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ON THE ORIGIN OF THE SCHWINGER ANOMALY

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It is now well known [1-3] that the Dirac sea may be viewed as an origin of the Schwinger terms in the commutators of currents. Actually, it is an infinite depth of the Dirac sea that proves to be a source of anomalous terms. Mathematically, this phenomenon is quite natural because filling in the Dirac sea implies the transition to a non-equivalent representation of canonical anticommutation relations. The aim of the present note is to study all this machinery in detail. My interest in this field was stimulated by some problems in quantum integrable systems where filling in the Dirac sea is also claimed to produce Schwinger terms [4].

We choose to deal with the simplest model where the phenomenon yet exists. Consider a free non-relativistic one-dimensional model of the one-component fermion field

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int dk\, e^{ikx}\, a_k \ . \tag{1}$$

The momenta could be even discrete, which corresponds to a finite box in x-space. However, I prefer to use continuous momenta because the discretization does not lead to any simplification. The anticommutation relations are standard canonical ones,

$$[a_k, a^+_{k'}]_+ = \delta(k - k'), \tag{2}$$

$$[\psi(x), \psi^+(y)]_+ = \delta(x - y). \tag{3}$$

To be a free one, our model must have the Hamiltonian which is bilinear in the fields or in a, a^+ - operators. For our purposes it is not even necessary to fix the Hamiltonian unambiguously. Let it be

$$H = \int dk\, a^+_k a_k \omega(k) \tag{4}$$

with a function $\omega(k)$ such that in a certain interval $\omega(k) < 0$. We fix this interval to be $-\Lambda < k < 0$. Just here the Dirac sea will reside. Λ is a positive number which eventually will go to $+\infty$.

Anticipating the result of our investigation we can formulate the final answer: The Schwinger terms appear if the interval of negative values of $\omega(k)$ is infinite ($\Lambda = \infty$) or, in the discrete version $H = \sum_k \omega_k a_k^+ a_k$, if the number of k's with ω_k negative is infinite. Both the cases can be interpreted as the presence of infinitely deep Dirac sea to fill in. It should be emphasized that the magnitude of $\omega(k)$ is not relevant, only the sign of it is.

Now let us describe the Fock space of the model. The vacuum $|0\rangle$ of operators a, a^+,

$$a_k | 0 \rangle = 0, \tag{5}$$

is a mathematical or false vacuum because it is not the state of the lowest energy. The true or physical vacuum $\|0\rangle$ emerges as a result of filling in the Dirac sea. In other words, $\|0\rangle$ is the vacuum of new operators b, b^+,

$$b_k \| 0 \rangle = 0, \tag{6}$$

defined by

$$\begin{aligned} a_k &= \theta_\Lambda(k) b_k + (1 - \theta_\Lambda(k)) b_k^+ , \\ a_k^+ &= \theta_\Lambda(k) b_k^+ + (1 - \theta_\Lambda(k)) b_k , \end{aligned} \tag{7}$$

where $\theta_\Lambda(k)$ generalizes the ordinary θ-function:

$$\theta_\Lambda(k) = \begin{cases} 1 \text{ for } k > 0 \\ 0 \text{ for } -\Lambda < k < 0 \\ 1 \text{ for } k < -\Lambda \end{cases} \tag{8}$$

so that

$$\theta_\Lambda(k) \xrightarrow[\Lambda \to \infty]{} \theta(k). \tag{9}$$

Operators b, b^+ obey canonical anticommutation relations

$$[b_k, b_{k'}^+]_+ = \delta(k - k'). \tag{10}$$

In terms of b, b^+ the Hamiltonian H is positive definite.

In what follows we shall study various operators in this new Fock space over the vacuum $\|0\rangle$. The operators a, a^+ when encountered are to be substituted by (7) with Λ finite or infinite.

Consider the current (or particle density)

$$I(x) = \psi^{+}(x)\psi(x) \tag{11}$$

and its commutator

$$[I(x), I(y)]_{-} = ? \tag{12}$$

Naively, due to (3), the latter seems to be equal to zero. But we shall see that for $\Lambda = \infty$ the answer will be quite different: the Schwinger term will appear. How this happens to be?

Let us begin with considering the Fourier transform of $I(x)$,

$$I(p) = \int_{-\infty}^{\infty} dk\, a^{+}_{k+p} a_{k} \; . \tag{13}$$

By our convention, we must treat a, a^{+} in terms of b, b^{+} using (7). If we do it, we see that $I(p)$ is not normal-ordered with respect to b's. It consists of 4 terms, and one of them is not ordered. However, operators similar to (13), when non-ordered, can prove not to be the true operators in the Fock space. This is due to the infinite interval of k-integration. For example, $\int_{-\infty}^{0} dk\, b_k b_k^{+}$ is not the true Fock-space operator.

Anticipating the limit $\Lambda \to \infty$, we have to deal with normal-ordered operators. Let us introduce the notation $:I: \equiv J$ (the ordering with respect to b-operators, of course). It is easy to obtain (for finite Λ)

$$I_{\Lambda}(p) = J_{\Lambda}(p) + \Lambda\delta(p), \tag{14}$$

$$[I_{\Lambda}(p), I_{\Lambda}(p')]_{-} = [J_{\Lambda}(p), J_{\Lambda}(p')]_{-} = 0 . \tag{15}$$

Up to now, no Schwinger terms.

However, consider the limit $\Lambda \to \infty$. $I_{\Lambda}(p)$ does not survive this limit, it diverges due to the $\Lambda\delta(p)$-term, whereas $\lim_{\Lambda\to\infty} J_{\Lambda}(p)$ is evidently a well-defined operator. To find it explicitly, one should substitute a's in $I(p)$, $I(p')$ by (7) using ordinary θ-functions and then perform the normal ordering of b's. The result is

$$\lim_{\Lambda\to\infty} J_{\Lambda}(p) \equiv J_{\infty}(p) =$$

$$= \int_{-\infty}^{\infty} dk\, [\theta(k)\theta(k+p) b^+_{k+p} b_k + \theta(k+p)\,\theta(-k) b^+_{k+p} b^+_k + \qquad (16)$$

$$+ \theta(k)\theta(-k-p) b_{k+p} b_k - \theta(-k)\,\theta(-k-p) b^+_k b_{k+p}].$$

One can straightforwardly show that /2/

$$[J_\infty(p),\ J_\infty(p')]_- = -p\,\delta(p+p'), \qquad (17)$$

which corresponds to

$$[J(x),\, J(y)]_- = \frac{\partial}{\partial x}\,\delta(x-y), \qquad (18)$$

the r.h.s. being the Schwinger term.

Returning to (17) we observe that

$$[\lim_{\Lambda\to\infty} J_\Lambda(p),\ \lim_{\Lambda\to\infty} J_\Lambda(p')]_- = -p\,\delta(p+p') \neq 0 \qquad (19)$$

whereas

$$\lim_{\Lambda\to\infty} [J_\Lambda(p),\ J_\Lambda(p')]_- = 0\,. \qquad (20)$$

So, taking the product and taking the limit do not commute. This is possible since the difference

$$J_\infty(p) - J_\Lambda(p) \qquad (21)$$

does not vanish in the operator sense as $\Lambda\to\infty$. Really, this difference is a sum of several terms, the typical of which are

$$\int_{-\infty}^{-\Lambda} dk\, b^+_{k+p} b_k\,, \quad \int_{-\Lambda-p}^{-\Lambda} dk\, b^+_{k+p} b^+_k\,, \ldots\,. \qquad (22)$$

All these terms are of the form $\int^{-\Lambda} dk \hat{A}(k,p)$ with a bilinear operator $\hat{A}$ and therefore vanish in the limit $\Lambda\to\infty$ when sandwiched between two given Fock states. However, the norm of an operator like (22) does not go to zero as $\Lambda\to\infty$. We can conclude that the mathematical nature of the Schwinger anomaly consists in that J_Λ does not converge to J_∞ in the operator sense when the depth of the Dirac sea goes to infinity.

Maybe, the most clarifying formula is

$$J_{\infty}(p) = I_{\Lambda}(p) - \Lambda\delta(p) + \int_{-\infty}^{-\Lambda} dk\,\hat{A}(k, p). \qquad (23)$$

We see that the "anomalous" current $J_{\infty}(p)$ differs from the "commuting" one $I_{\Lambda}(p)$ not by a mere c-number term $\Lambda\delta(p)$ as it were if $I_{\infty}(p)$ would exist, but also by a certain operator term (the last term in the r.h.s. of (23)). It is just this term that generates Schwinger anomalies. It resides deeply in the Dirac sea and, as $\Lambda \to \infty$, seems to be drowned. However, the c-number commutators produced by the $\hat{A}$-terms do not depend on Λ at all, do not drown, and eventually display themselves in the form of anomalies.

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