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TRANSVERSE-LONGITUDINAL PART OF A VECTOR POTENTIAL IN CLASSICAL ELECTRODYNAMICS

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1. Introduction

About 60 years ago Fock [1] and Weyl [2] proposed the principle of gauge symmetry for describing the interaction of charged particles with an external electromagnetic field. It consists in the requirement for the Schrödinger equation to be invariant under both local transformations of a phase of a wave function and simultaneous gradient transformations of electromagnetic potentials. In this approach, the interaction of an electromagnetic field with matter is described by the vector-potential and both electric and magnetic fields become its products. Then the principle of gauge symmetry was generalized to the nonabelian case by Yang and Mills/3/ and it becames one of fundamental principles of the modern elementary particle physics.

In classical physics the interaction of electromagnetic field with charged particles is described by Maxwell-Lorentz equations which are presented only in terms of both $\vec{E}$ (electric) and $\vec{B}$ (magnetic) fields, and potentials are auxiliary quantities. Thus, on the one hand, the passage to quantum mechanics demanded altering look on the physical meaning of a <u>classical</u> vector potential and, on the other hand, the potential concept turned into defining potentials with respect to field strengths after the proclaiming the principle of gauge symmetry, as the main cause of existing interaction of particles in nature.

However, in electrodynamics there exists an ambiguity in determining potentials up to a gradient of an arbitrary function. This initiated the discussion about a possibiltiy of observing such potentials configurations which generate no electromagnetic field [4]. To a great extent this discussion was stirred up by that a gauge was usually fixed for picking out physical degrees of freedom of a vector potential. But, on the other hand, gauge conditions can be chose in arbitrary ways. For example, the vector potential distributed about the solenoid with a constant internal magnetic field does not vanish everywhere in one gauge (Coulomb gauge), but in an other gauge it can be turned into zero everywhere except a certain plane [5]. If we want to keep the positions of locality and short-range interaction then the question arises. What (which a real field) does a charged particle being near by the solenoid interact with? Note that the formulation of electrodynamics in terms of strengths gives a non-local interaction of a charge with an external field in quantum mechanics [6].

In the present paper, based on the generalized Hamiltonian dynamics by Dirac [7] for systems with constraints, we show that in electrodynamics there exists a gauge-invariant degree of freedom of a vector-potential generating locally no electromagnetic strengths $\vec{E}$ and $\vec{B}$. It turns out that a quantum charged particle interacts locally just with this degree of freedom. Then we give both Lagrangian and Hamiltonian forms independing on a gauge choice for describing interaction of charged particles in an electromagnetic field, and that is more, our consideration does not contain Mandelstam path-depending integrals [8-10].

Based on the developed formalism we get the gauge-independing form of quantum theory in which it is shown that the found degree of freedom a vector-potential, as any physical field, can influence ~~physical field,~~ physical systems, in particular it induces an electric current in a superconductor.

2. The canonical formalism for electrodynamics with an external current

The electromagnetic action with an external current $J^\mu = (\rho, \vec{J})$ ($\mu = 0,1,2,3$), ρ is a charge density, $\vec{J}$ is a current density) has the form

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^2 - A_\mu J^\mu\right). \tag{2.1}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength tensor, the metric tensor $g_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$, $A_\mu = (A_0, \vec{A})$ are scalar and vector potentials, respectively. Action (2.1) is invariant under gauge transformations

$$A_\mu \to A_\mu - \partial_\mu \omega \quad , \qquad J_\mu \to J_\mu \tag{2.2}$$

therefore the 4-current is a local-conserving quantity

$$\partial_\mu J^\mu = \partial_t \rho + \mathrm{div}\, \vec{J} = 0. \tag{2.3}$$

The gauge symmetry (2.2) leads to constraints on dynamic variables of the theory [7] i.e., between canonical coordinates and conjugated momenta there are relations containing no time derivatives. For a consistent dynamic description of a system with constraints, it is necessary to use the Hamiltonian formalism generalized by Dirac [7]. For this purpose we pass from action (2.1) to the Lagrangian

$$L = \int d^3x \left(-\frac{1}{4} F_{\mu\nu}^2 - A_\mu J^\mu\right). \tag{2.4}$$

Defining canonical momenta $\pi_\mu = \delta L / \delta \dot{A}_\mu = F_{\mu 0}$ in a standard way we can see that there is a primary constraint $\pi_0 = 0$. in the theory and $\pi_n = -E_n = \dot{A}_n + \partial_n A_0$, $n = 1, 2, 3$, E_n is an electric field. The Hamiltonian of the system is

$$H = \int d^3x \left[\frac{1}{2}(\pi_n^2 + B_n^2) + A_0(\partial_n \pi_n + \rho) - J_n A_n\right], \tag{2.5}$$

where $B_n = \frac{1}{2} \varepsilon_{nkl} F_{kl}$ stands for a magnetic field. For a consistent dynamics the condition $\pi_0 = 0$ should hold at all moments of time [7], i.e.,

$$\dot{\pi}_0 = \{\pi_0, H\} = -\partial_n \pi_n - \rho = 0. \tag{2.6}$$

The Poisson brackets is defined in the following way

$$\{A^\nu(x), \pi_\mu(y)\} = \delta^\nu_\mu \, \delta^3(x-y). \tag{2.7}$$

Eq.(2.6) is a secondary constraint which sould be fulfilled at all moments of time

$$\{\partial_n \pi_n + \rho, H\} = \{\partial_n \pi_n, H\} = \partial_n J_n = 0. \tag{2.8}$$

As Eq.(2.8) does not contain dynamical variables, there are no restrictions on canonical coordinates A_μ and π_μ except (2.6) and $\pi_0 = 0$, and all constraints are of the first class [7].

A part from ordinary electrodynamic constraints we have got the additional condition on external current (2.8). From Eqs.(2.3) it follows that $\partial_t \rho = 0$. However, the Maxwell equations do not contain such a restriction on J_μ. Actually, $-\delta S / \delta \vec{A} = \mathrm{rot}\, \vec{B} - \partial_t \vec{E} - \vec{J} = 0$, and taking divergence of both sides of this equality we find that $\partial_t \,\mathrm{div}\, \vec{E} + \mathrm{div}\, \vec{J} = \partial_t \rho + \mathrm{div}\, \vec{J} = 0$ because of the conservation law (2.3). The question arises: is Eq.(2.8) either an artifact of the Hamiltonian formalism or is there Hamiltonian dynamics without restriction (2.8)? It turns out that we can get rid of restriction (2.8) by the redefinition of Lagrangian

(2.4) or action (2.1) so that the equations of motion will be equivalent to the initial ones.

The origin of additional restrictions on external parameters in theories with constraints is quite clear. The external parameters are given functions of coordinates and time and their Poisson brackets with the Hamiltonian always vanishes.

At the same time the evolution of canonical variables in time is determined by their nonvanishing Poisson brackets with the Hamiltonian. However, constraints can contain external parameters (see (2.6)). Then the requirement for constraints to fulfill at all moments of time can lead to some restrictions on the time dependence of external parameters, which has been found in passing from (2.6) to (2.8). Thus, to eliminate similar restrictions from a theory, it is enough to construct the Hamiltonian formalism in which constraints do not contain external parameters. In the general case this requirement is not necessary because secondary constraints depend on the Hamiltonian form, i.e., successive finding of all conditions of consistency cannot lead to restrictions on external parameters in spite of that a part of constraints may depend on them. However, this question requires a special investigation. Here we shall study it in the case of electrodynamics.

It is well known that the Lagrangian and hence the Hamiltonian rectricted from equations of motion have ambiguities, namely, the Lagrangian is always defined with an accuracy of a full divergence. Nevertheless, for systems with external parameters there is a supplementary freedom in choosing the action we shall demonstrate this for electrodynamics with an external current.

Consider the action

$$S' = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^2 - \rho\left(A_0 - \Delta^{-1}\partial_n \dot{A}_n\right) + J_n\left(A_n - \partial_n\partial_k \Delta^{-1} A_k\right)\right], \tag{2.9}$$

here Δ^{-1} is an operator inverse to the Laplace operator $\Delta = \partial_n\partial_n$ in the whole space $\mathbb{R}^3$. It is easy to test that by using the conservation law of current (2.3) notion (2.9) is that its corresponding Lagrangian density is gauge-invariant. Note that Lagrangian (2.4) is not invariant under gauge transformations. It gets an additional term that can be rewritten in the form of a total time-derivative by using the conservational law of current (2.3). Actions S' and S do not differ from each other by an integral of 4-divergence therefore their variations over A_μ give different equations of motion, which are, however, equivalent to each other if Eq.(2.3) takes place. This is the additional freedom in

choosing the action of a system with the external parameters. Really,

$$-\frac{\delta S}{\delta \vec{A}} = -\dot{\vec{E}} - \vec{J} + \mathrm{rot}\,\vec{B} = 0 \tag{2.10a}$$

$$-\frac{\delta S'}{\delta \vec{A}} = -\dot{\vec{E}} - \vec{J} + \mathrm{rot}\,\vec{B} + \mathrm{grad}\,\Delta^{-1}\left(\partial_\mu J^\mu\right) = 0\,. \tag{2.10b}$$

Variations of (2.1) and (2.9) over A_0 coincide and give Eq.(2.6).

Consider the Hamiltonian formalism for action (2.9). Obviously, canonical momenta are

$$\pi_0 = \frac{\delta L'}{\delta \dot{A}_0} = 0\,, \quad \pi_n = \frac{\delta L'}{\delta \dot{A}_n} = -E_n + \partial_n \Delta^{-1}\rho\,. \tag{2.11}$$

Hence, the Hamiltonian of the system has the following form

$$H = \int d^3x \left[\frac{1}{2}\left(\pi_n^2 + B_n^2 - \rho\Delta^{-1}\rho\right) + A_0\partial_n A_n + \rho\Delta^{-1}\partial_n\pi_n + J_n\left(-A_n + \partial_n\Delta^{-1}\partial_k A_k\right)\right]. \tag{2.12}$$

Now the condition of consistency of dynamics does not contain external parameters:

$$\{\pi_0, H\} = -\partial_n\pi_n = 0\,. \tag{2.13}$$

To pick out physical (gauge-invariant) degrees of freedom, it is necessary to accomplish a canonical transformation that diagonalizes constraint (2.13), i.e., converts the constraint into an equality of some generalized momentum to zero. Following papers [11] we introduce new variables

$$\alpha_n = A_n - \partial_n\partial_k\Delta^{-1}A_k\,, \quad \xi = \Delta^{-1}\partial_n A_n\,,$$
$$\varepsilon_n = \pi_n - \partial_n\partial_k\Delta^{-1}\pi_k\,, \quad \pi_\xi = -\partial_n\pi_n\,. \tag{2.14}$$

It is not difficult to check that both pairs α_n, ε_n and ξ, π_ξ are canonical conjugated quantities. Constraint (2.13) in new variables has the required form, $\pi_\xi = 0$. So, unphysical variables turn out to be ξ and A_0 as their momenta are equal to zero identically and there are only two physical degrees of freedom α_n, $\partial_n\alpha_n = 0$. It is well known that when passing to gauge-invariant variables in the Hamiltonian unphysical variables become cyclic i.e., the Hamiltonian does not depend on them. In fact,

$$H_{ph} = \int d^3x \left[\frac{1}{2}\left(\varepsilon_n^2 + B_n^2 - \rho\Delta^{-1}\rho\right) - J_n^{\perp}\alpha_n\right], \tag{2.15}$$

where $\mathcal{J}_n^{\perp} = \mathcal{J}_n - \partial_n \Delta^{-1} \partial_k \mathcal{J}_k$ is the tansverse part of a current. The last addend in(2.15) is equal to the last term in (2.12) as $\mathcal{J}_n = \mathcal{J}_n^{\perp} + \partial_n \Delta^{-1} \partial_k \mathcal{J}_k$ and $\partial_n \alpha_n = \partial_n \mathcal{J}_n^{\perp} \equiv 0$. The third term in (2.15) is the Coulomb energy. The Hamiltonian equations of motion in gauge-invariant variables have the form

$$\dot{\mathcal{E}}_n = \{ \mathcal{E}_n , H_{ph} \} = \mathcal{J}_n^{\perp} - \varepsilon_{nku} \partial_k B_u ; \quad \dot{\alpha}_n = \{ \alpha_n , H_{ph} \} = \mathcal{E}_n \qquad (2.16)$$

that coincides with Maxwell equation of the relation $\mathcal{E}_n = - E_n^{\perp} = - E_n + \partial_n \Delta^{-1} \partial_k E_k$ following from (2.14) and (2.11) is taken into account.

3. Nature of the vector-potential

As has been already noted, the postulation of the gauge principle, as a basic principle for describing the interaction of a field with matter, has initiated the discussion of the vector potential nature. Is $\vec{A}$ simply a convenient form for describing interactions of quantum particles with an electromagnetic field or is it a more fundamental physical concept than a field itself? A reason for a negative answer to this question is usually the ambiguity in determination $\vec{A}$ (see (2.2)). Really, the observed quantities, like $\vec{E}$ and $\vec{B}$, cannot depend on gauge arbitrariness. Below we shall demonstrate that guage-invariant degrees of freedom of the vector potential can be excited in some space regions in which, however, both electric and magnetic fields are absent. In this sense the concept of the potential already in classical theory turns out to be more fundamental than fields themselves.

First let us consider the question of a class of permissible functions in (2.2). Though the action S' is invariant with respect to transformations (2.2) with an arbitrary function ω, there are some restrictions on ω. Actually, calculate a circulation of the vector potential over an arbitrary closed contour C limiting an area

$$\oint_C (\vec{A}\, d\vec{\ell}) = \iint_{\sigma} (\mathrm{rot}\vec{A}, d\vec{\sigma}) = \iint_{\sigma} (\vec{B}\, d\vec{\sigma}) = \Phi . \qquad (3.1)$$

Essentially, the magnetic field flux Φ is a gauge-invariant. However we can replace in the left-hand side of Eq.(3.1) $\vec{A} \to \vec{A} + \vec{\nabla}\omega$, then we find that the circulation of $\vec{\nabla}\omega$ over an arbitrary closed coutour may vanish. This is the necessary and sufficient condition of one-valuedness of the function [12].

Every observed quantity $F_{ph}[A_\mu, \Pi_\mu ; J_\mu]$ depending on canonical variables and external parameters J_μ should be gauge--invariant and satisfy the following equations

$$\{ F_{ph} , \Pi_0 \} = \{ F_{ph} , \Pi_\xi \} = 0 \qquad (3.2)$$

since first class-constraints are generators of gauge transformations [7]. Eqs. (3.2) obviously, mean that F_{ph} does not depend on A_0 and that is a longitudinal component of the vector potential. Since $\Pi_0 = \Pi_\xi = 0$, the dependence of F_{ph} on Π_0 and Π_ξ is also absent. Hence

$$F_{ph} = F[\vec{\alpha}, \vec{\mathcal{E}}, \rho, \vec{J}], \qquad (3.3)$$

i.e., every quantity observed in experiments depends only on gauge-invariant variables α_n ; $\mathcal{E}_n$ and external currents.

Let us take the variable α_n itself as an observable and calculate it in space regions, where $\vec{E} = \vec{B} = 0$. If $\vec{E} = 0$, then we may set $\rho = 0$ and $\partial_t \vec{A} = 0$ for simplicity. It follows from $\vec{B} = 0$ that $\mathrm{rot}\vec{A} = 0$ and thus, $\vec{A} = \vec{\nabla}\chi$. If $\vec{B} = 0$ in the whole space $\mathbb{R}^3$, then substituting $\vec{A} = \vec{\nabla}\chi$ into (2.14) we find $\mathcal{L} = 0$. Another situation arizes if there exists a region V in which $\vec{B} \neq 0$. In this case we have

$$\vec{A}(\vec{x}) = \begin{cases} A^V(\vec{x}) & \vec{x} \in V , \ \mathrm{rot}\vec{A}^V = \vec{B} , \\ \vec{\nabla}\chi(\vec{x}) & \vec{x} \notin V , \ \vec{B} = 0 \end{cases} \qquad (3.4)$$

and there is the condition of countinuity on a boundary of a region ∂V : $\vec{\nabla}\chi = A^V$, $\vec{x} \in \partial V$, i.e. $\vec{B} = 0$ at $\vec{x} \in \partial V$. Apparently, a condition of that short does not restrict our considerations. A real current always is distributed over a spatially extended region. Two- and one-dimensional current carriers are ideal objects for which we must pay be the introduction of corresponding conditions for magnetic field components. Note also that the region $\mathbb{R}^3 \setminus V$ is multiconnected. Really if $\vec{B} \neq 0$ only in V, then there always exists a countour that is not in V, the magnetic flux through which is nonzero. For example, this coutour can be chosen on the surface ∂V ($\vec{B} = 0$, $\vec{x} \in \partial V$) as the lines of the magnetic field are always closed : $\mathrm{div}\,\vec{B} = 0$. Hence, the function χ in (3.4) should be multi-valued. Since a curculation of $\vec{\nabla}\chi$ over the coutour described avobe zero, does not vanish. So, in spite of a "pure gauge" from of the vector potential in $\mathbb{R}^3 \setminus V$ it cannot be reduced to zero by a gauge transformation as functions ω in (2.2) should be single-valued (see (3.1)). In other words $\vec{A}$ in $R^3 \setminus V$ has a gauge-invariant part. To pick out it explicitly, we calculate the functional $\alpha_n[\vec{A}]$ for configuration (3.4) at a point $\vec{x} \in V$. We find

$$\alpha_n = \partial_n \chi + \partial_n \partial_k \int_V \frac{\overset{\vee}{A}_k(\vec{y})}{4\pi|\vec{x}-\vec{y}|} d^3_y + \partial_n \partial_k \int_{R^3\setminus V} \frac{\partial_k \chi(\vec{y})}{4\pi|\vec{x}-\vec{y}|} d^3_y . \tag{3.5}$$

After integration by parts in the second integral in (3.5) we get

$$\vec{\alpha} = \vec{\nabla}\chi^{Ph}, \quad \chi^{Ph}[\vec{A}] = \partial_n \int \frac{\overset{\vee}{A}_n(\vec{y})}{4\pi|\vec{x}-\vec{y}|} d^3_y - \partial_n \oint_{\delta V} \frac{\nu_n(\vec{y})\chi(\vec{y})}{4\pi|\vec{x}-\vec{y}|} d^3_y , \tag{3.6}$$

where $\vec{x} \in V$, ∂V is the external normal to the surface ∂V, $d\sigma_y$ is an element of the area ∂V. Functional (3.6) is gauge-invariant $(\vec{A}^{\vee} \to \vec{A}^{\vee} + \vec{\nabla}\omega,\ \chi \to \chi + \omega)$ and moreover, is a harmonic function in the region $R^3 \setminus V$ $(\partial_n \alpha_n = \Delta \chi^{Ph} = 0)$ i.e. χ^{Ph} gives a solution of the corresponding Dirichlet problem for the Laplace equation in the multi-connected region.

For the simplest regions V formula (3.6) can be simplified. Since $\chi^{Ph}[A]$ is gauge-invariant, we can transform (3.6) in any appropriate gauge of $\vec{A}$ and χ (in the Coulomb gauge also) if of course, it is not singular for a given distribution of currents creating both $\vec{A}^{\vee}$ and χ i.e., it completely fixes the gauge arbitrariness and is regular defined at all points of the space $\mathbb{R}^3$. Acting proceeding in a similar way, we find for double-connected regions

$$\chi^{Ph}[A] = f(x) \oint_C (\vec{A}, d\vec{\ell}) = f(x)\Phi, \tag{3.7}$$

where Φ is the flux through the countour C that is in $\mathbb{R}^3 \setminus V$ and once encircles the region V, and the function $f(x)$ depends on the geometry of the region V. For example, for an infinite solenoid directed along the 0_z axis, $f(x) = \theta/2\pi$ where θ is the polar angle of the cylindrical system of coordinates.

Thus, we conclude that around the regions occupied by the magnetic field there is a real physical field distributed in vacuum, like both electric and magnetic fields. If charges and currents are sources for electric and magnetic fields, respectively, the source of the field χ^{Ph}, according to (3.7), is the magnetic field flux. A peculiarity of the picked out degree of freedom of the vector potential (3.6) consists in that $\alpha_n = \partial_n \chi^{Ph}$ is both longitudinal

and transverse simultaneously. Just the fact that an unphysical part of the longitudinal component of the vector potential ξ vanishes for configuration (3.4) and $\alpha_n \neq 0$ allows us to consider this longitudinal-transverse degree of freedom of the vector potential to be physical.

Expression (2.15) represents the total energy of an electromagnetic field with external sources. Let us take the current $\vec{j}$ in the form $\vec{j} \to \vec{j} + \vec{j}'$ where j' we shall call a trial current system and $\vec{j}$ is an external current. We shall assume currents $\vec{j}$ and $\vec{j}'$ independent i.e., they have not common carriers. Let also the configuration of $\vec{j}$ be such that there exists a magnetic field only in some region V and an electric field is generally, absent (compensated), i.e. only $\vec{\alpha}$, χ^{Ph} is distributed out of V. Put the trial current system $\vec{j}'$ out of V. Then the energy of this current system in the external field χ^{Ph} created by the current $\vec{j}$ is (see (2.15))

$$E = -\int_\Omega d^3x \left(\vec{j}'^{\perp}, \vec{\nabla}\chi^{Ph} \right) = 0, \tag{3.8}$$

where Ω is the volume in which currents flow. The equality of the integral to zero follows from the requirement that the current $\vec{j}'^{\perp}$ has no normal component on the surface $\partial\Omega$ (out of Ω $\vec{j}' = 0$). Generally, the integration region, according to (2.15) can be replaced by a larger region $\Omega' \supset \Omega$, $\Omega' \cap \Omega = 0$ since both $\vec{j}$ and $\vec{j}'$ have no common carrier. In this case Eq.(3.8) becomes obvious.

However, if Ω is broken into smaller parts then the energy of individual parts may turn out to be nonzero due to the surface contribution (on the surface of the part of Ω, $\vec{j}'^{\perp}$ can have a normal component). What is the physical meaning of this energy? To answer this question, one should make a certain assumption about the nature of the current $\vec{j}'$, i.e., to formulate its microscopic theory. So, in the framework of electrodynamics with an external current it is impossible to answer the question about physical actions of the field χ^{Ph} on current systems. This is a consequence of electrodynamics being incomplete [13].

4. Hamiltonian dynamics of a particle in an electromagnetic field

Let us assume that the current of a trial system is induced by moving classical particles (pointed). To describe the interaction of charged particles with the field $\mathcal{X}^{ph}$, one should find the gauge-invariant formulation of dynamics of the system of the electromagnetic field and charged particles. Usually, the Mandelstam formalism is used [9, 10]. However, in our opinion, this method of describing the interaction of charged particles with the electromagnetic field is deprived of a physical clarity in a sense*), moreover, particular calculations through this formalism are highly cumbersome [14], although they have explicit gauge invariance. We shall solve the task presented above by the Hamiltonian formalism where gauge-invariant variables have a simple physical meaning.

The standard Lagrangian of interaction of a charged particle with an external electromagnetic field is

$$L = \frac{1}{2} m v^2 + e\,(\vec{A}, \vec{v}) - e\,A_0\,, \qquad (4.1)$$

where $\vec{v} = \dot{\vec{r}}$, $\vec{r}$ is a radius-vector of the particle. Under gauge transformations (2.2) Lagrangian (4.1) is added with a total derivative, $L \to L + e\, d/dt\, \omega$ ($d/dt = \partial_t + (\vec{v}, \vec{\nabla})$ is a material derivative) therefore the Lagrangian equations of motion remain invariable under gauge transformations. Interaction in (4.1) is described by $\int d^3x\, (\vec{j}\vec{A} - \rho A_0)$, where $\vec{j} = e\vec{v}\, \delta^3(x - y)$, $\rho = e\,\delta^3(x - r)$, therefore, by analogy with (2.9) the gauge-invariant Lagrangian has the form

$$L' = L_{ext} + \sum_q \left\{ \frac{m_q v_q^2}{2} + e_q \vec{v}_q \left(\vec{A} - \vec{\nabla}\Delta^{-1}\partial_n A_n\right) - e_q\left(A_0 - \Delta^{-1}\partial_n \dot{A}_n\right) \right\}. \qquad (4.2)$$

Here L_{ext} is the Lagrangian of the electromagnetic field with external sources J_μ corresponding to action (2.9), and instead of one particle we take a set of particles enumerated by

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*) For example, in Ref. [9] authors propose to replace $\vec{A}$ by $\vec{A}' = \vec{A} - \vec{\nabla} \int_C (\vec{A}, d\vec{\ell})$ in the interaction Lagrangian for a charged particle with an electromagnetic field. The quantity $\vec{A}'$ is a gauge invariant depending on the contour C. Let $\vec{A} = \vec{\nabla}\mathcal{X}$ be out of V and C also be out of V, then $\vec{A}'[\nabla\mathcal{X}] = \vec{\nabla}\mathcal{X} - \vec{\nabla} \int_{C} (\vec{\nabla}\mathcal{X}, d\vec{\ell}) = \vec{\nabla}\mathcal{X} - \vec{\nabla}(\mathcal{X} + \mathrm{const}) = 0$, but the gauge-invariant variable α is nonzero (see (3.6)), i.e., in the suggested formalism [9] the degree of freedom $\mathcal{X}^{ph}$ is absent. We emphasize that the introduction of α as a canonical variable diagonalizes constraints, therefore α contains full information about physical degrees of freedom in electrodynamics.

the index q. These particles form the current J'_μ of a trial system.

Passing to the Hamiltonian formalism we determine canonical momenta

$$P_q^n = \frac{\partial L'}{\partial v_q^n} = m_q v_q^n + e_q\left(A_n - \partial_n \Delta^{-1} \partial_k A_k\right), \qquad (4.3a)$$

$$\Pi_0 = \frac{\delta L'}{\delta \dot{A}_0} = 0, \qquad (4.3b)$$

$$\Pi_n(x) = \frac{\delta L'}{\delta \dot{A}_n(x)} = -E_n + \partial_n \Delta^{-1} \rho - \sum_q \frac{\partial}{\partial x_n} \frac{e_q}{4\pi |\vec{r}_q - \vec{x}|}. \qquad (4.3c)$$

In addition to (2.7) we put $\{r_q^n, P_{q'}^m\} = \delta_{nm}\delta_{qq'}$. By usual rules one can find the Hamiltonian of the system and verify that secondary constraints are setted by Eq. (2.13). Then we pass to new canonical variables (2.14). As a result, we get the Hamiltonian written through physical gauge-invariant variables

$$H_{ph} = H_{ext} - \sum_q e_q \Delta^{-1}\rho + \frac{1}{2} \sum_{q \neq q'} \frac{e_q e_{q'}}{4\pi |\vec{r}_q - \vec{r}_{q'}|} + \frac{1}{2} \sum_q \frac{1}{m_q} \left(P_q - e_q \alpha\right)^2, \qquad (4.4)$$

where H_{ext} is Hamiltonian (2.15), the second term in (4.4) is the energy of Coulomb interaction of charged particles with external charges ρ, and what is more, $\Delta^{-1}\rho$ should be taken at the point r_q, the third term is the Coulomb energy of charged particles from which the infinite self-energy of every point-like charge is eliminated. The Hamiltonian equations of motion do not depend on gauge and have the form

$$\dot{\alpha}_n = \{\alpha, H_{ph}\} = \mathcal{E}_n\,, \qquad (4.5a)$$

$$\dot{\mathcal{E}}_n(x) = \{\mathcal{E}_n, H_{ph}\} = J_n^\perp(x) + \varepsilon_{nk\ell}\, \partial_k B_\ell(x) + \qquad (4.5b)$$

$$+ \sum_q \frac{e_q}{m_q} \delta_{nk}^\perp (\vec{x} - \vec{r}_q) \left(P_q^k - e_q \alpha^k\right),$$

$$\dot{\vec{r}}_q = \{\vec{r}_q, H_{ph}\} = \frac{1}{m_q}\left(\vec{P}_q - e_q \vec{\alpha}\right) \qquad (4.5c)$$

$$\dot{\vec{P}}_q = \{\vec{P}_q, H_{ph}\} = \dot{r}_q^n\, \vec{\nabla}\alpha_n - \vec{\nabla} V_{coul}, \qquad (4.5d)$$

where we have taken into account that $\{\alpha_k(x), \mathcal{E}_n(y)\} = \delta_{kn}^\perp(x - y) = (\delta_{kn} - \Delta^{-1}\partial_k\partial_n)\,\delta^3(x - y)$ and V_{coul} is a sum of the second and third terms in (4.4) and the configuration of $\vec{J}$.

Now let $\rho = 0$ be such that there exists a spatial region in which χ^{ph} is distributed. For example, consider the motion of charges in the field of the infinite or toroidal solenoid. In this case the term of interaction in (4.2) ($A_0 = 0, \partial_t \vec{A} = 0$) has the form of the total derivative $\sum_q e_q (\vec{v}_q, \vec{\nabla}) \chi^{ph} = d/dt \sum_q e_q \chi^{ph}(\vec{r}_q)$. Which can be omitted from Lagrangian (4.2)? This might be expected as the motion of classical particles in the external field is described by they Lorentz equation in which contains only $\vec{E}$, $\vec{B}$ and they vanish here.

From the viewpoint of the canonical formalism the essence consists in the following. Although by passing from (4.1) to (4.2) we have destroyed the gauge ambiguity of the particle momentum (4.3a) is gauge-invariant) nevertheless, in the definition of the canonical momentum there is an arbitrariness that is larger then the gauge freedom in electrodynamics. Really, if we alter the Lagrangian of a system, $L \to L + d/dt\, \Omega$, Ω is an arbitrary function of coordinates and time, then $\vec{p} \to \vec{p} + \vec{\nabla}\Omega$. Multivalued function Ω are also admissible here. This arbitrariness does not influence the Hamiltonian equations of motion although the Hamiltonian depends on Ω . Note, however, that when we pass to quantum mechanics, the multi-valuedness of Ω influences the form of wave functions. If we consider the wave function single-valued, then Ω should not enter into its phase. Thus, we have found that classical particles do not feel the field χ^{ph} , in such idealized consideration.

There is one remark here. We have neglected the influence of the trial system of charges on the source $\vec{j}$ of the external field which should remain stationary to remove both $\vec{E}$ and $\vec{B}$ field out of some region V . Obviously, a moving charge creates the emf of induction in contours where currents flow, therefore, $\vec{j}$ will no longer be stationary. Hence, in the region of motion of a charge there arises an "external" magnetic field which influences the charge. In other words, the charge influences itself by its own magnetic field by means of winding along which the external current flows (however, the existence of external current in winding is not already necessary for this effect).

Though a powe action of field χ^{ph} is absent on a classical particle nevertheless χ^{ph} has a clear physical meaning. The Coulomb field of a charged particle penetrates into the region therefore the electromagnetic momentum in V differs from zero

$$\vec{P}(\vec{r}_q) = \frac{1}{4\pi} \int_V d^3x \left[\vec{\nabla} \frac{e_q}{|\vec{r}_q - \vec{x}|} , \vec{B}(\vec{x}) \right], \qquad (4.6)$$

where $\vec{B}(\vec{x})$ is the magnetic field in the region V is the position vector of charge e_q ($\vec{r}_q \in V$) . After simple calculations one can be convined of [15] that

$$\vec{P}(\vec{r}_q) = e_q \vec{\nabla} \chi^{ph}(\vec{r}_q). \qquad (4.7)$$

For this we have integrated (4.6) by parts and have used Eq. (2.16) in the stationary case $\mathrm{rot}\, \vec{B} = -\Delta \vec{\mathcal{A}} = \vec{j}^{\perp}$, where $\vec{\mathcal{A}} = \vec{\nabla}\chi^{ph}$. So, in the classical theory χ^{ph} describes the momentum of the electromagnetic field in the system charge-stationary solenoid.

5. Quantum theory and superconductivity

Using the rules of canonical quantization for Lagrangian (4.1) we obtain the Schrödinger equation describing a quantum charged particle in an external electromagnetic field

$$i\hbar \partial_t \Psi = \left[\frac{1}{2m} \left(\vec{P} - \frac{e}{c} \vec{A} \right)^2 + e A_0 \right] \Psi. \qquad (5.1)$$

Here $\vec{P} = -i\hbar \vec{\nabla}$ is the momentum operator (for more clarity we restore the dependence on the ligh velocity c). Eq.(5.1) is invariant under gauge transformations (2.2) with simultaneous phase rotations of the wave function $\Psi \to \exp(i \frac{e}{\hbar c} \omega) \Psi$.

To answer the question about a possible influence of the external field χ^{ph} on a quantum trial particle, it is necessary, as in [4], to write Eq.(5.1) in terms of gauge-invariant variables. This could be made directly (without (5.1)) by quantizing the theory determined by the Lagrangian (4.2) written only for one particle

$$L' = \frac{1}{2} m v^2 + \frac{e}{c} \left(\vec{v}, \left(\vec{A} - \vec{\nabla} \Delta^{-1} (\vec{\nabla}, \vec{A}) \right) \right) + e \Delta^{-1} \rho, \qquad (5.2)$$

where we have used the constraint $\partial_n E_n - \rho = -\Delta A_0 - \partial_n \dot{A}_n - \rho = 0$ for transforming the last term in (4.2). Hence,

$$i\hbar \partial_t \Psi_{ph} = \left[\frac{1}{2m} \left(\vec{P} - \frac{e}{c} \vec{\mathcal{A}} \right)^2 - e \Delta^{-1} \rho \right] \Psi_{ph}. \qquad (5.3)$$

Eq.(5.3) can also be obtained from (5.1) by introducing the gauge-invariant wave function

$$\Psi_{ph} = \exp\left(-i \frac{e}{\hbar c} \Delta^{-1} \partial_n A_n \right) \Psi. \qquad (5.4)$$

Substituting Ψ into Eq.(5.1) we get Eq.(5.3) if the constraint $\partial_n E_n = \rho$ is taken into consideration for transforming the term of the Coulomb interaction. Substitution (5.3) corresponds to the known result that both the phase of a charged field and a longitudinal component of a vector field $\vec{A}$ are linear combinations of one physical and one unphysical degrees of freedom [11] . Put $\rho = 0$, $\vec{\mathcal{L}} = \nabla \chi^{Ph}$ in (5.3). It is well known that the spectrum of the Hamiltonian can depend on the solenoid field if the one-valuedness of a wave function is assumed [16] . Otherwise, the substitution

$$\Psi_{ph} = \exp\left(i\frac{e}{\hbar c}\chi^{Ph}\right)\psi \tag{5.5}$$

reduces Eq.(5.2) to the equation for a free particle. The energy spectrum does not depend on χ^{Ph} in this case and function (5.5) becomes multi-valued in view of the multi-valuedness of χ^{Ph} . For example, for a charged rotator in the field of the infinite solenoid [16] we have

$$H\Psi_{ph} = \frac{1}{2I}\left(L_z - \frac{e\Phi}{2\pi c}\right)^2\Psi_{ph} = \frac{\hbar^2}{2I}\left(\ell - \frac{e\Phi}{2\pi\hbar c}\right)^2\Psi_{ph} = E_\ell\Psi_{ph}. \tag{5.6}$$

Here I is a moment of inertia, ℓ is any integer, $L_z = -i\hbar\partial/\partial\theta$ is the operator of the angular momentum projection on the axis Oz in cylindrical coordinates, χ^{Ph} is taken from (3.7) where $f(x) = \theta/2\pi$. Eq.(5.6) remains correct if we assume that $\Psi_{ph}(\theta+2\pi) = \Psi_{ph}(\theta) \sim \exp i\ell\theta$. Then it follows from (5.6) that the rotator spectrum E_ℓ depends on χ^{Ph} at noninteger $e\Phi/2\pi\hbar c$.

The consideration of a rotator shows that the field χ^{Ph} changes the rotation energy of a quantum particle. Hence, the macroscopic manifestation of the field χ^{Ph} is possible only as a coherent quantum effect. The superconductivity gives the simplest and, apparently, most visual example of it.

The free energy of a superconductor of volume V in the external electromagnetic field is given by the Ginzburg-Landau functional

$$F_S = \int_V d^3x\left\{-a|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{1}{4m}\psi^*\left(-i\hbar\vec{\nabla} + \frac{2e}{c}\vec{A}\right)^2\psi\right\}, \tag{5.7}$$

where ψ is the complex order parameter of the wave function of the Cooper pair in the BCS model, m is the electron mass, e is its charge. Our further analysis will be qualitative. We shall not solve the Ginzburg-Landau equation exactly. A matter of principle for us is the following: can the field χ^{Ph} have a possibility to influence a real physical system or not, i.e., is there a way of observating χ^{Ph}? In other words, is a method to elucidate whether the magnetic field exists inside the solenoid without any manipulations with the solenoid itself?

Therefore we omitted the magnetic field energy in the superconductor, i.e., disregarded the depth of penetration of the field $\vec{B}$ into the superconductor.

The basic state of the superconductor is specified by the absolute minimum of its free energy. However, when switching an external field, relative energy minima can arise where the system can be kept for arbitrary time. If the external field is absent, then the absolute minimum is reached at $\psi = \psi_0 = const$, i.e., when the kinetic energy assumes zero values, and $|\psi_0|^2 = a/b$ (coefficients a and b are determined from thermodynamical properties of a superconductor). Usually ψ_0 is normalized as $|\psi_0|^2 = n_s/2$ where n_s is a number of paired electrons per unit volume (in general , $|\psi|^2$ is a density of Cooper pairs).

Consider a ring made from a superconducting material and put on the solenoid. If the ring temperature is $T > T_c$, where T_c is the critical temperature, the current in the ring dies down. Let us stabilize the flux of the magnetic field for $T > T_c$. Then, upon cooling the ring to $T < T_c$ it transforms into superconducting state. Let us find the wave function of the ground state of a superconductor in this case. Thus, using substitution (5.4) (where $e \to 2e$) we rewrite F_S through gauge-invariant variables $\mathcal{L}$ and ψ and, then, we pass to cylindric coordinates in the kinetic energy operator which becomes $\frac{\hbar^2}{4m}\left(-\partial^2/\partial z^2 - \frac{1}{r}\partial/\partial r\, r\,\partial/\partial r + 1/r^2\left(-i\partial/\partial\theta - \Phi/\Phi_0\right)^2\right)$, where we have substituted $\vec{\mathcal{L}} = \vec{\nabla}\chi = \vec{e}_\theta\Phi/2\pi r$, $\vec{e}_\theta$ is a basis vector of the cylindric coordinate system, $\Phi_0 = \pi\hbar c/e$ is the magnetic flux quantum (fluxon). The only difference of the kinetic energy operator, as compared with the case $\chi^{Ph} = 0$, consists in the charge of the rotation energy of the condensate, therefore, the minimum is reached on function Ψ_{ph} independing both of z and r , i.e.,

$$F_S = \int_V d^3x\left[-a|\Psi_{ph}|^2 + \frac{b}{2}|\Psi_{ph}|^4 - \frac{\hbar^2}{2mr^2}\Psi^*_{ph}\left(-i\frac{\partial}{\partial\theta} - \frac{\Phi}{\Phi_0}\right)^2\Psi_{ph}\right]. \tag{5.8}$$

Hence, the minimum of F_S is reached when $|\Psi_{ph}|^2 = |\psi_0|^2$. The phase of Ψ_{ph} is only altered (see (5.6)). Assuming wave function to be multi-valued we see that the function

$$\Psi_{ph} = \psi_0\exp\left(i\theta\Phi/\Phi_0\right) \tag{5.9}$$

realizes the minimum of F_S according to (5.5) the function is multi-valued at $\Phi \neq$ integer $\times\, \Phi_0$ therefore the angular momentum of the Cooper pair can take all real values. If the angular momentum is quantized with an integer, i.e., the wave function is one-valued, then the function

$$\psi_{ph} = \psi_0 \exp i l_0 \theta, \quad l_0 = \left[\Phi / \Phi_0 \right] \tag{5.10}$$

gives the minimum of F_S. Here l_0 is the integer to be chosen from the condition of minimal F_S, $[\Phi/\Phi_0]$ means rounding of Φ/Φ_0 to the nearest integer. Between solutions (5.9) and (5.10) there is an essential difference. The state with a nonzero current corresponds to solution (5.10), but there is no current for state (5.9). In fact, the density of a superconducting current is

$$j_S = \frac{e}{2m} \left[\psi_{ph}^* \left(-i\hbar\nabla - \frac{2e}{c} \mathcal{A} \right) \psi_{ph} + h.c. \right]. \tag{5.11}$$

The substitution (5.9) into (5.11) gives $j_S = 0$, but for (5.10) we find

$$\vec{j}_S = \vec{e}_\theta \frac{e n_s \hbar}{2m r_c} \left(l_0 - \Phi/\Phi_0 \right), \tag{5.12}$$

where r_0 is the ring radius. It follows from (5.12) that j_S can have different directions in dependence on l_0. If $\Phi = N\Phi_0 + \frac{1}{2}\Phi_0$ where N is integer, the system turns out to be in the state with an unsteady equilibrium as F_S is identical for both cases $m_0 = N$ and $m_0 = N+1$. After "throwing down" the system into one of minima the current j_S gets maximal in absolute value.

There is the Bloch theorem according to which F_S has a minimum at $j_S = 0$. We emphasize that the statement of this theorem relates to the absolute minimum of F_S. State (5.10) defines a relative minimum which turns out to be steady. The appearance of the current in the situation described above can be explained in a sufficiently simple way. The magnetic flux passing through the superconducting ring should be quantized, i.e., it should be divisible by Φ_0. Thereby, after a passage of the ring into a superconducting state the current arises in it, the magnetic field of which supplements the total magnetic flux through the ring to an integer number of quanta . Although (5.12) may only serve as the first approach to the solution of Ginzburg-Landau equations, nevertheless, it gives a correct qualitative picture of the phenomenon. But for quantitave estimates of the current, the known formula $I_S = c(\Phi_0 l_0 - l)/L$, where L is the contour inductance, can always be used. This formula is exact for superconductors [17] . We want to add the question to the presented reasoning: which real physical field makes the Cooper pairs to move thus creating a current? If we keep the field-theoretical interpretation of interating matter, then there is no other, <u>locally</u> interacting with matter field (see (5.3), (5.8)).except χ^{ph} .

In conclusion of this paragraph we note that the behaviour of a superconductor in the field of a solenoid was discussed in Ref. [18] in connection with quantization of the angular momentum. However, the authors of ref. [18] were not correct in their reasonings, for example, their use of the Coulomb gauge and also the use of singular gauge transformations (on the occasion of it see [19] were incorrect.

6. Conclusion

As a matter of fact, the field χ^{ph} can produce a mechanical influence. Consider two simple (but difficult in experimental realization) examples. Let the frame made from a suitable material be hung up in a vertical plane and the toroidal solenoid be run through the frame (the toroidal solenoid should be taken to exclude the return magnetic flux). Moreover, let the external constant homogeneous magnetic field B_0 be run through the frame. Put the frame so that the flux of B_0 through it would be quantized. Then, after cooling this frame to $T < T_c$ the frame begins to oscillate if the flux inside the solenoid is not quantized. The frequency of small oscillations is $\omega = (I_S \Phi_{ext} / I)^{1/2}$, where I_S is the superconducting current in the frame, Φ_{ext} is the flux of B_0 through the frame and I is its inertial moment with respect to the hanging axis (oscillations of the frame with a current in the external magnetic field). Besides, the turning scales can be used for observing mechanical recoil in the superconducting ring when a superconducting current appears after cooling. When the Cooper pairs begin to move coherently, the atomic frame of the ring gets a recoil in according to the conservation of angular momentum. So, oscillations of the turning scales will arise, the amplitude of which will be $2\pi I_S m_e / \omega_c e M$, M is the ring mass, ω_c is the natural frequency of the turning scales, m_e is the electron mass.

Certainly, in technical aspects it is more easy to observe electromagnetic displays of χ^{ph} by using SQWID's. Moreover, the Aharonov-Bohm effect should be interpreted as the result of scattering a charged particle by the field χ^{ph}, as follows from (5.3).

Based on the simplest idea of short-range interaction, i.e. the transfer of interaction by a field we should point out, in the analysis of the above-mentioned effects what physical (i.e., independent of gauge) field influences charged particles. We have demonstrated, using both the principle of gauge symmetry and postulates of quantum mechanics, that just the field χ^{Ph} can be the only transmitter of interaction. In this connection, we would like to recall the paper by T.T.Wu and C.N.Yang [20] in which they have introduced the "minimal" description of electromagnetic interactions in terms of a nonintegrable phase factor $\oint(\vec{A}\cdot d\vec{\ell})$. Again, keeping the idea of short-range interaction and locality, one should point out what field is responsible for the appearance of this phase factor in the wave function. In other words, how the stokes theorem works in electrodynamics, when by going around the solenoid over a closed contour we find the flux of a magnetic field inside the solenoid (see (3.1)). The reference to mathematical reasons (milti-valuedness of the functions χ beyond V in (3.4) of this does not turn out to be convincing from a physical view point because one can take a real physical system as the contour, for example, the superconducting ring described above. It seems to be incredible thatthis nonintegrable phase factor of a wave function of a charged particle appears by a jump, for example, when crossing same plane (such gauge can be chosen for the vector potential [5]). Thus, the existence of the physical (gauge-independent) field distributed in vacuum throws light upon these problems.

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