объединенный
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ядерных
исследований
дубна

E2-89-498

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QUANTUM GEOMETRY OF THE DIRAC FERMIONS

Definition of the Spinor Functional

Submitted to "Nuclear Physics B"

1989

Корчемский Г.П. E2-89-498

Квантовая геометрия дираковских фермионов
Определение спинорного функционала

Для описания дираковских фермионов, взаимодействующих с неабелевым калибровочным полем в D-мерном евклидовом пространстве-времени в работе развивается формализм бозонных интегралов по путям. Получены представления для эффективного действия и корреляционных функций фермионов в виде суммы по путям в комплексном проективном пространстве CP^{2d-1} $(d = 2^{[D/2]-1})$, в которых вся спинорная структура поглощается одномерным членом Весса-Зумино. Именно весс-зуминовский член обеспечивает все необходимые свойства фермионов при квантовании: квантованные значения спина, уравнение Дирака, Ферми-статистику и т.д.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1989

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Quantum Geometry of the Dirac Fermions.
Definition of the Spinor Functional

The bosonic path integral formalism is developed for Dirac fermions interacting with a nonabelian gauge field in the D-dimensional Euclidean space-time. The representation for the effective action and correlation functions of interacting fermions as sums over all bosonic paths on the complex projective space CP^{2d-1}, $d = 2^{[D/2]-1}$ is derived where all spinor structure is absorbed by the one dimensional Wess-Zumino term. It is the Wess-Zumino term that ensures all necessary properties of Dirac fermions under quantization, i.e., quantized values of spin, Dirac equation, Fermi statistics.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1989

1. Introduction

It was realized in the last years [1,2,3,4] that one of the novel properties of $D=2$ and $D=3$ dimensional quantum field theories is the transmutation of statistics of elementary particles, i.e., under nontrivial interaction bosons become fermions and vice versa. Recently Polyakov [2] has suggested that in a three-dimensional gauge CP^1-model with the Chern-Simons form for the kinetic term of the gauge field fermi-boson transmutation occurs. The original boson excitations are dressed by the gauge field to transform into free Dirac fermions. Earlier, similar theories with the Chern-Simons action were studied [5] and the appearance of the soliton vortices [6] with anomalous spin was noticed. Unlike the well-known two-dimensional nonabelian bosonization [1] the equivalence between three-dimensional interacting bosons and free Dirac fermions is not exact and it holds only in the limit of low momenta [2,4].

To carry out the proof Polyakov [2] has supposed that the propagator of the free Dirac electron in $D=3$ Euclidean space-time may be represented as a bosonic path integral (its explicit form is given below in sect.1.3.2) that turns out to coincide with the dressed soliton propagator in the gauge CP^1-model in the limit of low momenta.

Do similar phenomena exist in higher dimensions? There were attempts [7,8] to answer the question in the case $D=4$. The present paper is devoted to one aspect of this problem. Namely, only the fermionic part of the problem is analyzed and the final goal of the paper is to obtain the bosonic path integral representation for the correlation functions of Dirac fermions in D-dimensional Euclidean space-time.

We recall that in the standard fashion [9,10] to describe the Dirac fermions, bosonic as well as Grassmann variables have to be introduced to form the path integrals in a superspace corresponding to the spinning particle. This representation is very useful when the Neveu-Schwarz-Ramond string [11] is formulated as a string analogue of Dirac fermions [12] but it is not so when the correspondence with the interacting bosons is established. Therefore one would like to get rid of the Grassmann variables in the description of the Dirac fermions and to replace them by the bosonic degrees of freedom of a spinning particle.

It was Feynman [13] who first noted the possibility of the bosonic path integral representation for two-dimensional Dirac fermions. Later the use of the bosonic path integral formalism for two-dimensional Majorana fermions made it possible to solve the $D=2$ Ising model [14]. The spin correlation functions were expressed [15,16] as a sum over all paths on the sphere S^3. Using the geometric quantization approach (or the coherent state method) [17] the bosonic path integral representation for the propagator of the three-dimensional Dirac electron was found [2,3,8,18,19]. In all these cases the bosonic path integrals for the fermionic correlation functions contain the same additional term. This term is a geometric characteristic of paths [2] and it is well-known as a one-dimensional Wess-Zumino term obtained by Witten and Novikov [1,20]. It is a Wess-Zumino term that ensures all necessary properties of Dirac fermions under canonical quantization of the action of a spinning particle [2,16,21,22,23], i.e. Dirac equation, Fermi statistics, quantized values of spin and so on. Note that both approaches (the Grassmann and bosonic path integrals) give equivalent descriptions of a spinning particle and correspondence between them was established [22].

In the present paper, the formalism of the bosonic path integrals is developed for interacting Dirac fermions in D-dimensional Euclidean space-time. The paper is organized as follows. All necessary definitions and a brief review of existing results on the subject are given in the remaining part of sect.1. Here the representation for the effective action and propagator of fermions as a sum over all paths in x-space is derived where the spinor functional is the only unknown quantity. The spinor functional is analyzed in sect.2 after a proper regularization and approximation. It turns out that two space-time dimensions $D=2,3$ play a special role and the corresponding spinor functionals are found in sect.3.

In the case $D\geq 4$ the direct calculation of the spinor functional is reduced to the evaluation of an infinite product of Dirac matrices. To overcome the problem, the additional transformation of the spinor functional called dimensional extension is performed in ref.[24]. The space-time dimension is changed from D to the dimension of the $su(2d)$, ($d=2^{[D/2]-1}$) Lie algebra with Dirac matrices being its elements. Using some well-known properties of semisimple Lie algebras [25] it is possible to represent the spinor functional as a sum over all bosonic paths on the complex projective space CP^{2d-1}. In ref.[24] the transformation of the spinor functional inverse to the dimensional extension is done and the final bosonic path integral representation for the effective action and propagator of interacting Dirac fermions in D-dimensional Euclidean space-time is derived.

1.1. Dirac fermions in D-dimensional Euclidean space-time

Let us consider, in D-dimensional Euclidean space-time, Dirac fermions with mass M interacting with a nonabelian gauge field $A_\mu = A^a_\mu T^a$ where T^a are some generators of the gauge (or "color") group, whose explicit form is not essential for our purposes. We define the effective action and propagator of interacting fermions as follows:

$$W[A] = \log\det(\hat{D}+M) = \int d^Dx\, \langle x|\mathrm{Tr}\,\log(\hat{D}+M)|x\rangle \tag{1.1}$$

and

$$S(x,y;A) = \langle x|(\hat{D}+M)^{-1}|y\rangle = -i\langle x|(\hat{p}+g\hat{A}+iM)^{-1}|y\rangle \tag{1.2}$$

where $\hat{D} = D_\mu\gamma^\mu$, $D_\mu = \partial_\mu - igA_\mu$ is the covariant derivative, γ^μ are Dirac matrices in D-dimensional Euclidean space-time, Tr refers to color indices of the gauge field and spinor indices of Dirac matrices, $p_\nu = i\partial_\nu$, $[x_\mu, p_\nu] = -ig_{\mu\nu}$ and

$$\langle x|y\rangle = \delta^D(x-y), \quad \langle p|k\rangle = \delta^D(p-k), \quad \langle x|p\rangle = (2\pi)^{-D/2}e^{-i(px)}$$

Using the identity $A^{-1} = -i\int_0^\infty dT\,\exp(iTA)$ one transforms eqs.(1.1) and (1.2) to obtain two different expressions for the fermion propagator:

$$S(x,y;A) = \langle x|(\hat{D}+M)^{-1}|y\rangle = \langle x|\int_0^\infty dT\,\exp(iT(i\hat{D}+iM))|y\rangle \tag{1.3a}$$

and

$$\begin{aligned} S(x,y;A) &= \langle x|(\hat{D}-M)(\hat{D}^2-M^2)^{-1}|y\rangle \\ &= i\langle x|\int_0^\infty dT\,(\hat{D}-M)\exp(iT(-\hat{D}^2+M^2))|y\rangle \\ &= i\langle x|\int_0^\infty dT\,(\hat{D}-M)\exp\left(iT(-D_\mu^2+\frac{1}{2}gF_{\mu\nu}\sigma_{\mu\nu}+M^2)\right)|y\rangle \end{aligned} \tag{1.3b}$$

where $F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu]$ is the strength tensor of the gauge field and $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu,\gamma_\nu]$.

Matrix elements entering into the right-hand side of eqs.(1.3a) and (1.3b) have the form $\langle x|e^{iTH}|y\rangle$ and may be treated as matrix elements of the evolution operator of a spinning particle with T and H being the proper time and hamiltonian of the particle, respectively. The hamiltonian H has both spinor and color indices and, as a result, the spinning particle has a new quantum degree of freedom. Following Feynman [13] one can represent $\langle x|e^{iTH}|y\rangle$ as a path integral over the phase space of a spinning particle. It is well-known [26] that if one starts from eq.(1.3b), one finds in this way a representation for the fermion propagator as a sum over random paths in the superspace with even (or c-number) and odd (Grassman) coordinates [9,10]. We will demonstrate in this paper that it is just eq.(1.3a) that allows us to describe Dirac fermions without the use of Grassman variables.

1.2. Random walk in superspace

For the sake of simplicity we assume in this section that the gauge field is Abelian and rewrite eq.(1.3b) as a path integral in superspace.

Splitting the interval $[0,T]$ into N equal pieces and inserting the completeness relations

$$1 = \int d^Dp\,|p\rangle\langle p| = \int d^Dx\,|x\rangle\langle x| \tag{1.4}$$

in a proper manner one obtains from eq.(1.3b) that in the limit $N \to \infty$

$$\begin{aligned} S(x,y,A) = \int_0^\infty dT\, \exp(iTM^2) \int_y^x \mathcal{D}x_\mu \mathcal{D}p_\mu\, (\hat{p}(T) + g\hat{A}(x) - iM) \\ \times P\exp\left(i\int_0^T dt\Big(-p\dot{x} + (p+gA(x))^2 + \frac{1}{2}g(F\sigma)\Big)\right) \end{aligned} \tag{1.5}$$

where integration is performed over all x-space paths between the points $x_\mu(0) = y_\mu$ and $x_\mu(T) = x_\mu$ and all unrestricted p-space paths; P denotes the Dyson path ordering defined as

$$P\exp\left(\int_0^T dt\, O(t)\right) = \lim_{N\to\infty} e^{\tau O(N\cdot\tau)}\cdots e^{\tau O(2\cdot\tau)}e^{\tau O(1\cdot\tau)}, \qquad \tau = \frac{T}{N} \tag{1.6}$$

for an arbitrary operator O. The momentum path integral in (1.5) is Gaussian and is easily calculated. The integrand of the resulting expression is a matrix. To get rid of the matrix structure we introduce additional Grassman variables ψ_μ, ψ_5 and χ and a bosonic "einbein" field $e(t)$ [9,10] using identities like

$$\mathrm{Tr}\, P\exp\left(\frac{i}{2}g\int_0^T dt(\sigma F(t))\right) = \int \mathcal{D}\psi_\mu \exp\left(\int_0^T dt\psi_\mu\dot{\psi}_\mu - \frac{1}{2}g\int_0^T dt\,\psi_\mu\psi_\nu F_{\mu\nu}\right) \tag{1.7}$$

The final expression for the fermion propagator has the form [9,10]

$$\begin{aligned} S(x,y;A) = \int \mathcal{D}x_\mu\mathcal{D}\psi_\mu\mathcal{D}\chi\mathcal{D}e\mathcal{D}\psi_5 \exp\left(i\int_0^1 dt\Big(g\dot{x}_\mu A_\mu(x) + \frac{i}{2}geF_{\mu\nu}\psi_\mu\psi_\nu\Big)\right) \\ \times \exp\left(i\int_0^1 dt\left[\frac{\dot{x}^2}{4e} - i\psi_\mu\dot{\psi}_\mu - i\psi_5\dot{\psi}_5 + \frac{i}{2}\chi\psi_\mu\dot{x}_\mu - Me\chi\psi_5 + M^2e\right]\right) \end{aligned}$$

It describes a random walk of a spinning particle in superspace with the action invariant under both local supersymmetric and general coordinate transformations.

Let us investigate another representation, (1.3a), for the fermion propagator.

1.3. Evolution operator

We try to express the effective action and propagator of interacting fermions as a sum over all paths in some space with c-number coordinates. Natural questions arise: what are the properties of this space and what kind of action corresponds to a spinning particle in this space? There are two examples that allow us to answer the above questions for two special values of the space-time dimension: $D = 2$ and $D = 3$.

1.3.1. $D = 2$ Ising model

The first example is related to the two-dimensional Ising model on a square lattice [27]. It is well-known [14] that in the vicinity of the phase transition point $T = T_c$ the Ising model is equivalent to the theory of free Majorana fermions. As a result, we have two equivalent representations for the partition function of the model as $T \to T_c$:

$$\log Z_{2D\ IM}(M) = -\frac{1}{2}\mathrm{Tr}\,\log(\hat{\partial} + M) = -\sum_P \exp(-ML(P))(-1)^{\nu(P)+1} \tag{1.8}$$

where the sum is taken over all closed paths P on the lattice, L is the length of the path, ν is the number of self-intersections of P, M is the mass parameter proportional to $(T - T_c)$. Eq.(1.8) means that in the lattice approximation the effective action of free fermions is a sum over all loops in the x-space with an additional spinor factor.

1.3.2. $D = 3$ free fermions

The second example has been given by Polyakov [2] who has supposed that the propagator of a free Dirac electron in three-dimensional space-time is given in the lattice approximation by the following sum over paths between points y and x:

$$S(x-y) \propto \sum_{P_{xy}} \exp(-ML(P_{xy}))\, \exp\left(-\frac{i}{2}\Phi(P_{xy})\right) \tag{1.9}$$

where the spin factor is

$$\Phi(P_{xy}) = \int_0^L ds \int_0^1 du\, \mathbf{e}\cdot\left[\frac{\partial\mathbf{e}}{\partial u}\times\frac{\partial\mathbf{e}}{\partial s}\right]$$

and the interpolating field is introduced

$$\mathbf{e}(s,u) = \begin{cases} \mathbf{e}(s) & , \quad u = 1 \\ \text{const.} & , \quad u = 0 \end{cases}$$

with $\mathbf{e}(s)$ being the tangent field for path P_{xy}. In the continuum limit eq.(1.9) is expected to reduce to

$$S(x-y) \propto \int_0^\infty dL\, \exp(-ML)\int \mathcal{D}\mathbf{e}\,\delta(\mathbf{e}^2 - 1)\exp\left(-\frac{i}{2}\Phi(\mathbf{e})\right)\delta\left(\mathbf{x} - \mathbf{y} - \int_0^L ds\,\mathbf{e}\right)$$

where integration is performed over all paths on the sphere S^2. An analogous representation was suggested also in ref.[19].

With these examples in mind we return to eqs.(1.1) and (1.2) for the effective action and propagator of interacting fermions.

1.3.3. Evolution operator as a path integral

Let us consider the function [28]

$$U(x,y;T)=\langle x|e^{iT(i\hat{D})}|y\rangle$$

and treat it as a matrix element of the evolution operator of a particle with hamiltonian $H=i\hat{D}$. The function $U(x,y;T)$ is an amplitude for a particle to go from point y to point x in proper time T. The effective action and propagator of the fermion are expressed in terms of the evolution operator as follows

$$S(x,y;A)=\int_0^\infty dT\,e^{-TM}\,U(x,y;T) \tag{1.10}$$

and

$$W[A]=\int_0^\infty \frac{dT}{T}e^{-TM}\int d^Dx\,\mathrm{Tr}\,U(x,x;T) \tag{1.11}$$

We note that in D-dimensional Euclidean space-time $U(x,y;T)$ is a spinor matrix of order $2^{[D/2]}$ and, moreover, it has color indices.

Following ref.[28] let us obtain a representation for the evolution operator $U(x,y;T)$ as a path integral:

$$U(x,y;T)=\langle x|e^{iTH}|y\rangle=\langle x|\left(e^{i\frac{T}{N}H}\right)^N|y\rangle=\lim_{N\to\infty}\langle x|\left(e^{i\frac{T}{N}\hat{p}}e^{i\frac{T}{N}g\hat{A}}\right)^N|y\rangle$$

where the Trotter identity $\lim_{N\to\infty}\left(\exp(\frac{A}{N})\exp(\frac{B}{N})\right)^N=\exp(A+B)$ is used. Now we insert the completeness relations (1.4) into the right-hand side of the last equation

$$U(x,y;T)=\lim_{N\to\infty}\int d^Dx_1\ldots\int d^Dx_{N-1}\int d^Dp_1\ldots\int d^Dp_N\,(2\pi)^{-DN}$$
$$\times e^{i\frac{T}{N}\hat{p}_N}e^{-ip_N(x-x_{N-1})}e^{i\frac{T}{N}g\hat{A}(x_{N-1})}\ldots e^{i\frac{T}{N}\hat{p}_1}e^{-ip_1(x_1-y)}e^{i\frac{T}{N}g\hat{A}(y)}$$

In the limit $N\to\infty$ this expression turns into the path integral

$$U(x,y;T)=\int_y^x \mathcal{D}x_\mu\mathcal{D}p_\mu\exp\left(-i\int_0^T dt\,p(t)\dot{x}(t)\right)P\exp\left(i\int_0^T dt\left(\hat{p}(t)+g\hat{A}(x(t))\right)\right)$$

where integration is performed over all unrestricted momentum paths and all x-paths between the points $x_\mu(0)=y_\mu$ and $x_\mu(T)=x_\mu$

$$\int_y^x\mathcal{D}x_\mu=\int\mathcal{D}x_\mu\,\delta(x(0)-y)\,\delta(x(T)-x)$$

and the P-exponential defined in eq.(1.6) orders color and Dirac matrices. Now we shift the momentum integration variable $p_\mu\to p_\mu-gA_\mu$ to get the final expression for the evolution operator [28,29]

$$U(x,y;T)=\int_y^x\mathcal{D}x_\mu P\exp\left(ig\int_0^T dt\,\dot{x}_\mu(t)A_\mu(x)\right)\mathcal{M}_D[\dot{x}] \tag{1.12}$$

where the momentum functional integral is factorized

$$\mathcal{M}_D[\dot{x}]=\int\mathcal{D}p_\nu\exp\left(-i\int_0^T dt\,p(t)\dot{x}(t)\right)P\exp\left(i\int_0^T dt\,\hat{p}(t)\right) \tag{1.13}$$

$\mathcal{M}_D$ does not depend on a gauge field and fermion mass M. It accumulates all spinor structure of the evolution operator and is called the *spinor functional*.

The dependence of the evolution operator on the gauge field is contained completely in the path ordered exponential whose gauge transformation properties imply that

$$A_\mu(x)\;\to\;A_\mu^G(x)=G(x)(A_\mu(x)+\frac{i}{g}\partial_\mu)G^{-1}(x)$$
$$U(x,y;T)\;\to\;U^G(x,y;T)=G(x)U(x,y;T)G^{-1}(y)$$

Finally, we substitute (1.12) into eqs.(1.10) and (1.11) to derive the following representations [28,29,30]:

$$S(x,y;A)=\int_0^\infty dT\,e^{-TM}\int_y^x\mathcal{D}x_\mu\exp\left(ig\int_y^x dx_\mu A_\mu(x)\right)\mathcal{M}_D[\dot{x}] \tag{1.14}$$

and

$$W[A]=\int_0^\infty\frac{dT}{T}e^{-TM}\int\mathcal{D}x_\mu\,\delta(x(0)-x(T))\,\mathrm{Tr}\,P\exp\left(ig\oint dx_\mu A_\mu(x)\right)\mathrm{Tr}\,\mathcal{M}_D[\dot{x}] \tag{1.15}$$

Thus the calculation of the spinor functional is a final goal of the present paper.

2. Spinor functional

The spinor functional defined in eq.(1.13) has some unusual properties. Its formal calculation yields the matrix δ-function: $\delta(\dot{x}_\mu(t)-\gamma_\mu)$ but this result has been shown [28] to be wrong. Unlike the analogous integral in (1.5) the momentum functional integral in (1.13) is ill-defined and it must be properly regularized for large values of momenta. One of the regularization prescriptions proposed in [29] is the insertion of the cut-off factor

$$\exp\left(-\int_0^T dt\epsilon(t)\sqrt{p_\mu^2}\right),\quad \epsilon(t)\to 0$$

into the right-hand side of (1.13). Then the regularized spinor functional is

$$\mathcal{M}_D[\dot{x}]=\lim_{\epsilon(t)\to 0}\mathcal{M}_{reg}[\dot{x}]$$

where

$$\mathcal{M}_{reg}[\dot{x}]=\int\mathcal{D}p_\nu\exp\left(-i\int_0^T dt\,p(t)\dot{x}(t)-\int_0^T dt\,\epsilon(t)|p(t)|\right)P\exp\left(i\int_0^T dt\,\hat{p}(t)\right)$$

and $|p(t)|^2=p_\mu(t)p_\mu(t)$.

2.1. Approximation of the spinor functional

To calculate the spinor functional we split interval $[0,T]$ into N equal pieces $\tau=\frac{T}{N}$ and define $\mathcal{M}_D[\dot{x}]$ to be given by the following limiting procedure

$$\mathcal{M}_D[\dot{x}]=\lim_{\epsilon(t)\to 0}\lim_{N\to\infty}\mathcal{M}_{reg}(x_N)\cdots\mathcal{M}_{reg}(x_2)\mathcal{M}_{reg}(x_1) \tag{2.1a}$$

where

$$\mathcal{M}_{reg}(x) = \int d^D k \exp(-i(kx) + i\hat{k}\tau - \varepsilon|k|\tau) \tag{2.1b}$$

and $\tau \to 0$. $x_i = \tau\dot{x}(\tau i) \to 0$ in the limit $N \to \infty$ and $\dot{x}_\mu =$ fixed. The relative order of factors in (2.1a) is essential since $\mathcal{M}_{reg}(x)$ is a matrix of order $2^{[D/2]}$.

We rewrite eq.(2.1b) as

$$\mathcal{M}_{reg}(x) = \int d^D k \exp(-i(kx) - \varepsilon|k|\tau) \left[\cos(|k|\tau) + i\frac{\hat{k}}{|k|}\sin(|k|\tau)\right] \tag{2.2}$$

The regularization cuts off large values of momenta in this expression:

$$|k| < \frac{1}{\varepsilon\tau}$$

but despite the small values of τ the functions $\cos(|k|\tau)$ and $\sin(|k|\tau)$ are rapidly oscillating in this region and they cannot be approximated by a few first terms in the Taylor expansion. This is a reason for the discrepancy of results of refs.[8,18] and [2].

2.2. Erroneous results

In refs.[8,18] an attempt was made to calculate the spinor functional for two special values of space-time dimension: $D = 3$ and $D = 4$ based on the coherent state approach [17]. In the case $D = 3$ three Dirac matrices γ_μ coincide with Pauli matrices and they can be decomposed in the basis of coherent states for $SU(2)$ group as follows [1]

$$\sigma_a = \int \frac{d^3\mathbf{e}}{\pi} 3e_a |\mathbf{e}\rangle\langle\mathbf{e}|\, \delta(\mathbf{e}^2 - 1), \quad 1 = \int \frac{d^3\mathbf{e}}{\pi} |\mathbf{e}\rangle\langle\mathbf{e}|\, \delta(\mathbf{e}^2 - 1)$$

After substitution of these relations into eq.(2.2) one gets

$$\begin{aligned}
\mathcal{M}_{reg}(x) &= \int \frac{d^3\mathbf{e}}{\pi} |\mathbf{e}\rangle\langle\mathbf{e}|\, \delta(\mathbf{e}^2 - 1) \int d^3k\, e^{-ikx-\varepsilon|k|\tau} \left[\cos(|k|\tau) + 3i\frac{(ke)}{|k|}\sin(|k|\tau)\right] \\
&= \int \frac{d^3\mathbf{e}}{\pi} |\mathbf{e}\rangle\langle\mathbf{e}|\, \delta(\mathbf{e}^2 - 1) \int d^3k\, e^{-ikx-\varepsilon|k|\tau} [1 + 3i(ke)\tau + O(k^2\tau^2)] \\
&\neq \int \frac{d^3\mathbf{e}}{\pi} |\mathbf{e}\rangle\langle\mathbf{e}|\, \delta(\mathbf{e}^2 - 1)\, (2\pi)^3\delta(\mathbf{x} - 3\mathbf{e}\tau), \qquad \varepsilon \to 0
\end{aligned}$$

where the smallness of $O(k^2\tau^2)$ terms is assumed in the last expression proposed in refs.[8,18]. But this assumption is wrong and therefore the results of refs.[8,18] do not upset Polyakov's results.

2.3. Calculation of the spinor functional

Let us transform eq.(2.2) identically

$$\begin{aligned}
\mathcal{M}_{reg}(x) &= \frac{1}{2}\int d^D k\, e^{-ikx-\varepsilon|k|\tau} \left[e^{i|k|\tau}\left(1 + \frac{\hat{k}}{|k|}\right) + e^{-i|k|\tau}\left(1 - \frac{\hat{k}}{|k|}\right)\right] \\
&= \frac{1}{2}\int d^D k\, e^{-\varepsilon|k|\tau} \left[e^{-ikx+i|k|\tau} + e^{ikx-i|k|\tau}\right]\left(1 + \frac{\hat{k}}{|k|}\right)
\end{aligned}$$

[1]The detailed definitions and explicit form of the coherent states will be given in sect.3.3.1.

and single out radial and angular parts of the vector $k_\mu = \lambda e_\mu$

$$d^D k = d\lambda\, \lambda^{D-1}\, d^D e\, \delta(e_\mu^2 - 1), \quad \lambda \geq 0$$

to derive the following relations in terms of variables λ and e_μ:

$$\begin{aligned}
\mathcal{M}_{reg}(x) &= \int_0^\infty d\lambda\, \lambda^{D-1} e^{-\epsilon\lambda\tau} \int d^D e\, \delta(e^2 - 1) \left(e^{i\lambda\tau - i\lambda(ex)} + e^{-i\lambda\tau + i\lambda(ex)}\right) \frac{1 + \hat{e}}{2} \\
&= \int d^D e\, \delta(e^2 - 1) \frac{1 + \hat{e}}{2} \int_{-\infty}^\infty d\lambda |\lambda|^{D-1} e^{-\varepsilon|\lambda|\tau} e^{i\lambda\tau - i\lambda(ex)}
\end{aligned} \tag{2.3}$$

To get rid of the modulus in the last equation one has to consider $\mathcal{M}_{reg}(x)$ for two values of dimension: $D = 2\nu$ and $D = 2\nu + 1$. It follows from (2.1b) that they are related to each other by a simple equation

$$\mathcal{M}_{reg}(x_1, x_2, \ldots, x_{2\nu}) = \frac{1}{2\pi} \int dx_{2\nu+1}\, \mathcal{M}_{reg}(x_1, x_2, \ldots, x_{2\nu}, x_{2\nu+1}) \tag{2.4}$$

where $x_\mu = (x_1, x_2, \ldots, x_{2\nu})$.

For $D = 2\nu + 1$ a simple but tiresome calculation yields

$$\begin{aligned}
\mathcal{M}_{reg}(x) &= \int_{-\infty}^\infty d\lambda\, \lambda^{D-1} e^{-\epsilon|\lambda|\tau} \int d^D e\, \delta(e^2 - 1) \frac{1 + \hat{e}}{2}\, e^{i\lambda\tau - i\lambda(ex)} \\
&= (4\pi)^{\frac{D-1}{2}} \Gamma(\frac{D + 1}{2})[\varepsilon\tau + i(\tau + \hat{x})](x^2 - \tau^2 + 2i\varepsilon\tau^2)^{-\frac{D+1}{2}} + \text{ h.c.}
\end{aligned} \tag{2.5}$$

For $D = 2\nu$ we substitute this expression into eq.(2.4) to find for $\mathcal{M}_{reg}(x)$ the result coinciding with eq.(2.5) if one analytically continues it to odd values of D.

Now we perform the limit $\varepsilon \to 0$ in the regularized expression (2.5) and note that as $x^2 \neq \tau^2$

$$\mathcal{M}_{reg}(x) \propto \varepsilon \to 0 \tag{2.6a}$$

but for $x^2 = \tau^2$ and $D \neq 4\mathrm{Z} + 3$

$$\mathcal{M}_{reg}(x) \propto \varepsilon^{-\frac{D+1}{2}} \tag{2.6b}$$

This means that $\mathcal{M}_{reg}(x)$ has a δ-function singularity at $x^2 = \tau^2$ or $\dot{x}_\mu^2 = 1$. A detailed examination of the limit $\varepsilon \to 0$ of eq.(2.5) gives

$$\begin{aligned}
\mathcal{M}(x) = \lim_{\varepsilon \to 0} \mathcal{M}_{reg}(x) &= \int d^D k \exp(-ikx + i\hat{k}\tau) \\
&= \int_{-\infty}^\infty d\lambda\, \lambda^{D-1} e^{-\varepsilon|\lambda|\tau} \int d^D e\, \delta(e^2 - 1) \frac{1 + \hat{e}}{2} e^{i\lambda\tau - i\lambda(ex)} \\
&= 2^D \pi^{\frac{D+1}{2}} \tau(1 + \frac{\hat{x}}{\tau})\, \delta^{(\frac{D-1}{2})}(\tau^2 - x^2) \\
&= 2^D \pi^{\frac{D+1}{2}} \tau(1 + \frac{\hat{x}}{\tau}) \int_{-\infty}^\infty \frac{d\lambda}{2\pi} (i\lambda)^{\frac{D-1}{2}} e^{i\lambda(\tau^2 - x^2)}
\end{aligned}$$

where the last relation is in fact the definition of the derivative of the δ-function of noninteger order. The validity of these equations may be checked with the use of the following identity

$$\int d^D x\, e^{ipx} \mathcal{M}(x) = 2^D \pi^{\frac{D+1}{2}} \int d^D x\, e^{ipx} \tau(1 + \frac{\hat{x}}{\tau})\, \delta^{(\frac{D-1}{2})}(\tau^2 - x^2) = (2\pi)^D e^{i\hat{p}\tau}$$

Thus, we calculated one of the terms entering into expression (2.1a) for the spinor functional

$$\begin{aligned}\mathcal{M}(x) &= C\int d^De\,\delta(e^2-1)\frac{1+\hat{e}}{2}\delta^{(D-1)}(1-(e\dot{x})),\ \text{even D}\\ &= C\frac{1+\hat{\dot{x}}}{2}\delta^{(\frac{D-1}{2})}(1-\dot{x}_\mu^2),\ \text{arbitrary D}\end{aligned}\qquad(2.7)$$

where C is a normalization factor. After its substitution into eq.(2.1a) one gets the following expressions for the spinor functional in the limit $N\to\infty$

$$\mathcal{M}_D[\dot{x}] = \int \mathcal{D}e_\mu\,\delta(e^2-1)\,\delta^{(D-1)}(1-e\dot{x})\,I[e],\ \text{even D}\qquad(2.8a)$$

$$= \delta^{(\frac{D-1}{2})}(1-\dot{x}_\mu^2)\,I[\dot{x}],\ \text{arbitrary D}\qquad(2.8b)$$

where for an arbitrary D-dimensional vector $n_\mu(t)$, $t\in[0,T]$ we denote

$$I[n]=\lim_{N\to\infty}\frac{1+\hat{n}(N\tau)}{2}\cdots\frac{1+\hat{n}(2\tau)}{2}\frac{1+\hat{n}(\tau)}{2},\qquad \tau=T/N\qquad(2.9)$$

The right-hand side of this equation is an infinite product of matrices of order $2^{[D/2]}$ whose calculation is quite nontrivial without the use of Grassman variables.

Had we had a lattice in the x-space, the definition of the spinor functional would be easy because on the lattice the path

$$P_{xy}=\{x_\mu(t),\ t\in[0,T]|x(0)=y,x(T)=x\}$$

with T being the path length is formed by a finite number of links

$$\mathcal{M}_D[P_{xy}]=\prod_{P_{xy}}\frac{1+\hat{u}}{2}\qquad(2.10a)$$

Here the product is taken along the path $P_{xy}$and $\{u_\mu\}$ are unit vectors forming the path P_{xy} on the lattice. It may be easily recognized that eq.(2.10a) does identically coincide with the spinor factor that appears when one represents the propagator for a free fermion particle as the following sum over the paths on the lattice [31]

$$S(x-y)=\sum_{P_{xy}}\exp(-ML(P_{xy}))\,\mathcal{M}_D[P_{xy}]\qquad(2.10b)$$

where L is the length of the path P_{xy} connecting the points y and x on the lattice.

2.4. Spinor functional as a path integral in superspace

The function $I[n]$ depends on the dimension of the Euclidean space-time and its two values $D=2$ and $D=3$ play a special role. In order to prove it we will transform eq.(2.9) assuming that $n_\mu^2(t)=1$. This condition is fulfilled identically in eqs.(2.8a) and (2.10b). However eq.(2.8b) vanishes as $\dot{x}_\mu^2\neq 1$ but its action on $(\dot{x}^2-1)$ differs from zero

$$(1-\dot{x}^2)\delta^{(\frac{D-1}{2})}(1-\dot{x}^2)=-\frac{D-1}{2}\delta^{(\frac{D-3}{2})}(1-\dot{x}^2)$$

In other words, the strength of the singularity of $(\dot{x}^2-1)\mathcal{M}(x)$ at $\dot{x}^2=1$ (or the power of ε in eq.(2.6b)) is reduced by unity. Nevertheless following ref.[29] we suppose that after substitution of eq.(2.7) into (2.1a) only terms with the maximum order of derivatives of δ-functions give rise to path integrals for the effective action and propagator of fermions. It means that to calculate $I[n]$, it is possible to set $n^2(t)=1$ and neglect $O(n^2-1)$ terms.

The condition $n^2(t)=1$ implies that

$$n_\mu(t)\dot{n}_\mu(t)=0,\qquad \frac{1+\hat{n}(t)}{2}\hat{\dot{n}}(t)=\frac{1+\hat{n}(t)}{2}$$

where $\dot{n}_\mu(t)=\frac{\partial}{\partial t}n_\mu(t)$. Using these properties one transforms the product of two neighbouring terms in eq.(2.9) as

$$\begin{aligned}\frac{1+\hat{n}(t+\tau)}{2}\frac{1+\hat{n}(t)}{2} &= \frac{1+\hat{n}(t+\tau)\hat{n}(t)}{2}\frac{1+\hat{n}(t)}{2}=\left(1+\frac{\tau}{2}\hat{\dot{n}}(t)\hat{n}(t)\right)\frac{1+\hat{n}(t)}{2}+O(\tau^2)\\ &=\left(1+\frac{i}{2}\tau n_\mu(t)\dot{n}_\nu(t)\sigma^{\mu\nu}\right)\frac{1+\hat{n}(t)}{2}+O(\tau^2)=\exp\left(\frac{i}{2}\tau n_\mu(t)\dot{n}_\nu(t)\sigma^{\mu\nu}\right)\frac{1+\hat{n}(t)}{2}+O(\tau^2)\end{aligned}$$

The subsequent application of this relation leads to the following representation for $I[n]$[2]

$$\begin{aligned}I[n] &= \exp\left(\frac{i}{2}\int_0^T dt\,n_\mu(t)\dot{n}_\nu(t)\sigma^{\mu\nu}\right)\frac{1+\hat{n}(0)}{2}\\ &= \frac{1+\hat{n}(T)}{2}\exp\left(\frac{i}{2}\int_0^T dt\,n_\mu(t)\dot{n}_\nu(t)\sigma^{\mu\nu}\right)\end{aligned}\qquad(2.11)$$

It will be demonstrated below that eq.(2.11) is exactly calculable for $D=2$ and $D=3$ but for arbitrary odd values of D there is no other way to deal with P-exponentials than to introduce the Grassman variables according to identity (1.7)

$$\begin{aligned}\mathrm{Tr}\,P\exp\left(\frac{i}{2}\int_0^T dt\,\sigma^{\mu\nu}\omega_{\mu\nu}(t)\right) &= \int\mathcal{D}\psi_\mu\exp\left(\int_0^T dt\,\psi_\mu\dot{\psi}_\mu-\frac{1}{2}\int_0^T dt\,\psi_\mu\psi_\nu\omega_{\mu\nu}\right)\\ &= \left[\det\left(g_{\mu\nu}\frac{\partial}{\partial t}-\frac{1}{2}\omega_{\mu\nu}\right)\right]^{1/2}\end{aligned}$$

where $\omega_{\mu\nu}=\frac{1}{2}(n_\mu\dot{n}_\nu-n_\nu\dot{n}_\mu)\equiv n_{[\mu}\dot{n}_{\nu]}$.

As a result, for odd D we derive the following expression for the spinor functional (2.8b)

$$\mathrm{Tr}\,\mathcal{M}_D[\dot{x}]=\left[\det\left(g_{\mu\nu}\frac{\partial}{\partial t}-\frac{1}{2}\dot{x}_{[\mu}\ddot{x}_{\nu]}\right)\right]^{1/2}\delta^{(\frac{D-1}{2})}(1-\dot{x}^2)$$

Analogous representations were proposed in ref.[12].

[2] Note that the following relation holds

$$\exp\left(\frac{i}{2}\int_0^T dt n_\mu(t)\dot{n}_\nu(t)\sigma^{\mu\nu}\right)=I[n]+I[-n]$$

3. Spinor functional for $D = 2$ and $D = 3$

3.1. Special case: $D = 2$

In the $D = 2$ case Dirac matrices coincide with Pauli matrices σ_1, σ_2 and

$$dt\, \omega_{\mu\nu}(t)\sigma^{\mu\nu} = dt\, \frac{d\varphi}{dt}\sigma_3$$

where $d\varphi$ is an infinitesimal angle between the vectors $n_\mu(t)$ and $n_\mu(t + dt)$. Therefore we have

$$I[n] = \exp\left(\frac{i}{2}\sigma_3 \int_0^T dt\, \frac{d\varphi}{dt}\right) \frac{1 + \hat{n}(0)}{2} = \left(\cos\frac{\varphi(T,0)}{2} + i\sigma_3 \sin\frac{\varphi(T,0)}{2}\right) \frac{1 + \hat{n}(0)}{2} \tag{3.1}$$

where $\varphi(T, 0)$ is the total rotation of the vector $n_\mu(T)$ with respect to vector $n_\mu(0)$. If $n_\mu(t)$ is the tangent field for a closed path: $n_\mu(t) = \dot{x}_\mu(t)$, $x_\mu(T) = x_\mu(0)$, then the above relation reduces to [12,29]

$$\mathrm{Tr}\, I[\dot{x}] = \cos\frac{\varphi(T,0)}{2} = (-1)^{\nu+1} \tag{3.2}$$

where ν is the total rotation of the tangent or the number of self-intersections of the path $x_\mu(t)$. Hence, for $D = 2$ we obtain from eqs.(1.15),(2.8b) and (3.2) the expression for the effective action of interacting fermions as a sum over paths in x-space [29]:

$$W[A] = \int_0^\infty \frac{dT}{T}\, e^{-TM} \int \mathcal{D}x_\mu\, \delta(x(0) - x(T))\, \delta^{(\frac{1}{2})}(1 - \dot{x}^2)\, (-1)^{\nu+1}\, \mathrm{Tr}\, P \exp\left(ig \oint dx_\mu A_\mu(x)\right)$$

This equation may be thought of as a continuous limit of the lattice expression (1.8).

3.2. Outline of the approach to arbitrary D

Now we have two different representations for the function $I[n]$ as an infinite product of matrices (2.9) and as a path-ordered exponential (2.11). P-exponentials turn out to be exactly calculable for $D = 2$ and $D = 3$. To understand the reason for this property, let us consider the initial eq.(2.9) in D-dimensional Euclidean space-time and introduce the notation

$$P(n) = \frac{1 + \hat{n}}{2}, \quad n_\mu^2 = 1 \tag{3.3}$$

where n_μ is an arbitrary D-dimensional unit vector and then

$$I[n] = \lim_{N\to\infty} P(n(N\tau)) \cdots P(n(2\tau))P(n(\tau)), \quad \tau = T/N \tag{3.4}$$

The matrix $P(n)$ possesses the following properties

$$P(n)P(n) = P(n), \quad \mathrm{Tr}\, P(n) = \frac{1}{2}\mathrm{Tr}\, 1 = 2^{[D/2]-1}$$

from which it follows that $P(n)$ is decomposed over $2^{[D/2]-1}$ orthogonal projection operators

$$P(n) = \sum_i |i, n\rangle\langle i, n|, \quad \langle i, n|j, n\rangle = \delta_{ij}, \quad i, j = 1, 2, ..., 2^{[D/2]-1} \tag{3.5}$$

The states involved in this equation are the eigenstates of operator $n_\mu\gamma^\mu$ corresponding to the degenerate eigenvalue $(+1)$

$$\hat{n}|i, n\rangle = |i, n\rangle, \quad i = 1, 2, ..., 2^{[D/2]-1} \tag{3.6}$$

The degree of degeneracy equals $2^{[D/2]-1}$ and it differs from unity unless dimension D has two special values $D = 2$ or $D = 3$. This means that for $D = 2$, 3 the right-hand side of (3.3) contains only one projection operator

$$P_{D\le 3}(n) = \frac{1 + \hat{n}}{2} = |+, n\rangle\langle +, n| \tag{3.7}$$

After substitution of eq.(3.7) into (3.4) the infinite product of matrices is replaced by the scalar products

$$I[n] = |+, n(T)\rangle\langle +, n(0)| \lim_{N\to\infty} \prod_{i=1}^{N} \langle +, n(i\tau)|+, n((i-1)\tau)\rangle, \quad \tau = T/N \tag{3.8}$$

that can be easily calculated if the solutions of eq.(3.6) are known.

For $D \geq 4$ the function $I[n]$ is a sum of projection operators whose substitution into eq.(3.4) leads again to the infinite product of matrices. The main idea of the proposed approach to the calculation of $I[n]$ for arbitrary D is the following. Eq.(3.6) does not fix the set of states $|i, n\rangle$ uniquely since any linear combination of these states satisfies the equation. Using the ambiguity it is possible to choose such solutions of eq.(3.6) that the following gauge symmetry holds:

$$|i, n(t)\rangle = |j, n^{(ij)}(t)\rangle \tag{3.9}$$

where $n^{(ij)}(t)$ is the D-dimensional vector obtained from $n(t)$ by the orthogonal gauge transformation

$$n^{(ij)}(t) = \Lambda^{(ij)}(t)n(t)$$

The gauge group of these transformations is in fact the Weyl group acting in the root space of the $su(2^{[D/2]})$ Lie algebra [25].

It follows from eqs.(3.5) and (3.9) that

$$P(n) = \sum_i |1, n^{(i,1)}\rangle\langle 1, n^{(i,1)}|, \quad i = 1, 2, ..., 2^{[D/2]-1}$$

If $f_a(n)$ is an arbitrary function of vector n_μ then due to the gauge symmetry (3.9) the integral $I(a) = \int d^D n\, f_a(n)P(n)$ can be transformed to the form when the integrand contains only one projection operator

$$\begin{aligned} I(a) &= \sum_i \int d^D n\, f_a(n)\, |1, n^{(i,1)}\rangle\langle 1, n^{(i,1)}| = \sum_i \int d^D n^{(i,1)} f_a(n)\, |1, n^{(i,1)}\rangle\langle 1, n^{(i,1)}| \\ &= |n \to (\Lambda^{(i,1)})^{-1} n| = \int d^D n\, |1, n\rangle\langle 1, n| \sum_i f_a((\Lambda^{(i,1)})^{-1} n) \end{aligned}$$

As a result, in the infinite product of integrals $\prod_a I(a)$ the product of matrices is replaced by the scalar products $\langle 1, n|1, n'\rangle$.

Let us begin to realize this program starting from the $D = 3$ case.

3.3. Calculation of the spinor functional for $D = 3$

To calculate the spinor functional at $D = 3$ we have to solve eq.(3.6) and then find the scalar products in eq.(3.8).

Let us consider eq.(3.6) for $D = 3$

$$\hat{n}\,|\pm,n\rangle = \pm\,|\pm,n\rangle, \qquad \langle\alpha,n|\beta,n\rangle = \delta_{\alpha\beta}, \qquad \alpha,\beta = +,- \tag{3.10}$$

Its solutions are related by

$$|+,n\rangle = |-,-n\rangle \tag{3.11}$$

This equation is a special case of (3.9). If any state $|\pm,n\rangle$ is represented by a point on the sphere S^2, then the last relation means that points corresponding to states $|+,n\rangle$ and $|-,n\rangle$ are antipodes.

To find the explicit form of states, one uses the parameterization of vector n_μ in terms of two angles on the sphere

$$n_\mu = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \qquad 0 \le \theta < \pi, 0 \le \varphi \le 2\pi$$

Solving equation (3.10) we obtain that up to a phase factor

$$|+,n\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}, \qquad |-,n\rangle = \begin{pmatrix} -e^{-i\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \tag{3.12}$$

In this notation the states corresponding to the north pole $N_\mu = (0,0,1)$ are

$$|+,N\rangle = |+\rangle, \qquad |-,N\rangle = |-\rangle$$

where vectors $|\pm\rangle$ are the eigenstates of Pauli matrix σ_3

$$\sigma_3|\pm\rangle = \pm|\pm\rangle$$

In terms of the complex coordinates $(z,\bar{z})$ of the stereographic projection

$$z = \tan\frac{\theta}{2}\,e^{i\varphi}, \qquad \bar{z} = z^*$$

the states $|\pm,n\rangle$ look like

$$|+,n\rangle = \frac{1}{(1+\bar{z}z)^{1/2}}\begin{pmatrix} 1 \\ z \end{pmatrix}, \qquad |-,n\rangle = \frac{1}{(1+\bar{z}z)^{1/2}}\begin{pmatrix} -\bar{z} \\ 1 \end{pmatrix} \tag{3.13}$$

We recall that the states (3.12) and (3.13) are defined by eq.(3.10) up to the phase factor. In the next section some important properties of states $|\pm,n\rangle$ will be formulated.

3.3.1. Coherent states for the $SU(2)$ group

It may be easily verified that states $|\pm,n\rangle$ have the form

$$|\pm,n\rangle = D(n)|\pm\rangle \tag{3.14a}$$

where

$$D(n) = \exp\left(\frac{i}{2}\theta\,(m\cdot\sigma)\right), \qquad 0 \le \theta < \pi \tag{3.14b}$$

and $m_\mu = (\sin\varphi, -\cos\varphi, 0)$ is a unit vector orthogonal to vectors n_μ and $N_\mu = (0,0,1)$. With matrices $E_\alpha = \frac{1}{2}(\sigma_1 + i\sigma_2)$ and $E_{-\alpha} = \frac{1}{2}(\sigma_1 - i\sigma_2)$ one rewrites $D(n)$ as

$$D(n) = \exp(\xi E_\alpha - \bar{\xi}E_{-\alpha}), \qquad \xi = -\frac{\theta}{2}e^{-i\varphi} \tag{3.14c}$$

The states defined in this manner are well-known as coherent states for the $SU(2)$ group [17]. The sets of states $|+,n\rangle$ and $|-,n\rangle$ form, for arbitrary vectors n_μ, two systems of coherent states and relation (3.11) sets up the correspondence between them. The coherent states (3.14a) are characterized by the ground states $|\pm\rangle$, that is the eigenstates of Pauli matrix σ_3. The stationary subgroup of states $|\pm\rangle$ is the Abelian $U(1)$ group and it consists of elements $e^{i\sigma_3\psi}$. Therefore the coherent states (3.14a) may be parameterized by points of the coset space [17]

$$G/H = SU(2)/U(1) \simeq S^2 \simeq CP^1$$

that is by the angles (θ,φ) on the sphere S^2 or by the complex variables z on the complex projective space CP^1.

The action of the stationary subgroup on the coherent states is defined as

$$D(n) \to D(n)\,e^{i\psi(n)\sigma_3}, \qquad |\pm,n\rangle \to |\pm,n\rangle\,e^{\pm i\psi(n)} \tag{3.15}$$

Now we briefly formulate the main properties of coherent states $|+,n\rangle$ [17]:

1.It follows from eqs.(3.14a) - (3.14c) that

$$\hat{n} = |+,n\rangle\langle+,n| - |-,n\rangle\langle-,n| = D(n)\sigma_3 D^{-1}(n) \tag{3.16}$$

2.The system of coherent states $|+,n\rangle$ is overcomplete.

3.The coherent states are not orthogonal to each other

$$\langle+,n_1|+,n_2\rangle = \exp\left(-\frac{i}{2}\Phi(N,n_2,n_1)\right)\left(\frac{1+(n_1\cdot n_2)}{2}\right)^{\frac{1}{2}} = \frac{1+\bar{z}_1 z_2}{((1+\bar{z}_1 z_1)(1+\bar{z}_2 z_2))^{\frac{1}{2}}} \tag{3.17}$$

where the expression

$$\Phi(N,n_2,n_1) = i\log\frac{1+\bar{z}_1 z_2}{1+\bar{z}_2 z_1}$$

is equal to the area of a triangle on the sphere S^2 shown in fig.1 with vertices being placed at points N, n_2 and n_1. z_1 and z_2 stand for complex coordinates of points n_1 and n_2. For the infinitesimal form when $n_2 = n(t)$ and $n_1 = n(t+\tau)$ we get

$$\langle+,n(t+\tau)|+,n(t)\rangle = \exp\left(-\frac{i}{2}\frac{d\Phi}{dt}\tau\right) + O(\tau^2) \tag{3.18}$$

where

$$d\Phi = \dot{\varphi}(1-\cos\theta)\,dt \tag{3.19a}$$

$$= i\frac{\dot{\bar{z}}z - \bar{z}\dot{z}}{1+\bar{z}z}\,dt \tag{3.19b}$$

$$= i\left(\dot{\bar{z}}\frac{\partial f(z,\bar{z})}{\partial\bar{z}} - \dot{z}\frac{\partial f(z,\bar{z})}{\partial z}\right)dt, \tag{3.19c}$$

and $f(z,\bar{z}) = \log(1+\bar{z}z)$. These relations are easily verified with the use of eqs.(3.12) and (3.13). From eqs.(3.18) and (3.14a) we conclude that

$$\frac{d\Phi}{dt} = -2i\langle +,n|\frac{d}{dt}|+,n\rangle = -2i\langle +|D^{-1}(n)\frac{d}{dt}D(n)|+\rangle = -i\,\mathrm{Tr}\left(\sigma_3 D^{-1}(n)\frac{d}{dt}D(n)\right) \qquad (3.19d)$$

4.There is the completeness relation

$$\mathrm{Tr}\,\mathbf{1}\cdot\int d\mu_0(n)\,|+,n\rangle\langle +,n| = 1$$

where the integration measure has in different coordinates the form

$$\begin{aligned} d\mu_0 &= \frac{1}{2\pi}d^3n\,\delta(n^2-1) & (3.20a)\\ &= \frac{1}{4\pi}d\theta\,d\varphi\,\sin\theta & (3.20b)\\ &= \frac{1}{2i\pi}\frac{dz\,d\bar{z}}{(1+\bar{z}z)^2} = \frac{1}{2i\pi}\frac{\partial^2 f(z,\bar{z})}{\partial z\,\partial\bar{z}}dz\,d\bar{z} & (3.20c)\end{aligned}$$

and the function $f(z,\bar{z})$ was defined in (3.19c). The last expression has a deep meaning. It reflects the fact that the coset space $SU(2)/U(1)$ is the Kähler manifold with $f(z,\bar{z})$ being the Kähler potential [17]. Eq.(3.20a)-(3.20c) is really a G-invariant measure on this manifold.

It is evident that analogous properties hold for the second system $|-,n\rangle$ of coherent states.

3.3.2. Calculation of the spinor functional

We substitute the scalar product (3.18) into eq.(3.8) and search the limit $N \to \infty$ or $\tau = T/N \to 0$

$$I_{D=3}[n] = |+,n(T)\rangle\langle +,n(0)|\exp\left(-\frac{i}{2}\int_0^T dt\frac{d\Phi}{dt}\right)$$

The exponent of this expression has a simple geometric meaning. If vector $n_\mu(t)$ walks from point $n(0)$ to point $n(T)$ on the sphere S^2 along some path C_1, then $\int_0^T dt\frac{d\Phi}{dt}$ is equal to the area enclosed by C_1 and by two main circles (that is, the ones lying in the meridian plane) connecting points $n(0)$ and $n(T)$ with the north pole and shown in fig.2a. To eliminate the dependence of $I[n]$ on the special point on the sphere (the north pole in our case), the last equation is transformed to

$$\begin{aligned} |+,n(T)\rangle\langle +,n(0)| &= |+,n(T)\rangle\langle +,n(T)|+,n(0)\rangle\langle +,n(0)|\left(\langle +,n(T)|+,n(0)\rangle\right)^{-1}\\ &= \frac{1+\hat{n}(T)}{2}\frac{1+\hat{n}(0)}{2}\exp\left(\frac{i}{2}\Phi(N,n(0),n(T)\right)\left(\frac{1+(n(T)\cdot n(0))}{2}\right)^{-1/2}\end{aligned}$$

where eq.(3.17) is used. Now Φ equals the area enclosed by the contour shown in fig.1. The function $I[n]$ is given by

$$I[n] = \frac{1+\hat{n}(T)}{2}\frac{1+\hat{n}(0)}{2}\left(\frac{1+(n(T)\cdot n(0))}{2}\right)^{-1/2}\exp\left(\frac{i}{2}\Phi(N,n(0),n(T) - \frac{i}{2}\int_0^T dt\frac{d\Phi}{dt}\right)$$

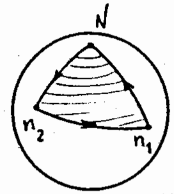

Fig.1:The function $\Phi(N,n_2,n_1)$ is equal to the area of a shaded triangle on the sphere S^2 with vertices being placed at the north pole N and points n_2 and n_1.

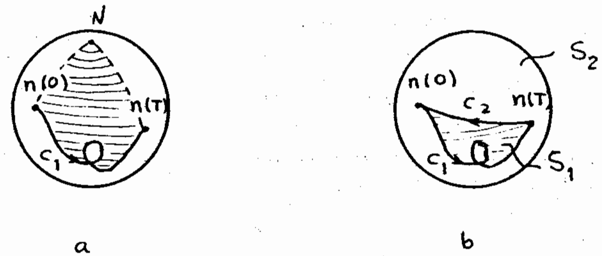

Figure 2: (a) If a particle walks from point $n(0)$ to point $n(T)$ on the sphere S^2 along some path C_1 then $\int dt\frac{d\Phi}{dt}$ is equal to the shaded area; (b) The function $\Phi(C)$ appearing in eqs.(3.21) and (3.22) equals the area S_1 enclosed by contour $C = C_1 + C_2$.

or

$$I[n] = \frac{1+\hat{n}(T)}{2}\frac{1+\hat{n}(0)}{2}\left(\frac{1+(n(T),n(0))}{2}\right)^{-1/2}\exp\left(-\frac{i}{2}\Phi(C)\right) \quad (3.21)$$

where $\Phi(C)$ is equal to the area S_1 shown in fig.2b with boundary $C = C_1 + C_2$ and C_2 being the main circle joining the end points of C_1. We note that there are many different choices of C_2 when the end points of C_1 are antipodes on the sphere S^2. This ambiguity is compensated by the preexponential factor in eq.(3.21): $\frac{1-\hat{n}(0)}{2}\frac{1+\hat{n}(0)}{2} = 0$. For a closed path C_1 the circle C_2 is replaced by a point and we get

$$I[n] = \frac{1+\hat{n}(0)}{2}\exp\left(-\frac{i}{2}\Phi(C_1)\right)$$

The explicit form of the function $\Phi(C)$ will be given in the next section.

3.3.3. One-dimensional Wess-Zumino term

The contour C in eq.(3.21) is shown in fig.2b. We parameterize it as

$$C = \{e_\mu(t), 0 \le t \le 2\pi | e_\mu(0) = n_\mu(0), e_\mu(\pi) = n_\mu(T), e_\mu(0) = e_\mu(2\pi)\}$$

Using eqs.(3.19) one gets the following relations

$$\Phi(C) = \int_0^{2\pi} dt\,\dot{\varphi}(1-\cos\theta) \quad (3.22a)$$

$$= i\int_0^{2\pi} dt\,\frac{\dot{\bar{z}}z - \bar{z}\dot{z}}{1+\bar{z}z} \quad (3.22b)$$

$$= i\int_0^{2\pi} dt\left(\dot{\bar{z}}\frac{\partial f(z,\bar{z})}{\partial \bar{z}} - \dot{z}\frac{\partial f(z,\bar{z})}{\partial z}\right), \quad f(z,\bar{z}) = \log(1+\bar{z}z) \quad (3.22c)$$

$$= -i\int_0^{2\pi} dt\,\mathrm{Tr}\left(\sigma_3 D^{-1}(e)\frac{d}{dt}D(e)\right) \quad (3.22d)$$

where (θ,φ) and $(z,\bar{z})$ are different coordinates of points $e_\mu(t)$. To express $\Phi(C)$ in terms of $e_\mu(t)$ we have to introduce the interpolating field $e_\mu(t,u)$

$$e(t,u) = \begin{cases} e(t) & , \quad u = 1 \\ \text{const.} & , \quad u = 0 \end{cases}$$

and the boundary values are the points of contour C. Then the function $\Phi(C)$ being equal to the area S_1 in fig.2b is expressed as

$$\Phi(C) = \int_0^{2\pi} dt \int_0^1 du\,\varepsilon_{\mu\nu\rho} e_\mu(t,u)\frac{\partial}{\partial u}e_\nu(t,u)\frac{\partial}{\partial t}e_\rho(t,u) \quad (3.22e)$$

Eqs.(3.22) are well-known as different forms of the one-dimensional Wess-Zumino term [1,20]. One of the properties of this term is the quantization of the physical parameters as a consequence of a consistency condition of underlying quantized dynamics. To check this property in our case,

we recall that there is a gauge ambiguity in the definition of the phase of coherent states. Under $U(1)$ gauge transformations defined in eq.(3.15) the function $\Phi(C)$ changes as

$$\begin{aligned} D(e) &\to D(e)e^{i\psi(e)\sigma_3} \\ \Phi(C) &\to -i\int_0^{2\pi} dt\,\mathrm{Tr}\left[\sigma_3\left(D^{-1}(e)\frac{d}{dt}D(e) + i\psi(e)\sigma_3\right)\right] \\ &= \Phi(C) + 2[\psi(e(2\pi)) - \psi(e(0))] \\ &= \Phi(C) + 4\pi k, k \in \mathbf{Z} \end{aligned} \quad (3.23)$$

since the boundary condition $e(2\pi) = e(0)$ implies that $e^{i\psi(e(2\pi))} = e^{i\psi(e(0))}$. Therefore demanding invariance for the phase exponential of the action $e^{-i\kappa\Phi(C)}$ enforces a quantization of the parameter κ that governs the gauge variation

$$2\kappa \in \mathbf{Z} \quad (3.24)$$

Eq.(3.21) implies that $\kappa = \frac{1}{2}$ and the action $\Phi(C)$ is nonmanifestly gauge invariant under gauge transformations (3.15). The physical reason of the quantization condition is the quantized value of spin of fermions. The case $\kappa = \frac{1}{2}$ corresponds to spin $\frac{1}{2}$ but for an arbitrary spin J the additional factor J appears in eq.(3.22) leading to $\kappa = J$ [17].

The quantization condition has a simple geometric meaning. The function $\Phi(C)$ equals the area enclosed by contour C on the sphere S^2 shown in fig.2b. But there are two such areas in fig.2b: S_1 and S_2. The quantization condition (3.24) may be expressed as the independence of the particular choice of the area

$$e^{-i\kappa\Phi(C)} = e^{-i\kappa S_1} = e^{i\kappa S_2}, \quad S_1 + S_2 = 4\pi$$

There are different interpretations of the phase factor $\Phi(C)$. For instance, the integrand of eq.(3.22a) is the torsion of the curve C [2]. Eq.(3.22a) may be written as [32]

$$\Phi(C) = \oint_C dz_\mu A_\mu(z)$$

where $A_\mu(z)$ is the Dirac potential of a magnetic monopole placed at the centre of the sphere S^2. Eq.(3.22d) was rediscovered as Berry's phase [33]. Eq.(3.22b) was discussed in connection with the geometric quantization of spin [16].

4. Summary

Substituting eq.(3.21) into (2.8) we derive the final expressions for the spinor functional for $D = 3$:

$$\mathcal{M}_{D=3}[\dot{x}] = \int \mathcal{D}e\,\delta(e^2-1)\,\delta''(1-(e\cdot\dot{x}))\exp\left(-\frac{i}{2}\Phi(e)\right)S(e(T),e(0)) \quad (4.1a)$$

$$= \delta'(1-\dot{x}^2)\exp\left(-\frac{i}{2}\Phi(\dot{x})\right)S(\dot{x}(T),\dot{x}(0)) \quad (4.1b)$$

where $S(n,m) = \frac{1+\hat{n}}{2}\frac{1+\hat{m}}{2}\left(\frac{1+(n\cdot m)}{2}\right)^{-1/2}$

The spinor functional $\mathcal{M}_D$ for $D = 2$ may be found as a special case of these relations. The dimensions of Dirac matrices coincide for $D = 2$ and $D = 3$ and therefore $I_{D=2}[n]$ is equal to

$I_{D=3}[n]$ with path $C = \{n_\mu(t)\}$ restricted to lie in the equatorial plane $n_3 = 0$. Then the path C is the equator and

$$\Phi(C) = 2\pi N$$

where N is the number of total rotations of the tangent vector $n_\mu(t) = \dot{x}_\mu(t)$, $n(0) = n(T)$ or with the number ν of self-intersections of the path $x_\mu(t)$

$$N = \nu + 1 \pmod 2$$

So we get

$$I_{D=2}[n] = \frac{1+\hat{n}(0)}{2}(-1)^{\nu+1}$$

and after the replacement $\varphi(T,0) = 2\pi(\nu+1)$ one obtains eq.(3.1).

Now we have the desired representations (1.14) and (1.15) for the effective action and propagator of interacting fermions as integrals over all paths in the x-space and explicit expressions for spinor functionals for two special cases of the space-time dimension $D = 2$ and $D = 3$. Hence, at $D = 3$ we get

$$S(x,y;A) = \int_0^\infty dT e^{-TM} \int_y^x \mathcal{D}x_\mu \,\delta'(1-\dot{x}^2) \exp\left(-\frac{i}{2}\Phi(\dot{x})\right) \times S(\dot{x}(T),\dot{x}(0))\, P\exp\left(ig\int_y^x dx_\mu A_\mu(x)\right) \tag{4.2}$$

and

$$W[A] = \int_0^\infty \frac{dT}{T} e^{-TM} \int \mathcal{D}x_\mu \,\delta(x(0)-x(T))\delta'(1-\dot{x}^2)\exp\left(-\frac{i}{2}\Phi(\dot{x})\right) \left(\frac{1+(\dot{x}(T)\cdot\dot{x}(0))}{2}\right)^{1/2} \mathrm{Tr}\, P\exp\left(ig\oint dx_\mu A_\mu(x)\right) \tag{4.3}$$

It is interesting to note that in the lattice approximation the above-mentioned representations are reduced to

$$S(x,y;A) \propto \sum_{P_{xy}} \exp(-ML(P_{xy}))\exp\left(-\frac{i}{2}\Phi(P_{xy})\right) P\exp\left(ig\int_{P_{xy}} dx_\mu A_\mu(x)\right)$$

and identically coincide (for $A_\mu(x) = 0$) with those in eqs.(1.8) and (1.9) proposed in refs.[2,31].

Acknowledgements

I would like to thank A.V.Efremov, N.P.Ilieva, E.A.Ivanov, D.I.Kazakov, E.A.Kochetov, S.O.Krivonos, V.M.Leviant, A.A.Migdal, H.Panagopoulos,O.K.Pashaev, V.N.Plechko, A.V.Radyushkin, O.V.Teryaev for numerous helpful discussions. I am also grateful to V.G.Kadyshevsky for his stimulating interest in the subject and support.

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Received by Publishing Department
on June 30, 1989.