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QUANTUM GEOMETRY OF THE DIRAC FERMIONS
Dimensional Extension of the Spinor Functional

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Квантовая геометрия дираковских фермионов

Размерное расширение спинорного функционала

Для описания дираковских фермионов, взаимодействующих с неабелевым калибровочным полем в D-мерном евклидовом пространстве-времени в работе развивается формализм бозонных интегралов по путям. Получены представления для эффективного действия и корреляционных функций фермионов в виде суммы по путям в комплексном проективном пространстве CP^{2d-1} $(d = 2^{[D/2]-1})$, в которых вся спинорная структура поглощается одномерным членом Весса-Зумино. Именно весс-зуминовский член обеспечивает все необходимые свойства фермионов при квантовании: квантованные значения спина, уравнение Дирака, Ферми-статистику и т.д.

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Quantum Geometry of the Dirac Fermions.

Dimensional Extension of the Spinor Functional

The bosonic path integral formalism is developed for Dirac fermions interacting with a nonabelian gauge field in the D-dimensional Euclidean space-time. The representation for the effective action and correlation functions of interacting fermions as sums over all bosonic paths on the complex projective space CP^{2d-1}, $d = 2^{[D/2]-1}$ is derived where all spinor structure is absorbed by the one-dimensional Wess-Zumino term. It is the Wess-Zumino term that ensures all necessary properties of Dirac fermions under quantization, i.e. quantized values of spin, Dirac equation, Fermi statistics.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. Introduction

In the previous papers [1,2] the formalism of the bosonic path integrals was developed for interacting Dirac fermions in D-dimensional Euclidean space-time. Representations (I.1.14) and (I.1.16)[1] for the effective action and propagator of interacting D-dimensional Dirac fermions as sums over all paths in the x-space were obtained

$$S(x,y;A) = \int_0^\infty dT e^{-TM} \int_y^x \mathcal{D}x_\mu \, P\exp\left(ig\int_y^x dx_\mu A_\mu(x)\right) \mathcal{M}_D[\dot{x}]$$

and

$$W[A] = \int_0^\infty \frac{dT}{T} e^{-TM} \int \mathcal{D}x_\mu \, \delta(x(0)-x(T)) \, \mathrm{Tr}\, P\exp\left(ig\oint dx_\mu A_\mu(x)\right) \mathrm{Tr}\, \mathcal{M}_D[\dot{x}]$$

and for the spinor functional $\mathcal{M}_D[\dot{x}]$ for arbitrary D expression (I.2.8) was derived. This expression contains the function $I[n]$ defined in eq.(I.2.9). It is equal to the infinite product of Dirac matrices that was calculated in sect.3 of ref.[1] only for two values of the space-time dimension $D = 2,3$.

The purpose of this paper is to generalize the above result: we will calculate the spinor functional for arbitrary values of the space-time dimension.

2. Dimensional extension of the spinor functional

To evaluate $\mathcal{M}_D[\dot{x}]$ for $D \geq 4$, let us consider the original expression (I.1.13) for the spinor functional in D-dimensional Euclidean space-time

$$\mathcal{M}_D[\dot{x}] = \int \mathcal{D}p_\nu \exp\left(-i\int_0^T dt\, p(t)\dot{x}(t)\right) P\exp\left(i\int_0^T dt\, \hat{p}(t)\right)$$

and perform on it a transformation called the *dimensional extension.*

The γ_μ-matrices are traceless, hermitian matrices of order $2d$:

$$2d = 2^{[D/2]}$$

(those properties are formulated in the Appendix of ref.[2]). In particular, matrices $\Gamma^a = \{\gamma_\mu, \gamma_{\mu\nu}, \ldots\}$, $a = 1, \ldots, 4d^2-1$ are elements of the $su(2d)$ Lie algebra [3]. Let us transform eq.(I.1.13) to complete the exponent to an arbitrary element of this algebra: $k_a\Gamma^a$, where k_a is some $(4d^2-1)$-dimensional vector. To this end the dimensional extension is performed. We introduce the additional coordinates k_α, x_α, $\alpha = D+1, \ldots, 4d^2-1$ and identically transform eq.(I.1.13) as follows:

$$\mathcal{M}_D[\dot{x}] = \int \mathcal{D}\dot{x}_\alpha \, \mathcal{M}_{2d}[\dot{x}], \quad \alpha = D+1, \ldots, 4d^2-1 \tag{2.1}$$

where

$$\mathcal{M}_{2d}[\dot{x}] = \int \mathcal{D}k_a \exp\left(-i\int_0^T dt\, k_a\dot{x}_a\right) P\exp\left(i\int_0^T dt\, k_a\Gamma^a\right), \quad a = 1, \ldots, 4d^2-1$$

The components $\dot{x}_\alpha(\tau)$ are the Lagrange multipliers in eq.(2.1). The integration is performed over all paths in the k-space. At $D=3$ (or $d=1$) the dimensional extension is unnecessary since

[1] Henceforth eqs.(I.X.Y) and (II.X.Y) should be understood as equation (X.Y) of refs.[1] and [2], respectively.

the dimension of the $su(2)$ Lie algebra coincides with the space-time dimension. Moreover, the relation analogous to (2.1):

$$\mathcal{M}_D[\dot{x}] = \int \mathcal{D}\dot{x}_{D+1} \, \mathcal{M}_{D+1}[\dot{x}] \tag{2.2}$$

allows us to restrict further consideration to the case of odd values of D. Thus we will determine the spinor functional $\mathcal{M}_{2d}[\dot{x}]$ and then using eqs.(2.1) and (2.2) one will be able to find $\mathcal{M}_D[\dot{x}]$ for odd and even D.

We define $\mathcal{M}_{2d}[\dot{x}]$ as the following limit:

$$\mathcal{M}_{2d}[\dot{x}] = \lim_{N\to\infty} \mathcal{M}_{2d}(x(N\tau)) \cdots \mathcal{M}_{2d}(x(2\tau))\mathcal{M}_{2d}(x(\tau)), \quad \tau = T/N \tag{2.3a}$$

where

$$\mathcal{M}_{2d}(x(t)) = \int d^{4d^2-1}k \exp(-ik_a x_a(t) + ik_a\Gamma^a\tau), \quad x_a(t) = \dot{x}_a(t)\tau \tag{2.3b}$$

The integrand of $\mathcal{M}_{2d}(x(t))$ is an element of the $SU(2d)$ group. To deal with it, the well-known properties of the $su(2d)$ Lie algebra are formulated in sect.2.1.

2.1. $su(N)$ Lie algebra for $SU(N)$ group [3]

Traceless, unitary and unimodular matrices of order N form the $SU(N)$ group. The dimension of this group is equal to

$$\dim(SU(N)) = N^2 - 1$$

It is well-known that in the $su(N)$ Lie algebra corresponding to the $SU(N)$ group the orthogonal Cartan-Weyl basis consisting of operators $\{H_i, E_\alpha\}$ may be chosen:

$$\mathrm{Tr}\,(H_i, H_j) = \delta_{ij}, \quad \mathrm{Tr}\,(E_\alpha, E_\beta) = \delta_{\alpha,-\beta}, \quad \mathrm{Tr}\,(H_i, E_\alpha) = 0$$

and the following commutation relations are fulfilled:

$$\begin{aligned} &[H_i, H_j] = 0, \quad i,j = 1, \ldots, N-1 \\ &[H_i, E_\alpha] = \alpha^i E_\alpha \\ &[E_\alpha, E_\beta] = N_{\alpha\beta}E_{\alpha+\beta} \text{ if } \alpha+\beta \in \Phi \text{ or } 0 \text{ if } \alpha+\beta \notin \Phi \\ &[E_\alpha, E_{-\alpha}] = \alpha^j H_j \end{aligned} \tag{2.4}$$

The abelian subalgebra of $su(N)$ generated by the operators H_i is called the *Cartan subalgebra* $\mathcal{H}$ and its dimension is the rank of the algebra:

$$\dim(\mathcal{H}) = \mathrm{rank}(su(N)) = N-1$$

The real variables α^i are collected to form vector $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^{N-1})$ in the space $\mathbf{R}^{N-1}$, called the *root*. The operator E_α corresponding to the root α is referred to as the *step* operator. The set of all roots of $su(N)$ is denoted by $\Phi = \{\alpha\}$. Φ is a *root system* and it consists of $N(N-1)$ roots. To describe the explicit form of the root system, one introduces the orthonormal basis in the space $\mathbf{R}^N$:

$$(e_i, e_j) = \delta_{ij}, \quad (e_i)_A = \delta_{iA}, \quad i, A = 1, 2, \ldots, N$$

The root system Φ of the $su(N)$ Lie algebra is the set of the following vectors:

$$\alpha \in \Phi, \quad \alpha = e_i - e_j, \quad i \neq j, i,j = 1,2,\ldots,N$$

that lie in the $(N-1)$-dimensional subspace orthogonal to vector e:

$$e = \sum_{i=1}^{N} e_i, \quad (\alpha, e) = 0, \quad \alpha \in \Phi$$

The root system Φ possesses the important symmetry property: it is invariant under transformations of the Weyl group. A more detailed definition of this group will be given below.

In the root system Φ the subset Δ consisting of $(N-1)$ roots called simple roots may be chosen:

$$\Delta = \{\alpha_i | \alpha_i = e_i - e_{i+1}, \ i = 1, 2, \ldots, N-1\} \tag{2.5}$$

the use of which enables one to represent an arbitrary root $\alpha \in \Phi$ as:

$$\alpha = \sum_{i=1}^{N} \alpha_i n_i, \quad \text{where } (n_i \geq 0 \ \forall i) \text{ or } (n_i \leq 0 \ \forall i) \tag{2.6}$$

The sets of roots Φ_+ and Φ_-

$$\Phi_+ = \{\alpha = e_i - e_j, \ i < j\}, \quad \Phi_- = \{\alpha = e_i - e_j, \ i > j\}$$

are called systems of positive and negative roots, respectively, and they are denoted by:

$$\alpha > 0 \ \text{ if } \ \alpha \in \Phi_+ \ \text{ or } \ \alpha < 0 \ \text{ if } \ \alpha \in \Phi_-$$

The step operators entering into eq.(2.4) may be chosen in the form:

$$(E_\alpha)_{AB} = \delta_{iA}\delta_{jB}, \quad \text{if } \alpha = e_i - e_j \tag{2.7}$$

Once the definitions are given, we set $N = 2d$ in all the above relations and consider an arbitrary element $(k_a\Gamma^a)$ of the $su(2d)$ Lie algebra appearing in eq.(2.1).

2.2. The Weyl group as a gauge group

Let k_a be some $(4d^2-1)$-dimensional vector. Then a powerful theorem of Lie algebra states that an element $k_a\Gamma^a$ of $su(2d)$ may be obtained by the gauge transformation from the Cartan subalgebra:

$$k_a\Gamma^a = D(z)(y, H)D^{-1}(z) \tag{2.8a}$$

where $(y, H) \equiv \sum_{i=1}^{2d-1} y_i H_i$ is an element of the Cartan subalgebra $\mathcal{H}$ and y is some vector in the space $\mathbf{R}^{2d-1}$. The unitary matrix $D(z)$ is given by:

$$D(z) = \exp\left(\sum_{\alpha>0}(z_\alpha E_\alpha - \bar{z}_\alpha E_{-\alpha})\right), \quad \bar{z}_\alpha = z_\alpha^* \tag{2.8b}$$

where the sum runs over all positive roots defined in (2.6), z_α are complex variables whose number $(d(2d-1))$ is equal to half of the number of step operators. The special case of eq.(2.8a) for the $SU(2)$ group was used early in eq.(I.3.16).

Eq.(2.8) relates $4d^2-1$ variables k_a to $2d-1$ variables y_i and $d(2d-1)$ complex variables z_α. This fact is expressed as:

$$k_a = k_a(y, z_\alpha) \tag{2.9}$$

As only y and z_α are known, the vector k_a is determined unambiguously from eq.(2.8a). But the reverse statement is wrong. There is the gauge ambiguity in the solutions of equations $y = y(k_a)$, $z_\alpha = z_\alpha(k_a)$ and the corresponding gauge group is the Weyl group.

2.2.1. Definition of the Weyl group

To prove the above statement, we rewrite eq.(2.8a):

$$k_a\Gamma^a = (D(z)U)(U^{-1}(y, H)U)(D(z)U)^{-1}$$

where U is a unitary matrix of order $2d$. Let it be chosen as:

$$U = S_\beta \equiv \exp\left(\frac{\pi}{2}(E_\beta - E_{-\beta})\right) \tag{2.10}$$

It follows from commutation relations (2.4) that

$$S_\beta^{-1}(y, H)S_\beta = (\sigma_\beta(y), H), \quad D(z)S_\beta = D(z_{,\beta})\exp(i(\phi, H)) \tag{2.11}$$

where variables $z_{,\beta}$ and the $(2d-1)$-dimensional vector ϕ both depend on z_α, y and β. The linear operator $\sigma_\beta(\cdot)$ is defined for an arbitrary vector y and root β as

$$\sigma_\beta(y) = y - 2\beta\frac{(y, \beta)}{(\beta, \beta)} \tag{2.12}$$

It has a simple geometric meaning. Acting on vector y operator $\sigma_\beta(y)$ reflects it in the hyperplane orthogonal to root β. After substitution of eqs.(2.10) and (2.11) we have

$$k_a\Gamma^a = D(z_{,\beta})(\sigma_\beta(y), H)D^{-1}(z_{,\beta}) \tag{2.13}$$

for an arbitrary root β. Comparing eqs.(2.8a) and (2.13) one concludes that dependence (2.9) is invariant under *discrete* transformations of variables y and z_α:

$$k_a = k_a(y, z_\alpha) = k_a(\sigma_\beta(y), (z_{\alpha,\beta})), \quad \alpha, \beta \in \Phi \tag{2.14}$$

These transformations form a *finite* group known as the *Weyl group* $\mathcal{W}$ [3]. Operator $\sigma_\alpha(\cdot)$ is called the *Weyl reflection*. The number of gauge invariant relations (2.14) is equal to the dimension of the Weyl group and for the $su(2d)$ Lie algebra it is

$$\dim(\mathcal{W}) = 2d - 1$$

Therefore for eq.(2.8a) to have a unique solution, the gauge condition for the Weyl group must be fixed.

2.2.2. Gauge fixing for the Weyl group

To determine the allowed form of the gauge condition, let us consider the action of the Weyl group on an arbitrary vector y in the root space $\mathbf{R}^{2d-1}$:

$$\mathcal{W}: \quad y \to y' = \sigma_\alpha(y), \quad \alpha \in \Phi \tag{2.15}$$

It is convenient to decompose vector y over the basis in the space $\mathbf{R}^{2d}$:

$$y = \sum_{i=1}^{2d} e_i y_i \tag{2.16}$$

where y_i are coordinates of the vector. The vector y lies in the subspace orthogonal to vector $\sum_{i=1}^{2d} e_i$ and therefore the coordinates are restricted by the condition:

$$(y, \sum_{i=1}^{2d} e_i) = \sum_{i=1}^{2d} y_i = 0 \tag{2.17}$$

Since $\sigma_\alpha(y)$ is a linear operator, it is sufficient to find its action on the basis vectors e_i. For root $\alpha = e_i - e_j \in \Phi$ and basis vector e_k one gets:

$$\sigma_\alpha(e_k) = \begin{cases} e_j \ , & \text{if } k = i \\ e_i \ , & \text{if } k = j \\ e_k \ , & \text{if } k \neq i, j \end{cases}$$

that is, the Weyl reflection acting on the basis interchanges vectors e_i and e_j. As a consequence, for an arbitrary vector y operator $\sigma_\alpha(y)$ permutes coordinates y_i and y_j:

$$\sigma_\alpha(y_1, \ldots, y_i, \ldots, y_j, \ldots, y_{2d}) = (y_1, \ldots, y_j, \ldots, y_i, \ldots, y_{2d}), \quad \alpha = e_i - e_j \tag{2.18}$$

Thus the Weyl group acts on the components of vector y as the permutation group.

It follows from eq.(2.18) that the root space R^{2d-1} is split into nonoverlapping regions called the *Weyl chambers* C_i under the action of the Weyl group [3]. The Weyl chamber for the $su(2d)$ Lie algebra is defined by the set of conditions: $(y_i \geq y_j)$ or $(y_i \leq y_j)$, $(i, j = 1, 2, ..., 2d)$ and two neighboring chambers have a common boundary. Their number is equal to $(2d)!$ and

$$R^{2d-1} = C_1 \cup C_2 \cup \cdots \cup C_{(2d)!} \tag{2.19}$$

Let vector y lie in a Weyl chamber C_i. Then the Weyl reflection $\sigma_\alpha(y)$ sends it from one Weyl chamber to another. It is essential that for any two Weyl chambers C_i and C_j the gauge transformation

$$y \in C_i, \quad y' = \sigma_\alpha(y) \in C_j$$

is unique [3] unless vector y belongs to the boundary of the Weyl chamber. Hence, the gauge invariance of eqs.(2.14) and (2.15) may be fixed demanding vector y to lie within the Weyl chamber:

$$y \in C_1 \tag{2.20a}$$

Let C_1 be the *fundamental Weyl chamber* in the last relation. Then the gauge condition (2.20) is, in fact, the definition of the fundamental Weyl chamber [3]:

$$(y, \alpha_i) \geq 0, \quad \alpha_i \in \Delta \tag{2.20b}$$

or

$$y_1 \geq y_2 \geq \ldots \geq y_{2d} = -y_1 - y_2 - \cdots - y_{2d-1} \tag{2.20c}$$

where α_i are simple roots defined in eq.(2.5). With this choice of the gauge condition, vector y belong to its own Weyl chamber in each of the gauge-equivalent sets (2.14).

Nevertheless, there are problems with gauge (2.20). Condition (2.20) does not fix the gauge at the boundary of the fundamental Weyl chamber:

$$(y, \alpha_i) = 0 \quad \text{or} \quad y_i = y_{i+1} \tag{2.21a}$$

because in that case vector y is invariant under transformations

$$\mathcal{W}: \quad y \to y' = \sigma_{\alpha_i}(y) = y \tag{2.21b}$$

and therefore there is a residual gauge ambiguity in eqs.(2.14) and (2.15) analogous to the Gribov copies [4]. To overcome the problem, one has to examine the action of the Weyl group on variables z_α defined in eq.(2.8b) and then fix the gauge at the boundary of the fundamental Weyl chamber by imposing additional constraints on the variables z_α. This program will be completed in ref.[2].

For the special case $D = 3$ one easily derives from eq.(2.20) that the gauge condition is:

$$y_1 \geq y_2 = -y_1 \quad \text{or} \quad y_1 \geq 0 \tag{2.22}$$

and it is really fulfilled in eq.(I.2.3) due to the positive definiteness of the radial part of the vector.

2.3. Decomposition over the projection operators

In the previous section an arbitrary element of the $su(2d)$ Lie algebra was decomposed, in eq.(2.8), in the Cartan-Weyl basis and the condition was found under which it is unique. At the same time Γ^α, H_i, E_α are matrices of order $2d$ that act in the space of the fundamental representation of the $SU(2d)$ group with dimension $2d$ called the minimal fundamental representation.

2.3.1. Minimal fundamental representation[3]

There exists a highest weight state in the representation space defined as:

$$H_i|\lambda_1\rangle = (\lambda_1)_i|\lambda_1\rangle, \quad E_\alpha|\lambda_1\rangle = 0, \quad \forall\alpha > 0$$

where λ_1 is a vector in the root space called the highest weight. In our case the highest weight is a fundamental weight, that is the one obeying the equation: $2(\lambda_1, \alpha_i)/(\alpha_i, \alpha_i) = \delta_{1i}$. The basis in the representation space consists of the highest weight state $|\lambda_1\rangle$ and states $|\lambda_i\rangle = E_{-\alpha}E_{-\beta}\cdots E_{-\gamma}|\lambda_1\rangle$ obtained from $|\lambda_1\rangle$ under the action of step operators corresponding to the negative roots:

$$|\lambda_2\rangle = E_{-\alpha_1}|\lambda_1\rangle, \quad |\lambda_3\rangle = E_{-\alpha_1}E_{-\alpha_2}|\lambda_1\rangle, \ldots, |\lambda_{2d}\rangle = E_{-\alpha_1}\cdots E_{-\alpha_{2d-1}}|\lambda_1\rangle \tag{2.23}$$

where $\alpha_i \in \Delta$ are simple roots. All vectors $|\lambda\rangle$ are simultaneously the eigenstates of operators from the Cartan subalgebra:

$$H|\lambda_i\rangle = \lambda_i|\lambda_i\rangle, \quad i = 1, 2, ..., 2d \tag{2.24}$$

with vectors λ_i called *weights*. For the minimal fundamental representation weights λ_i are related to one another by equations: $\lambda_i = \lambda_{i-1} - \alpha_i$ and they may be represented as:

$$\lambda_i = e_i - \frac{1}{2d}\sum_{j=1}^{2d} e_j \tag{2.25}$$

where $\{e_i\}$ is the basis in the space R^{2d}. Weights λ_i, $(i = 1, ..., 2d)$ have the following properties:

$$(\lambda_i, \sum_{j=1}^{2d} e_j) = 0, \quad (\lambda_i, \lambda_j) = (\lambda_i, e_j) = \delta_{ij} - \frac{1}{2d}$$

One concludes from eq.(2.24) that operators from the Cartan subalgebra may be decomposed as:

$$H = \sum_{i=1}^{2d} \lambda_i |i\rangle\langle i|$$

where $|i\rangle \equiv |\lambda_i\rangle$.

2.3.2. Decomposition of the spinor functional

After substitution of the last relation into eq.(2.8a) with the use of eqs.(2.16) and (2.25) we get:

$$\begin{aligned} k_a\Gamma^a &= \sum_{i=1}^{2d}(y,\lambda_i)D(z)|i\rangle\langle i|D^{-1}(z) \\ &= \sum_{i=1}^{2d} y_i P_i(z) \\ &= \sum_{i=1}^{2d} y_i|i,z\rangle\langle i,z| \end{aligned} \quad (2.26)$$

where states

$$|i,z\rangle = D(z)|i\rangle, \quad i=1,2,\dots,2d \quad (2.27)$$

and projection operators onto these states $P_i(z) = |i,z\rangle\langle i,z|$

$$P_i(z)P_j(z) = \delta_{ij}P_i(z) \quad (2.28)$$

are introduced. It follows from eq.(2.26) that $|i,z\rangle$ is the eigenstate of operator $k_a\Gamma^a$ corresponding to an eigenvalue equal to the coordinate of vector y_i:

$$(k_a\Gamma^a)|i,z\rangle = y_i|i,z\rangle$$

In all the above equations we denoted through (z) the dependence of the corresponding quantities on the variables $(z_\alpha, \bar z_\alpha)$ entering into eq.(2.8b).

Using eq.(2.26) one finds the relation between variables k_a and (y, z_α):

$$k_a = \frac{1}{2d}\sum_{i=1}^{2d} y_i e_a^{(i)}(z) \quad (2.29)$$

where the orthogonality condition for Γ^a matrices: $\mathrm{Tr}(\Gamma^a\Gamma^b) = 2d\,\delta^{ab}$ is taken into account and the following notation is introduced:

$$e_a^{(i)}(z) = \langle i,z|\Gamma^a|i,z\rangle = \mathrm{Tr}\left(P_i(z)\Gamma^a\right) \quad (2.30)$$

where $i = 1,\dots,2d$ and $a = 1,\dots,4d-1$. The functions $e_a^{(i)}(z)$ thus defined possess the properties:

$$\sum_{i=1}^{2d} e_a^{(i)}(z) = \mathrm{Tr}\,\Gamma_a = 0 \quad (2.31a)$$

$$\sum_{a=1}^{4d^2-1} e_a^{(i)}(z)e_a^{(j)}(z) = 2d\,\delta^{ij} - 1 \quad (2.31b)$$

$$P\left(e^{(i)}(z)\right) = \frac{1}{2d}\left(1 + e_a^{(i)}(z)\Gamma^a\right) = P_i(z) \quad (2.31c)$$

where for an arbitrary vector k_a we denote: $P(k_a) = \frac{1+k_a\Gamma^a}{2d}$. To prove the last two relations, the Fierz identity (II.A.5) is used.

We conclude from eq.(2.29) that the dependence of k_a on the variables $(z_\alpha, \bar z_\alpha)$ is contained entirely in functions $e_a^{(i)}(z)$ and, hence, the number of independent components of $e_a^{(i)}(z)$ is equal

to the number of step operators: $2d(2d-1)$. A part of constraints on $e_a^{(i)}(z)$ are expressed in eqs.(2.31) but the remaining ones may be easily found after substitution of eq.(2.31c) into orthogonality conditions (2.28) of projection operators. The variables y_i involved in eq.(2.29) are restricted by eq.(2.17) and the gauge condition (2.20).

For the special case $D = 3$ we get from eqs.(2.22) and (2.31a)

$$k_a = \frac{1}{2}(y_1 e_a^{(1)}(z) + y_2 e_a^{(2)}(z)) = y_1 e_a^{(1)}(z), \quad y_1 \geq 0 \quad (2.32)$$

and this expression is, in fact, the decomposition of vector into radial and angular parts.

The substitution of expression (2.26) for vector k_a in terms of variables y and $e_a^{(i)}(z)$ into eq.(2.3b) yields:

$$\begin{aligned} \mathcal{M}_{2d}(x) &= \int d^{4d^2-1}k \exp\left(-\frac{i}{2d}\sum_{i=1}^{2d} y_i e_a^{(i)}(z)x_a\right)\sum_{j=1}^{2d} P_j(z)e^{iy_j\tau} \\ &= \sum_{j=1}^{2d}\int d^{4d^2-1}k \exp\left(-\frac{i}{2d}\sum_{i=1}^{2d} y_i e_a^{(i)}(z)x_a + iy_j\tau\right)P(e^{(j)}(z)) \end{aligned} \quad (2.33)$$

where eqs.(2.28) and (2.31c) are used. We recall that variables y_j and $e_a^{(j)}(z)$ are functions of k_a whose explicit form may be found by solving eq.(2.8a) with the additional gauge condition (2.20).

2.4. Gauge invariance of the spinor functional

At $D = 3$ we find from eq.(2.31a) that $e_a^{(1)}(z) = -e_a^{(2)}(z)$ and after substitution of eq.(2.32) into eq.(2.33) the resulting spinor functional coincides with eq.(I.2.3) obtained earlier. The integrand of (I.2.3) contains only one projection operator, and it was the property that enabled us to calculate the infinite product of factors in eq.(I.3.8).

For $D \geq 4$ we have the old problem stressed in sect.3.2 of ref.[1]: there is a sum of projection operators in the integrand of eq.(2.33) that does not allow us to calculate the infinite product (2.3a). In this section it will be demonstrated that there is a simple relation between the projection operators $P(e^{(j)}(z))$ in eq.(2.33) that enables us to transform the spinor functional to the desired form when the integrand contains only one projection operator.

This relation is based on the gauge invariance of vector k_a and spinor functional $\mathcal{M}_{2d}(x)$ under transformations of the Weyl group.

With the use of eqs.(2.14) and (2.29) the action of the Weyl group on the vector k_a may be represented as:

$$k_a = \frac{1}{2d}\sum_{i=1}^{2d} y_i e_a^{(i)}(z) = \frac{1}{2d}\sum_{i=1}^{2d}(\sigma_\alpha(y), e_i)e_a^{(i)}(z_{,\alpha}) \quad (2.34)$$

This relation is fulfilled for arbitrary values y and α. Therefore assuming $\alpha = e_i - e_j$ and with eq.(2.18) one compares coefficients of variables y_i and finds the relations between the functions $e_a^{(j)}(z)$:

$$e_a^{(i)}(z) = e_a^{(j)}(z_{,\alpha}), \quad e_a^{(j)}(z) = e_a^{(i)}(z_{,\alpha}), \quad e_a^{(k)}(z) = e_a^{(k)}(z_{,\alpha}), \quad k \neq i,j$$

In particular, for $\alpha = e_1 - e_j$ we have

$$e_a^{(i)}(z) = e_a^{(1)}(z_{,i}), \quad i \geq 2 \quad (2.35)$$

where $z_{,i} \equiv z_{,\alpha=e_1-e_i}$. It is evident that the projection operators $P(e^{(i)}(z))$ satisfy analogous relations. Thus the expression for the spinor functional is:

$$\mathcal{M}_{2d}(x) = \sum_{j=1}^{2d} \int d^{4d^2-1}k \exp\left(-\frac{i}{2d}\sum_{i=1}^{2d} y_i e_\alpha^{(i)}(z)x_\alpha + iy_j\tau\right) P(e^{(1)}(z_{,j})) \tag{2.36}$$

where $e^{(1)}(z_{,1}) \equiv e^{(1)}(z)$.

Let us express the integration measure over momenta $d^{4d^2-1}k$ in terms of the variables y, z_α and $\bar{z}_\alpha$. Since the functions $y = y(k_\alpha)$, $z_\alpha = z_\alpha(k_\alpha)$ and $\bar{z}_\alpha = \bar{z}_\alpha(k_\alpha)$ may be found by solving eq.(2.8a) under gauge condition (2.20), the general structure of the measure is:

$$d^{4d^2-1}k = d\mu(y,z) \prod_{\alpha_i\in\Delta} \theta((y,\alpha_i)) \tag{2.37}$$

where α_i are simple roots defined in eq.(2.5) and θ-functions take into account the gauge condition. The explicit form of the measure $d\mu(y,z)$ will be derived in sect.2.7 but now it is sufficient to establish some properties of $d\mu(y,z)$.

The spinor functional is a gauge invariant quantity and it does not depend on the explicit form of the gauge condition. Hence the measure $d\mu(y,z)$ is unchanged under transformations (2.34) of the Weyl group:

$$\mathcal{W}: \quad d\mu(y,z) = d\mu(\sigma_\alpha(y),(z_{,\alpha})) \tag{2.38}$$

for an arbitrary root α. With this property and expression (2.37) the spinor functional (2.36) is given by:

$$\begin{aligned}\mathcal{M}_{2d}(x) &= \sum_{j=1}^{2d}\int d\mu(y,z)\prod_{\alpha_i\in\Delta}\theta((y,\alpha_i))\exp\left(-\frac{i}{2d}\sum_{i=1}^{2d}y_i e_\alpha^{(i)}(z)x_\alpha + iy_j\tau\right)P(e^{(1)}(z_{,j}))\\ &= \sum_{j=1}^{2d}\int d\mu(\sigma_j(y),(z_{,j}))\prod_{\alpha_i\in\Delta}\theta((y,\alpha_i))\\ &\times \exp\left(-\frac{i}{2d}\sum_{i=1}^{2d}y_i e_\alpha^{(i)}(z)x_\alpha + i(\sigma_j(y),e_1)\tau\right)P(e^{(1)}(z_{,j}))\end{aligned} \tag{2.39a}$$

where

$$\sigma_j(y) \equiv \begin{cases}\sigma_{\alpha=e_1-e_j}(y) & , \quad j\geq 2\\ y & , \quad j=1\end{cases} \tag{2.39b}$$

and the identity $y_j = (\sigma_j(y),e_1)$ is used. Let us perform the inverse Weyl transformation:

$$\mathcal{W}^{-1}: \quad (\sigma_j(y),z_{,j}) \to (y,z)$$

in the j-th item of the sum and take into account the gauge invariance (2.34) of vector k_α to derive, with the use of equality $\sigma_\alpha\sigma_\alpha = 1$, the following relation:

$$\mathcal{M}_{2d}(x) = \int d\mu(y,z)\exp\left(-\frac{i}{2d}\sum_{i=1}^{2d}y_i e_\alpha^{(i)}(z)x_\alpha + i(y,e_1)\tau\right)P(e^{(1)}(z))\sum_{j=1}^{2d}\prod_{\alpha_i\in\Delta}\theta((\sigma_j(y),\alpha_i)) \tag{2.40}$$

Comparing eqs.(2.36) and (2.40) one concludes that it is the gauge invariance of the spinor functional that enables us to get rid of the sum of projection operators in the integrand of (2.36). The

final expression (2.40) for the spinor functional contains only one projection operator onto state $|1,z\rangle$, defined in eq.(2.27) and the sum of the products of θ-functions is really the sum over gauge conditions. Indeed, the sum may be rewritten as:

$$\sum_{j=1}^{2d}\prod_{\alpha_i\in\Delta}\theta((\sigma_j(y),\alpha_i)) = \sum_{j=1}^{2d}\prod_{\alpha_i\in\Delta}\theta((y,\sigma_j(\alpha_i))) \tag{2.41}$$

The first item at $j=1$ is the definition of the gauge condition (2.20) that restricts vector y to belong to the fundamental Weyl chamber C_1, defined in eq.(2.20b). At $j\geq 2$ vector y lies in the region of the space $\mathbf{R}^{2d-1}$, formed by $(2d-1)$ Weyl chambers C_j obtained from the fundamental Weyl chamber under reflection transformations σ_α, $(\alpha = e_1 - e_j)$. Thus eq.(2.41) determines the following region:

$$\Omega = C_1 \cup C_2 \cup C_3 \cup \cdots \cup C_{2d} \tag{2.42}$$

where the Weyl chamber C_j is defined as:

$$C_j: \quad (y,\sigma_j(\alpha_i)) \geq 0, \quad \alpha_i \in \Delta \tag{2.43a}$$

or with the use of eqs.(2.18) and (2.39b):

$$\begin{aligned} C_1: &\quad y_1 \geq y_2 \geq \cdots \geq y_j \geq \cdots \geq y_{2d}\\ C_j: &\quad y_j \geq y_2 \geq \cdots \geq y_1 \geq \cdots \geq y_{2d}, \quad j\geq 2\end{aligned} \tag{2.43b}$$

and $y_{2d} = -y_1 - y_2 - \cdots - y_{2d-1}$. We note that region Ω does not coincide with the space $\mathbf{R}^{2d-1}$ formed by $(2d)!$ Weyl chambers (2.19). However at $D=3$ the region Ω is the unification of the two Weyl chambers:

$$\Omega = C_1 \cup C_2 = (y_1 \geq y_2 = -y_1) \cup (y_2 \geq y_1 = -y_2)$$

and vector $y = (y_1,y_2)$ can take any value in that case.

Thus we derive the following expression for the spinor functional:

$$\begin{aligned}\mathcal{M}_{2d}(x) &= \int d\mu(y,z)\exp\left(-\frac{i}{2d}\sum_{i=1}^{2d}y_i e_\alpha^{(i)}(z)x_\alpha + iy_1\tau\right)P(e^{(1)}(z))\,\theta(y\in\Omega)\\ &= \int_\Omega d\mu(y,z)\exp\left(-\frac{i}{2d}\sum_{i=1}^{2d}y_i e_\alpha^{(i)}(z)x_\alpha + iy_1\tau\right)|1,z\rangle\langle 1,z|\end{aligned} \tag{2.44}$$

After its substitution into eq.(2.3a) the infinite product of matrices occurring in eq.(I.3.4) is replaced by the scalar products:

$$|1,z(T)\rangle\langle 1,z(0)| \lim_{N\to\infty}\prod_{i=1}^{N}\langle 1,z(i\frac{T}{N})|1,z((i-1)\frac{T}{N})\rangle \tag{2.45}$$

where index i numbers different factors in eq.(2.3a).

The only undetermined quantities involved in expression (2.44) are the state $|1,z\rangle$ and integration measure $d\mu(y,z)$ defined in eqs.(2.27) and (2.37), respectively. In the next section the properties of the state $|1,z\rangle$ are studied and the integration measure will be calculated in sect.2.7.

2.5. The coherent states for the $SU(2d)$ group

We recall the definition of the state $|i,z\rangle$:

$$|i,z\rangle = D(z)|i\rangle, \quad i = 1,2,\ldots,2d \tag{2.46a}$$

where the unitary matrix $D(z)$ is:

$$D(z) = \exp\left(\sum_{\alpha>0}(z_\alpha E_\alpha - \bar{z}_\alpha E_{-\alpha})\right), \quad \bar{z}_\alpha = z^*_\alpha \tag{2.46b}$$

and $|i\rangle$ is an eigenstate of operators from the Cartan subalgebra corresponding to the weight λ_i. For $i = 1,2,\ldots,2d$ the states $|i,z\rangle$ form an orthonormal basis in the representation space:

$$\langle i,z|j,z\rangle = \delta_{ij} \tag{2.46c}$$

For a fixed i the state $|i,z\rangle$ is known as a coherent state for the group $G = SU(2d)$ [5]. For different values of i the states $|i,z\rangle$ form $2d$ systems of coherent states. It is well-known [5] that all properties of these states depend on the structure of the stationary subgroup of the reference state $|i\rangle$.

2.5.1. Stationary subgroup

The stationary subgroup of any weight vector $|i\rangle$ contains the Cartan subgroup $U(1)\otimes U(1)\cdots U(1)$ (here $U(1)$ enters $(2d-1)$-times):

$$\begin{aligned} \mathcal{H}: \quad D(z) &\rightarrow D(z)\exp(i(\phi,H)) \\ |i,z\rangle &\rightarrow |i,z\rangle\exp(i(\phi,\lambda_i)) \end{aligned} \tag{2.47}$$

Besides, the minimal fundamental representation of the $SU(2d)$ group is degenerate: the highest weight λ_1, defined in eq.(2.25) is orthogonal to some of the roots and therefore the stationary subgroup is larger [5]. For each of the weight vectors the stationary subgroup is $H = U(2d-1)$. To prove this, one uses eq.(2.25) and notes that:

$$(\lambda_i,\alpha) = 0 \quad \text{for } \alpha = e_k - e_j, \quad k,j \neq i \tag{2.48}$$

The roots α, satisfying this equation lie in the subspace $\mathbf{R}^{2d-2}$ orthogonal to vectors e_i and $\sum_{i=1}^{2d} e_i$. Then it follows from commutation relations (2.4) and eq.(2.48) that

$$(\alpha,H)\,E_\alpha\,|i\rangle = 2E_\alpha\,|i\rangle$$

At the same time eq.(2.24) implies that the eigenvalues of operator (α,H) are $(\alpha,\lambda_i) = \pm 1$ and hence $E_\alpha|i\rangle$ is a null-vector:

$$E_\alpha|i\rangle = E_{-\alpha}|i\rangle = 0 \quad \text{for} \quad \alpha = e_k - e_j, \quad k,j\neq i \tag{2.49}$$

Thus the stationary subgroup of the weight vector $|i\rangle$ is generated by $(2d-1)$ operators from the Cartan subalgebra and $(2d-1)(2d-2)$ step operators (2.49) and therefore H is

$$H = U(1)\otimes SU(2d-1) = U(2d-1)$$

As a result, each coherent state is characterized by a point of the coset space [5]

$$G/H = SU(2d)/U(2d-1) \simeq CP^{2d-1}$$

where the complex projective space CP^{2d-1} is obtained from the sphere $S^{4d-1} = \{\sum_{i=1}^{4d-1}|z_i|^2 = 1\}$ by identifying the points: $z \sim e^{i\varphi}z$ [6]. To find the explicit form of the coherent states, we introduce the harmonic coordinates on the space $SU(2d)/U(2d-1)$.

2.5.2. Harmonic coordinates on the space $SU(2d)/U(2d-1)$[7]

The harmonic coordinates $u^{(i)}_j$ on the space $SU(2d)/U(2d-1)$ are defined as the following matrix elements:

$$u^{(i)}_j(z) = \langle j|i,z\rangle = \langle j|D(z)|i\rangle = \langle j|\exp\left(\sum_{\alpha>0}(z_\alpha E_\alpha - \bar{z}_\alpha E_{-\alpha})\right)|i\rangle$$

Really the harmonics $u^{(i)}_j$ are the weights of the expansion of the coherent state $|i,z\rangle$ in the basis of the weight vectors:

$$|i,z\rangle = \sum_{j=1}^{2d} u^{(i)}_j(z)|j\rangle \tag{2.50}$$

Using the unitarity and unimodularity properties of the matrix $D(z)$ one gets relations for harmonics $u^{(i)}$ and $\bar{u}^{(i)} = (u^{(i)})^*$:

1) Unitarity: $D(z)D^\dagger(z) = D^\dagger(z)D(z) = 1$

$$\begin{aligned} \sum_{i=1}^{2d}\bar{u}^{(i)}_j(z)u^{(i)}_k(z) &= \delta_{jk} \\ \sum_{j=1}^{2d}\bar{u}^{(i)}_j(z)u^{(k)}_j(z) &= \delta^{ik} \end{aligned} \tag{2.51}$$

2) Unimodularity: $\det D(z) = 1$

$$\epsilon^{i_1 i_2\ldots i_{2d}}\, u^{(1)}_{i_1}u^{(2)}_{i_2}\cdots u^{(2d)}_{i_{2d}} = 1$$

or

$$\bar{u}^{(2d)}_{i_{2d}} = \epsilon^{i_1 i_2\ldots i_{2d}}\, u^{(1)}_{i_1}u^{(2)}_{i_2}\cdots u^{(2d-1)}_{i_{2d-1}} \tag{2.52}$$

The last equation relates the components of harmonic $u^{(2d)}$ to the components of other harmonics. For fixed i the harmonic $u^{(i)}(z)$ is the row consisting of $2d$ elements that can be chosen as:

$$u^{(i)}(z) = N_i\begin{pmatrix} z^{(i)}_1 \\ z^{(i)}_2 \\ \vdots \\ z^{(i)}_{2d}\end{pmatrix}, \quad z^{(i)}_i = 1 \tag{2.53a}$$

where $z^{(i)}_j$ are complex variables and N_i is a normalization factor:

$$N_i\bar{N}_i = \left(1+\sum_{j\neq i}\bar{z}^{(i)}_j z^{(i)}_j\right)^{-1} \equiv (1+\bar{z}^{(i)}z^{(i)})^{-1} \tag{2.53b}$$

According to eqs.(2.53) and (2.50) the complex variables

$$z^{(i)} = (z^{(i)}_1, z^{(i)}_2, \ldots, z^{(i)}_{i-1}, z^{(i)}_{i+1}, \ldots, z^{(i)}_{2d})$$

are local coordinates of the point $z^{(i)}$ on the manifold $SU(2d)/U(2d-1)$ corresponding to the coherent state $|i,z\rangle$. Each system of coherent states is characterized by its own points $z^{(i)}$ whose coordinates $z_j^{(i)}$ are not independent. The relations (2.51), (2.52) put severe restrictions on $z_j^{(i)}$ and the number of the independent components $z_j^{(i)}$ is equal to the number $(2d(2d-1))$ of roots :

$$z_j^{(i)} = z_j^{(i)}(z_\alpha, \bar{z}_\alpha)$$

The harmonic coordinates $z_j^{(i)}$ are very useful to study the properties of coherent states.

2.5.3. The properties of coherent states[5]

Let us consider the action of the group on coherent states. For G to be an arbitrary element of the minimal fundamental representation of the $SU(2d)$ group, the state

$$G: \quad G|i,z\rangle = |i, z_G\rangle e^{i\phi(G,z)}$$

is the coherent state of the same system. After substitution of eqs.(2.50) and (2.53) into this relation one finds that the group acts on the space $SU(2d)/U(2d-1)$ as a group of projective transformations:

$$G: \quad z^{(i)} \to (z_G^{(i)})_\alpha = \frac{G_{\alpha i} + G_{\alpha\beta} z_\beta^{(i)}}{G_{ii} + G_{i\beta} z_\beta^{(i)}}, \quad z^{(i)} \in SU(2d)/U(2d-1) \tag{2.54}$$

It is well-known that the space $SU(2d)/U(2d-1)$ is a Kähler manifold. This means that it is a complex manifold and there exists a Riemann metric on it that can be written in terms of the local coordinates $z_j \equiv z_j^{(i)}$, $j \neq i$ as follows:

$$ds^2 = g^{i\bar{j}}(z,\bar{z})\, dz_i\, d\bar{z}_j, \quad g^{i\bar{j}}(z,\bar{z}) = \frac{\partial^2 F(z,\bar{z})}{\partial z_i \partial \bar{z}_j} \tag{2.55a}$$

where the function

$$F(z,\bar{z}) = \log\left(1 + \sum_{i=1}^{2d-1} \bar{z}_i z_i\right) \equiv \log(1 + \bar{z}z) \tag{2.55b}$$

is the Kähler potential. Metric (2.55a) is invariant under transformations (2.54) of $SU(2d)$ since

$$G: \quad F(z^{(i)}, \bar{z}^{(i)}) \to F(z_G^{(i)}, \bar{z}_G^{(i)}) = F(z^{(i)}, \bar{z}^{(i)}) - \log(G_{ii} + G_{ij} z_j^{(i)}) - \log(\bar{G}_{ii} + \bar{G}_{ij} \bar{z}_j^{(i)})$$

The G-invariant measure on the manifold $SU(2d)/U(2d-1)$ normalized by the condition $1 = \int d\mu_0(z)$ has the form:

$$d\mu_0(z) = (2d-1)! \prod_{i=1}^{2d-1} \frac{dz_i\, d\bar{z}_i}{2\pi i} \frac{1}{(1+\bar{z}z)^{2d}} \tag{2.56}$$

and

$$G: \quad d\mu_0(z) \to d\mu_0(z_G) = d\mu_0(z)$$

Moreover, the closed G-invariant 2-form may be built on the manifold $SU(2d)/U(2d-1)$:

$$\omega(z,\bar{z}) = 2i \frac{\partial^2 F(z,\bar{z})}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j = d\theta(z,\bar{z}) \tag{2.57}$$

where

$$\theta(z,z) = i \sum_{i=1}^{2d-1} \left(d\bar{z}_i \frac{\partial}{\partial \bar{z}_i} - dz_i \frac{\partial}{\partial z_i} \right) F(z,\bar{z})$$

and the sign $\wedge$ denotes the exterior product $(dz_i \wedge d\bar{z}_j = -d\bar{z}_j \wedge dz_i)$.

Now we briefly formulate the main properties of the coherent states $|i,z\rangle$, $(i = 1,2,\ldots,2d)$.

1. The coherent states are not orthogonal to each other and according to eqs.(2.50) and (2.53):

$$\langle i, z_1 | i, z_2 \rangle = \frac{1 + \bar{z}_1^{(i)} z_2^{(i)}}{\left((1 + \bar{z}_1^{(i)} z_1^{(i)})(1 + \bar{z}_2^{(i)} z_2^{(i)})\right)^{1/2}} \tag{2.58}$$

In the infinitesimal form this relation reduces to

$$\begin{aligned} \langle i, z + dz | i, z \rangle &= 1 + \frac{1}{2} \frac{d\bar{z}^{(i)} z^{(i)} - \bar{z}^{(i)} dz^{(i)}}{1 + \bar{z}^{(i)} z^{(i)}} \\ &= 1 - \frac{i}{2} \theta(z^{(i)}, \bar{z}^{(i)}) \end{aligned} \tag{2.59}$$

where the 1-form θ was defined in eq.(2.57) and $\bar{z}dz \equiv \sum_{i=1}^{2d} \bar{z}_i dz_i$.

2. There is the completeness relation:

$$2d \int d\mu_0(z^{(i)})\, |i,z\rangle\langle i,z| = 1$$

where $d\mu_0(z^{(i)})$ is G-invariant measure (2.56) on the manifold $SU(2d)/U(2d-1)$.

3. The coherent states from different systems are related to one another by the transformations of the Weyl group:

$$|i,z\rangle = |j, z_{,\alpha}\rangle e^{i(\phi,\lambda_j)}, \quad \alpha \in \Phi \tag{2.60}$$

where variables $z_{,\alpha}$ are defined in eq.(2.11) and ϕ is a vector in the space $\mathbf{R}^{2d-1}$. Indeed, due to eq.(2.23) the weight vector $|i\rangle$ may be represented as

$$|i\rangle = E_\alpha |j\rangle = \exp\left(\frac{\pi}{2}(E_\alpha - E_{-\alpha}\right)|j\rangle \equiv S_\alpha |j\rangle$$

where $\alpha = e_i - e_j$. Hence, one concludes with the use of eqs.(2.11) and (2.27) that

$$|i,z\rangle = D(z) S_\alpha |j\rangle = D(z_{,\alpha}) \exp(i(\phi, H))|j\rangle = |j, z_{,\alpha}\rangle \exp(i(\phi,\lambda_j))$$

We note also that the validity of relation (2.60) follows from eqs.(2.35) and (2.30).

2.6. One-dimensional Wess-Zumino term

Let us apply the properties of the coherent states to calculate the limit of the infinite product (2.45) $(t = T/N)$

$$\begin{aligned} &|1, z(T)\rangle\langle 1, z(0)| \lim_{N\to\infty} \prod_{i=1}^{N} \langle 1, z(it) | 1, z((i-1)t)\rangle \\ &= |1, z(T)\rangle\langle 1, z(0)| \exp\left(-\int_0^T dt \langle 1, z(t)| \frac{d}{dt} |1, z(t)\rangle\right) \\ &\equiv |1, z(T)\rangle\langle 1, z(0)| \exp\left(-\frac{i}{2}\Phi(C)\right) \end{aligned} \tag{2.61}$$

that appears after substitution of eq.(2.44) into the definition of the spinor functional (2.3a). We get from eqs.(2.46a),(2.50) and (2.59)

$$\begin{aligned} \Phi(C) &= -2i\int_0^T dt\,\langle 1|D^{-1}(z)\frac{d}{dt}D(z)|1\rangle && (2.62a)\\ &= -2i\int_0^T dt\,\bar u^{(1)}(z)\frac{d}{dt}u^{(1)}(z) && (2.62b)\\ &= i\int_0^T dt\,\frac{\dot{\bar z}z-\bar z\dot z}{1+\bar z z} && (2.62c)\\ &= \int\theta(z,\bar z) && (2.62d)\end{aligned}$$

where $z=(z_2^{(1)},z_3^{(1)},\ldots,z_{2d}^{(1)})$ is a point of the complex projective space CP^{2d-1}. The points $z(t)$, $t\in[0,T]$ form a curve on the complex projective space denoted by C in eqs.(2.61) and (2.62).

Eqs.(2.62) coincide with the definition of the one-dimensional Wess-Zumino term [8,9] whose special case at $D=3$ was obtained in eq.(I.3.22).

Let us examine the transformation properties of the spin factor $\Phi(C)$ under the action of group $G=SU(2d)$ on the space CP^{2d-1} defined in eqs.(2.47) and (2.54). One finds from eq.(2.62a) that $\Phi(C)$ changes as

$$G:\quad \begin{aligned} D(z) &\to GD(z)\\ \Phi(C) &\to \Phi(C)'=\Phi(C)\end{aligned} \qquad (2.63)$$

and

$$\begin{aligned}\mathcal{H}:\quad D(z) &\to D(z)\exp(i(\phi,H))\\ \Phi(C) &\to \Phi(C)'=\Phi(C)+2\int_0^T dt\frac{d}{dt}(\phi(t),\lambda_1)\\ &= \Phi(C)+2(\phi(T),\lambda_1)-2(\phi(0),\lambda_1)\end{aligned} \qquad (2.64)$$

Thus the spin factor is changed under the action of the stationary subgroup. Analogously to eq.(I.3.23) this property leads to the quantization condition of the spin of fermions. To prove it, one transforms eq.(2.61) with the use of eqs.(2.31c) and (2.58) as

$$|1,z(T)\rangle\langle 1,z(T)|1,z(0)\rangle\langle 1,z(0)|\exp\left(-\frac{i}{2}\Phi(C)\right)\left(\langle 1,z(T)|1,z(O)\rangle\right)^{-1}$$

$$=P\left(e^{(1)}(z(T))\right)P\left(e^{(1)}(z(0))\right)\exp\left(-\frac{i}{2}\Phi(\bar C)\right)|\langle 1,z(T)|1,z(0)\rangle|^{-1}$$

$$=\frac{1}{4d^2}\left(1+e_a^{(1)}(z(T))\Gamma^a\right)\left(1+e_b^{(1)}(z(0))\Gamma^b\right)\exp\left(-\frac{i}{2}\Phi(\bar C)\right)|\langle 1,z(0)|1,z(T)\rangle|^{-1}$$

where

$$\Phi(\bar C)=\Phi(C)+i\log\frac{1+\bar z^{(1)}(T)z^{(1)}(0)}{1+\bar z^{(1)}(0)z^{(1)}(T)}$$

Now $\bar C$ is a closed curve on the space CP^{2d-1}

$$\bar C=\{z(\tau),\tau\in[0,1];z(0)=z(1)\}$$

Under the gauge transformations (2.64) the spin factor $\Phi(\bar C)$ changes to

$$\mathcal{H}:\quad \Phi(\bar C)\to\Phi(\bar C)' = \Phi(\bar C)+2(\phi(1),\lambda_1)-2(\phi(0),\lambda_1) = \Phi(\bar C)+4\pi k,\quad k\in \mathbf{Z}$$

since for closed paths eq.(2.47) implies

$$\exp\left(i(\phi(1),\lambda_1)\right)=\exp\left(i(\phi(0),\lambda_1)\right)$$

Therefore the phase exponential of the action $\exp(-iJ\Phi(\bar C))$ is nonmanifestly gauge invariant provided that the quantization condition

$$2J\in\mathbf{Z} \qquad (2.65)$$

is fulfilled. Indeed eq.(2.61) implies that the spin J of the Dirac fermions is one half, $J=\frac{1}{2}$.

2.7. Integration measure in terms of harmonic coordinates

To complete the calculation of the spinor functional (2.44), one has to determine the integration measure $d\mu(y,z)$, defined in eq.(2.37). The measure $d\mu$ and vector k_μ depend on the variables y and z_α, $\bar z_\alpha$. It turns out to be more useful to replace the variables $z_\alpha,\bar z_\alpha$ by the variables $z_j^{(i)},\bar z_j^{(i)}$, $i\neq j$ introduced in sect.2.5.2.To calculate the measure $d\mu(y,z)$ in terms of variables $z_j^{(i)}$, we try to represent the metric in the space $\mathbf{R}^{4d^2-1}$ in the form

$$ds^2=\frac{1}{2d}\mathrm{Tr}\,(dk_a\Gamma^a)^2=g^{AB}d\xi_A d\xi_B,\quad A,B=1,2,\ldots,4d^2-1 \qquad (2.66)$$

where $\xi_A=\xi_A(y,z_j^{(i)})$ are independent curved coordinates and g^{AB} is the metric in these coordinates. Then the integration measure is expressed as

$$d^{4d^2-1}k=g^{1/2}\prod_{A=1}^{4d^2-1}d\xi_A$$

where $g=\det|g_{AB}|$.

After substitution of (2.8a) into eq.(2.66) one gets:

$$ds^2=\frac{1}{2d}\left(\mathrm{Tr}\,(dy,H)^2-\mathrm{Tr}\,[iD^{-1}(z)dD(z),(y,H)]^2\right)$$

The Hermitian matrix $iD^{-1}(z)dD(z)$ is an element of the $su(2d)$ Lie algebra and it may be decomposed in the Cartan-Weyl basis as

$$iD^{-1}(z)dD(z)=\sum_{\alpha>0}(d\xi_\alpha E_\alpha+d\bar\xi_\alpha E_{-\alpha})+(d\zeta,H)$$

With the last relation we have:

$$ds^2=\frac{1}{2d}\left((dy,dy)+2\sum_{\alpha>0}(\alpha,y)\,d\xi_\alpha\,d\bar\xi_\alpha\right) \qquad (2.67)$$

Comparing eqs.(2.66) and (2.67) one concludes that the curved coordinates are:

$$\xi_A = (y, \xi_\alpha, \bar{\xi}_\alpha), \qquad \alpha > 0$$

The metric g^{AB} has a block structure in these coordinates and the integration measure is:

$$\begin{aligned} d^{4d^2-1}k &= \text{const}\left(dy \prod_{\alpha>0}(\alpha, y)^2\right)\left(\prod_{\alpha>0} d\xi_\alpha\, d\bar{\xi}_\alpha\right) \\ &\equiv \text{const}\; d\mu(y)\, d\mu(z) \end{aligned} \tag{2.68}$$

The curved coordinates ξ_α, $\bar{\xi}_\alpha$, $(\alpha = e_i - e_j)$ obey equations:

$$\begin{aligned} d\bar{\xi}_\alpha &= \text{Tr}\,(iD^{-1}(z)dD(z)E_\alpha) = \langle j|iD^{-1}(z)dD(z)|i\rangle \\ d\xi_\alpha &= \text{Tr}\,(iD^{-1}(z)dD(z)E_{-\alpha}) = \langle i|iD^{-1}(z)dD(z)|j\rangle \end{aligned} \tag{2.69}$$

where the explicit form (2.7) of the step operators

$$E_\alpha = |i\rangle\langle j| \qquad \text{for } \alpha = e_i - e_j$$

is taken into account. Eq.(2.69) sets up the connection between variables ξ_α, $\bar{\xi}_\alpha$ and coordinates $z_j^{(i)}$ of the harmonics $u^{(i)}$. Before resolving this connection, consider the properties of the integration measure.

2.7.1. The properties of the integration measure

There is an important consequence of eq.(2.68): the measure $d\mu(y,z)$ is a product of the integration measures over variables y and $z_j^{(i)}$. The measure $d\mu(y)$ is expressed as:

$$d\mu(y) = dy \prod_{\alpha>0}(\alpha, y)^2 = d^{2d}y\,\delta\left((y, \sum_{i=1}^{2d} e_i)\right) \prod_{\alpha>0}(\alpha, y)^2 = dy_1 \cdots dy_{2d}\,\delta(y_1 + \cdots + y_{2d}) \prod_{i\neq j}(y_i - y_j) \tag{2.70}$$

where the δ-function takes into account that vector y lies in the subspace $\mathbf{R}^{2d-1}$ orthogonal to the vector $\sum_{i=1}^{2d} e_i$. With eq.(2.69) we have for the measure $d\mu(z)$:

$$d\mu(z) = \prod_{\alpha>0} d\xi_\alpha\, d\bar{\xi}_\alpha = \prod_{i\neq j}\langle i|iD^{-1}(z)dD(z)|j\rangle = \prod_{i\neq j} \bar{u}_k^{(i)}\, du_k^{(j)} \tag{2.71}$$

It is important for us that there is a group of transformations of variables y and $z_j^{(i)}$ that retains measures $d\mu(y)$ and $d\mu(z)$ unchanged. First of all, the measures are invariant under transformations of the Weyl group according to eq.(2.38). This property may be easily verified with the use of eqs.(2.70) and (2.71).

Expression (2.70) is manifestly invariant under the permutation of variables y_i and y_j, and with eq.(2.18) this means that:

$$\mathcal{W}: \quad d\mu(y) \to d\mu(\sigma_\alpha(y)) = d\mu(y), \qquad \alpha = e_i - e_j \tag{2.72a}$$

The consideration of measure $d\mu(z)$ is similar:

$$\mathcal{W}: \quad d\mu(z) \to d\mu(z_{,\alpha}) = \prod_{i\neq j}\langle i|S_\alpha^{-1}\, iD^{-1}(z)dD(z)\, S_\alpha|j\rangle = d\mu(z) \tag{2.72b}$$

since at $\alpha = e_i - e_j$ an element of the Weyl group S_α acting on the weight vectors permutes the states $|i\rangle$ and $|j\rangle$.

Besides the Weyl group there are two subgroups of transformations that leave the measure $d\mu(z)$ unchanged. The integration measure $d\mu(z)$ is invariant under the gauge transformations of the Cartan subgroup:

$$\begin{aligned} \mathcal{H}: \quad D(z) &\to D(z)\exp(i(\phi(z), H)) \\ d\mu(z) &\to d\mu(z) \end{aligned} \tag{2.73}$$

where $\phi(z)$ is an arbitrary vector in the space $\mathbf{R}^{2d-1}$, whose components depend on the variables $z_j^{(i)}$. We have from eq.(2.69):

$$\mathcal{H}: \quad d\xi_\alpha \to d\xi'_\alpha = d\xi_\alpha \exp(i(\phi, \lambda_j - \lambda_i)), \qquad \alpha = e_i - e_j$$

and $d\mu(z)$ is invariant due to the measure being real-valued.

The integration measure $d\mu(z)$ is invariant under the following transformations:

$$\begin{aligned} \mathcal{G}: \quad D(z) &\to GD(z) \\ d\mu(z) &\to d\mu(z) \end{aligned} \tag{2.74}$$

since curved coordinates ξ_α are unchanged in that case.

2.7.2. Preliminary calculation of the measure

Had we had the expression for measure $d\mu(z)$ in terms of variables $z_j^{(i)}$, the invariance property (2.74) and eq.(2.54) would imply that $d\mu(z)$ as a function of $z_j^{(i)}$ is invariant under the projective transformations:

$$\begin{aligned} G: \quad z^{(i)} &\to (z_G^{(i)})_\alpha = \frac{G_{\alpha i} + G_{\alpha\beta} z_\beta^{(i)}}{G_{ii} + G_{i\beta} z_\beta^{(i)}} \\ d\mu(z^{(i)}) &\to d\mu(z_G^{(i)}) = d\mu(z^{(i)}), \qquad i = 1, 2, \ldots, 2d \end{aligned} \tag{2.75}$$

This is, in fact, the functional equation for $d\mu(z_j^{(i)})$. To solve it, one has to keep in mind relations (2.51)-(2.53) among variables $z_j^{(i)}$. If these variables are independent, the solution of eq.(2.75) normalized by condition $1 = \int d\mu(z^{(i)})$ has the form:

$$d\mu(z^{(i)}) = \prod_{i=1}^{2d} d\mu_0(z^{(i)}) \tag{2.76}$$

where $d\mu_0$ is a G-invariant measure on the manifold $SU(2d)/U(2d-1)$, defined in eq.(2.56). After the resolution of constraints (2.51) and (2.52) the number of integration variables in eq.(2.76) is reduced from $4d(2d-1)$ to $2d(2d-1)$. The same result may be achieved by the insertion of the additional δ-functions whose arguments are constraints into the right-hand side of eq.(2.76):

$$d\mu(z^{(i)}) = \prod_{i=1}^{2d-1} d\mu_0(z^{(i)}) \prod_{i\neq j=1}^{2d-1} \delta(\bar{u}^{(i)}u^{(j)}) \tag{2.77}$$

There are no variables $z_j^{(2d)}$ in eq.(2.77) since it follows from eq.(2.52) that they are functions of the remaining variables $z_j^{(i)}$, $i \leq 2d-1$.

It may be verified that expression (2.77) obeys eqs.(2.73) and (2.74). We will prove in the next section that the integration measure is given by eq.(2.77).

2.7.3. The proof of eq.(2.77)

Let us single out the following factor from the general expression (2.71) for measure $d\mu(z)$:

$$d\mu_1(z) = \prod_{j=2}^{2d} d\xi_{\alpha=e_1-e_j} d\bar{\xi}_{\alpha=e_1-e_j} \tag{2.78}$$

The variables $d\bar{\xi}_\alpha$ are found from eq.(2.69):

$$d\bar{\xi}_{\alpha=e_1-e_j} = i\bar{u}_i^{(j)} du_i^{(1)} = iN_1 \bar{u}_\alpha^{(j)} dz_\alpha^{(1)}, \quad \alpha = 2,3,\ldots,2d$$

where eq.(2.53) is used. With this relation one can replace the integration variables ξ_α, $\bar{\xi}_\alpha$ in eq.(2.78) by $z_\alpha^{(1)}$, $\bar{z}_\alpha^{(1)}$ to obtain for measure $d\mu_1(z)$ the expression:

$$d\mu_1(z) = (N_1\bar{N}_1)^{2d-1} \prod_{\alpha=2}^{2d-1} dz_\alpha^{(1)} d\bar{z}_\alpha^{(1)} \det|\bar{u}_\beta^{(\alpha)}| \det|u_\delta^{(\gamma)}|$$

where the determinant is taken from the matrices whose elements are equal to $\bar{u}_\beta^{(\alpha)}$, $\alpha,\beta \geq 2$ and $u_\delta^{(\gamma)}$, $\gamma,\delta \geq 2$, respectively. After simple transformations one has:

$$\det|\bar{u}_\beta^{(\alpha)}| \det|u_\delta^{(\gamma)}| = \det|\bar{u}_\beta^{(\alpha)} u_\delta^{(\alpha)}| = \det|\delta_{\beta\delta} - \bar{u}_\beta^{(1)} u_\delta^{(1)}|, \quad \beta,\delta \geq 2$$

$$= 1 - \sum_{\beta=2}^{2d} \bar{u}_\beta^{(1)} u_\beta^{(1)} = N_1\bar{N}_1 = (1 + \bar{z}^{(1)} z^{(1)})^{-1}$$

where eqs.(2.51) and (2.53) are taken into account.

Thus the normalized factor $d\mu_1(z)$ entering into expression (2.71) for measure $d\mu(z)$ is given by:

$$d\mu_1(z) = (2d-1)! \prod_{i=2}^{2d} \frac{dz_i^{(1)} d\bar{z}_i^{(1)}}{2\pi i} \frac{1}{(1+\bar{z}^{(1)} z^{(1)})^{2d}} = d\mu_0(z^{(1)})$$

and it is identical with the G-invariant measure (2.56) on the manifold $SU(2d)/U(2d-1)$.

In an analogous manner the factor $d\mu_i(z)$ may be calculated differing from eq.(2.78) only by the replacement of $\alpha = e_1 - e_j$ and $z^{(1)}$ by $\alpha = e_i - e_j$ and $z^{(i)}$, respectively. We note that the integration measure $d\mu(z)$ is not equal to the product $\prod_{i=1}^{2d-1} d\mu_i(z)$ since

$$\prod_{i=1}^{2d-1} d\mu_i(z) = \prod_{i=1}^{2d-1} \prod_{i\neq j=1}^{2d} d\xi_{ij} d\bar{\xi}_{ij} = \left(\prod_{j>i=1}^{2d} d\xi_{ij} d\bar{\xi}_{ij}\right) \left(\prod_{i>j=1}^{2d-1} d\xi_{ij} d\bar{\xi}_{ij}\right) \tag{2.79}$$

where $d\xi_{ij} = d\xi_{\alpha=e_i-e_j}$ and there is an extra factor in the right-hand side of this relation. To get rid of it, eq.(2.79) is multiplied by $2d(2d-1)$ additional δ-functions:

$$d\mu(z) = \prod_{i=1}^{2d-1} d\mu_i(z) \left(\prod_{i>j=1}^{2d-1} \delta(\xi_{ij} - \bar{\xi}_{ji}) \delta(\bar{\xi}_{ij} - \xi_{ji})\right)$$

To understand the meaning of δ-functions, one considers their arguments:

$$d(\xi_{ij} - \bar{\xi}_{ji}) = i\bar{u}^{(i)} du^{(j)} + id\bar{u}^{(i)} u^{(j)} = id\left(\bar{u}^{(i)} u^{(j)}\right)$$

and therefore

$$\xi_{ij} - \bar{\xi}_{ji} = i\bar{u}^{(i)} u^{(j)} + \text{const.}$$

With an arbitrary constant chosen equal to zero the arguments of the δ-functions coincide with the orthogonality condition of harmonics (2.51) and then for the integration measure $d\mu(z)$ we get expression (2.77).

Now we substitute eqs.(2.70) and (2.77) into eq.(2.68) to obtain the final expression for the integration measure $d^{4d^2-1}k$ in terms of variables y and $z_j^{(i)}$:

$$d^{4d^2-1}k = \text{const } d^{2d}y\, \delta\left((y, \sum_{i=1}^{2d} e_i)\right) \prod_{\alpha>0} (\alpha, y)^2 \prod_{i=1}^{2d-1} d\mu_0(z^{(i)}) \prod_{i\neq j=1}^{2d-1} \delta(\bar{u}^{(i)} u^{(j)}) \tag{2.80}$$

where $d\mu_0(z^{(i)})$ is defined in eq.(2.56).

In the special case $D = 3$ (or $d = 1$) eq.(2.80) reduces to the well-known expression for integration measure in terms of the coordinates of stereographic projection:

$$d^3k = \text{const } dy_1\, dy_2\, \delta(y_1 + y_2)\, (y_1 - y_2)^2 \frac{dz\, d\bar{z}}{(1+\bar{z}z)^2} = 4\pi\, dy_1\, y_1^2 \frac{dz\, d\bar{z}}{2\pi i} \frac{1}{(1+\bar{z}z)^2}$$

3. Summary

Now we have all necessary relations (2.44),(2.68) and (2.61) to evaluate the dimensionally extended spinor functional (2.3a)

$$\mathcal{M}_{2d}[\dot{x}] = \int_\Omega \mathcal{D}\mu(y)\, \mathcal{D}\mu(z)\, |1, z(T)\rangle\langle 1, z(0)| \exp\left(-\frac{i}{2d} \int_0^T dt \sum_{i=1}^{2d} y_i e_a^{(i)}(z) \dot{x}_a + i\int_0^T dt\, y_1 - \frac{i}{2}\Phi(C)\right) \tag{3.1}$$

where the spinor factor $\Phi(C)$ is defined in eqs.(2.62) and the integration measure is

$$\mathcal{D}\mu(y)\, \mathcal{D}\mu(z) = \lim_{N\to\infty} \prod_{i=1}^{N} d\mu(y(iT/N))\, d\mu(z(iT/N))$$

and measures $d\mu(y)$ and $d\mu(z)$ are given by eqs.(2.70) and (2.77). Eq.(3.1) expresses the spinor functional $\mathcal{M}_{2d}[\dot{x}]$ as a sum over all y-paths on the region Ω of the root space $\mathbf{R}^{2d-1}$ and all z-paths on the space CP^{2d-1}. At $D = 3$ the analogous relation (3.25a) has been obtained where due to the isomorphism $CP^1 \simeq S^2$ the summation is taken over all the paths on the sphere S^2.

Let us examine gauge invariant properties of the spinor functional. Note that the transformation properties of the Wess-Zumino term and integration measures were found in eqs.(2.63),(2.64), (2.72),(2.73) and (2.74). It follows from eqs.(2.73),(2.47) and (2.30) that under the action of the stationary subgroup the integration measures and functions $e_a^{(i)}$ are both invariant but the Wess-Zumino term is nonmanifestly invariant provided that the spin of fermions has quantized values (2.65). At the same time the integration measures and the Wess-Zumino term are invariant but function $e_a^{(i)}$ is not invariant under action (2.63) of the group $SU(2d)$. As a result, the integration in (3.1) over complex variables simply extracts the singlet component of the integrand.

Comparing eqs.(2.3) and (3.1) we conclude that all the spinor structure of the original expression (2.3) for the spinor functional $\mathcal{M}_{2d}[\dot{x}]$ is absorbed by the one-dimensional Wess-Zumino term. Moreover eq.(3.1) may be easily obtained from eq.(2.3) after replacement of momentum k_a and Γ_a-matrices by expression (2.29) and the c-number functions $e_a^{(i)}(z)$ defined in eq.(2.30), respectively, and addition of the one-dimensional Wess-Zumino term into the exponent of (2.3b).

There exists a classical mechanics on the space $CP^{2d-1} \simeq SU(2d)/U(2d-1)$ [6] with the action being equal to the spin factor $\Phi(C)$. The Poisson bracket for this mechanics is defined by the closed 2-form (2.57) and in terms of the local coordinates $z^{(1)}$ it is [10]

$$\{\ ,\ \}_{P.B.} = 2ig_{i\bar{j}}(z,\bar{z})\left(\frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_j} - \frac{\partial}{\partial \bar{z}_j}\frac{\partial}{\partial z_i}\right)$$

where $z \equiv z^{(1)}$, $\bar{z} \equiv \bar{z}^{(1)}$ and metric $g_{i\bar{j}}(z,\bar{z})$ is inverse to the metric defined in eq.(2.55a). As a result, under the geometrical quantization [10] the commutation relations for the variables $e_a^{(1)}(z)$ reproduce the commutation relations of the $su(2d)$ Lie algebra of Γ^a matrices and the consistency condition (2.65) of the underlying quantized dynamics leads to the quantized values for the spin of fermions. Thus the appearance of the one-dimensional Wess-Zumino term in the exponent of eq.(3.1) is by no means accidental and it is one of the effects of the quantum geometry of Dirac fermions.

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