)\rangle\exp\left(-\frac{i}{2}\Phi(C)\right)\mathrm{Tr}\,P\exp\left(ig\oint dx_\mu A_\mu(x)\right) \end{aligned} \qquad (3.7)$$

where $\langle 1,z(0)|1,z(T)\rangle = \frac{1+\bar{z}(0)z(T)}{((1+\bar{z}(0)z(0))(1+\bar{z}(T)z(T)))^{1/2}}$. We recall that $|1,z\rangle$ is a coherent state for the $SU(2d)$ group defined in sect.2.5 of ref.[2] . C is the path $z=z(t)$, $t\in[0,T]$ on the complex projective space CP^{2d-1}. $\Phi(C)$ is the one-dimensional Wess-Zumino term given by eq.(II.2.62). $\mathcal{D}\mu_0(z)$ is the G-invariant measure on the manifold $SU(2d)/U(2d-1)$ defined in eq.(II.2.56). Vector $e^{(1)}_\mu(z)$ was introduced in eq.(2.3) and is equal to the matrix element of the Dirac matrix $\langle 1,z|\gamma^\mu|1,z\rangle$. The functions $\varphi_\alpha(z,\bar{z})$, $(\alpha=1,...,n(D))$ are constraints (2.46a) on complex variables z_i, $(i=1,...,2d-1)$ imposed by the dimensional reduction equation (2.2).

To evaluate the spinor functional for even values of the space-time dimension we note that eqs.(I.2.8b) and (3.4) are fulfilled for arbitrary D. At the same time to perform the transformation (3.3) of eq.(3.4) for even D we identically transform eq.(3.4)

$$\int \mathcal{D}\dot{x}_{D+1}\,\mathcal{D}\dot{x}_\mu\,\delta(\dot{x}_{D+1})\,\delta\left(x-y-\int_0^T dt\dot{x}\right)\mathcal{M}_D[\dot{x}]\,P\exp\left(ig\int_0^T dt\,\dot{x}_\mu A_\mu(x(t))\right)$$

where $\mu=1,...,D$. Now the dimension $(D+1)$ of the $\dot{x}$-space has odd value and the use of the transformation (3.3) is justified. As a result, for even values of the space-time dimension the final representations for spinor functional, spinor propagator and effective action differ from eqs.(3.5),(3.6) and (3.7) by the following factor

$$\delta(\dot{x}_{D+1}) \rightarrow \delta(e^{(1)}_{D+1}(z))$$

Thus, for even D the summation in eqs.(3.5)-(3.7) is performed over all the paths on the complex projective space CP^{2d-1} for which $(D+1)$-dimensional vectors $e^{(1)}_\mu(z)$ lie at the subspace $e^{(1)}_{D+1}(z) = 0$.

At $D = 2, 3$ representations (3.6) and (3.7) coincide with expressions (2.26) and (2.27) proposed earlier.

Thus in the present paper we derived the representations (3.6) and (3.7) for the effective action and propagator of Dirac fermions interacting with a nonabelian gauge field in D-dimensional Euclidean space-time as sums over all bosonic paths on the complex projective space CP^{2d-1}, $(d = 2^{[D/2]-1})$. Now several important questions are raised:

1) How the well-known renormalization properties of the effective action and propagator follow from eqs.(3.6) and (3.7)?

2) The effective action of interacting fermions for odd D contains the Weyl anomaly. It is interesting to reproduce this anomaly contribution starting with eq.(3.7).

3) What kind of bosonic theory leads to the same path integral representations (3.6) and (3.7)? The answer to this question will allow us to establish the Bose-Fermi correspondence in higher dimensions.

All these problems will be the subject of forthcoming papers.

Acknowledgements

I would like to thank A.V.Efremov, N.P.Ilieva, E.A.Ivanov, D.I.Kazakov, E.A.Kochetov, S.O.Krivonos, V.M.Leviant, A.A.Migdal, H.Panagopoulos,O.K.Pashaev, V.N.Plechko, A.V.Radyushkin, O.V.Teryaev for numerous helpful discussions. I am also grateful to V.G.Kadyshevsky for his stimulating interest in the subject and support.

Appendix. Dirac matrices in D-dimensional Euclidean space-time

Dirac matrices are defined as solutions of the matrix equation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \quad \mu, \nu = 1, 2, ..., D \tag{A.1}$$

Let us introduce the matrices $\gamma_{\mu_1\mu_2\cdots\mu_n}$:

$$\gamma_{\mu_1\mu_2\cdots\mu_n} = i^{n(n-1)/2}\gamma_{[\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_n]} \tag{A.2}$$

where $[\cdots]$ denotes antisymmetrization. We briefly dwell on the properties of Dirac matrices:

1. The order of γ-matrices is equal to $2^{[D/2]}$.

2. Any two systems of γ-matrices being solutions of eq.(A.1) are related by the following transformation

$$\gamma'_\mu = T^{-1}\gamma_\mu T$$

where T is some nondegenerate matrix of order $2^{[D/2]}$.

3. Solutions of eq.(A.1) may be chosen to be hermitian, unitary and traceless matrices

$$\gamma^\dagger_\mu = \gamma_\mu, \quad \gamma^\dagger_{\mu_1\mu_2\cdots\mu_n} = \gamma_{\mu_1\mu_2\cdots\mu_n}$$
$$\gamma_\mu\gamma_\mu = \gamma_{\mu_1\mu_2\cdots\mu_n}\gamma_{\mu_1\mu_2\cdots\mu_n} = 1$$
$$\mathrm{Tr}\,\gamma_\mu = \mathrm{Tr}\,\gamma_{\mu_1\mu_2\cdots\mu_n} = 0$$

4. Under the change of the space-time dimension from $D = 2\nu$ to $D = 2\nu + 1$ the order of γ_μ-matrices is not changed but their total number is increased by unity and

$$\gamma_{2\nu+1} = i^{\nu(2\nu-1)}\gamma_1\gamma_2\cdots\gamma_{2\nu} \tag{A.3}$$

5. At $\nu = [D/2]$ the system of $(2^{2\nu} - 1)$ matrices

$$\Gamma^a = \{\gamma_{\mu_1}, \gamma_{\mu_1\mu_2}, \ldots, \gamma_{\mu_1\mu_2\cdots\mu_{2\nu}}\}, \qquad \mu_1 < \mu_2 < \cdots < \mu_{2\nu} \tag{A.4}$$

is linearly independent and forms the basis of all traceless matrices of order 2^ν. It means that any traceless matrix of order 2^ν may be decomposed over the basis with the use of the Fierz identity

$$\sum_{a=1}^{2^\nu-1}(\Gamma^a)_{AB}(\Gamma^a)_{CD} = 2^\nu\delta_{BC}\delta_{AD} - \delta_{AB}\delta_{CD} \tag{A.5}$$

6. An explicit form of γ_μ-matrices for arbitrary D is determined by means of recursion relations: At $D = 2$ the Dirac matrices coincide with the Pauli matrices

$$\gamma_1 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_2 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

At $D = 3$ the third matrix (A.3) is added

$$\gamma_3 = i\gamma_1\gamma_2 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

At $D = 2\nu$ we have $\dim\gamma^\mu = 2^\nu$ and γ_μ-matrices have the block structure

$$\gamma_\mu = \begin{pmatrix} 0 & -i\hat{\gamma}_\mu \\ i\hat{\gamma}_\mu & 0 \end{pmatrix}, \quad \gamma_{2\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mu = 1, 2, ..., 2\nu - 1 \tag{A.6}$$

where γ_μ are the Dirac matrices of order $2^{\nu-1}$ in the $D = 2\nu - 1$ dimensional Euclidean space-time.
At $D = 2\nu + 1$ one has to add the matrix (A.3)

$$\gamma_{2\nu+1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.7}$$

to matrices (A.6).

References

1. G.P.Korchemsky, Quantum geometry of the Dirac fermions: I.Definition of the spinor functional, preprint JINR E2-89-498.

2. G.P.Korchemsky, Quantum geometry of the Dirac fermions: II.Dimensional extension of the spinor functional, preprint JINR E2-89-499.

3. L.D.Faddeev and V.M.Popov, Phys.Lett.25B (1967) 29.

4. A.Strominger, Phys.Lett.101B (1981) 271.

Received by Publishing Department
on June 30, 1989.