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ON THE DYNAMICAL GENERATING FUNCTIONAL IN THE STOCHASTIC QUANTIZATION METHOD

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1. Introduction

In recent years, there has been a considerable interest in a new method proposed by Parisi and Wu /1/ for quantizing of Euclidean field theory. This is due to the fact that stochastic quantization (SQ) scheme seems, first, to offer some advantages /2-7/ over conventional ones, particularly in dealing with constrained systems, and second, to give a new vision of the problem of quantum behaviour. The basic idea of SQ is to consider the Euclidean path integral "measure" $\exp(-S[x])d[x]$ ($S[x]$ - Euclidean action) as the stationary distribution of a stochastic process (SP) governed by Langevin equation (LE). The system relaxes to stationary state for large fictions "time" τ . The prescription of the SQ method formulated by Parisi and Wu /1/ is the following:

I. One supplements the field $\varphi(x)$ with an additional fictions axis of "time" τ .

II. In the classical field equation

$$\frac{\delta S}{\delta \varphi}(x\tau) = \int dx' dt' \left[\frac{\delta \mathcal{L}}{\delta \varphi(x\tau)} - \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi(x\tau))} \right] = 0 \tag{1}$$

two new forces, a Gaussian white noise $\eta(x\tau)$ and a "friction" force $\frac{\partial}{\partial \tau}\varphi(x\tau)$, have been introduced:

$$\frac{\partial}{\partial \tau}\varphi(x\tau) + \frac{\delta S}{\delta \varphi}(x\tau) = \eta(x\tau), \tag{2}$$

where $\mathcal{L}(\varphi(x\tau), \varphi_{;}(x\tau))$ is the Lagrangian density.
The white noise

$$\langle \eta(x\tau) \rangle_\eta = 0$$
$$\langle \eta(x\tau)\eta(x'\tau') \rangle_\eta = 2\hbar\, \delta(\tau-\tau')\, \delta(x-x') \tag{3}$$

modulates nonequilibrium quantum fluctuations while the "friction" force is responsible, first, for evolution to equilibrium distribution, and, second, for the fact that nonstationary fluctuations are a result of stochastic forces at all preceding "time" superposition /4/.

III. Given some initial conditions, one has to solve the LE (2). To emphasize the noise dependence explicitly the solution is indicated by φ_η. Many "time" stochastical correlation functions of $\varphi_\eta(x\tau)$ are defined by performing Gaussian average over noise η :

$$\langle \varphi_\eta(x_1\tau_1)\ldots\varphi_\eta(x_n\tau_n)\rangle_\eta \quad . \tag{4}$$

IV. The central assertion in the SQ method is that the Euclidean Green functions may be obtained from large "time" limit of single "time" average (4):

$$\lim_{\tau\to\infty}\langle\varphi_\eta(x_1\tau)\ldots\varphi_\eta(x_n\tau)\rangle_\eta = \frac{\int d[\varphi]\exp(-S[\varphi])\,\varphi(x_1)\ldots\varphi(x_n)}{\int d[\varphi]\exp(-S[\varphi])} \,. \tag{5}$$

The paper is organized as follows: in Section 2 we consider the single "time" average to show the leading part of the single "time" probability distribution density (PDD) in solving SP ergodicity problem. In Section 3 we briefly review the works by Gozzi /8/. In Section 4 we propose a pure probabilistic interpretation of the generating-functional building problem.

2. The SP "ergodicity" in the SQ method. Fokker-Plank formulation

In this section, the single "time" stochastical correlation functions will be considered. They are formally obtained with respect to functional integration over a single "time" probability distribution density (PDD) $P(\varphi,\tau)$ /12/

$$\langle\varphi_\eta(x_1\tau)\ldots\varphi_\eta(x_n\tau)\rangle_\eta = \int d[\varphi]\,P(\varphi,\tau)\,\varphi(x_1)\ldots\varphi(x_n)\,, \tag{6}$$

where

$$P(\varphi,\tau) = \langle\delta[\varphi-\varphi_\eta(\tau)]\rangle_\eta \,. \tag{7}$$

The single "time" functional δ -function implies, for example, lattice space approximation at "time" τ :

$$\delta[\varphi-\varphi_\eta(\tau)] = \prod_x \delta\left(\varphi(x)-\varphi_\eta(x\tau)\right). \tag{8}$$

Single "time" PDD satisfies the Fokker-Plank equation (FPE):

$$\frac{\partial}{\partial\tau}P(\varphi,\tau) = \int dx\,\frac{\delta}{\delta\varphi(x)}\left\{\frac{\delta}{\delta\varphi(x)} + \frac{\delta S}{\delta\varphi(x)}\right\}P(\varphi,\tau). \tag{9}$$

Additionally, we have to give some initial conditions

$$\rho(\varphi, 0) = \delta[\varphi - c], \tag{10}$$

where c are constants. It is obvious that there are two equivalent formulations of the SQ method based on LE (2) and on FPE (9). We may reformulated in the FPE language the main assertion of the SQ method implying consideration of the Euclidean path measure as the stationary distribution of the SP (3) /2/

$$\lim_{\tau \to \infty} P(\varphi, \tau) = N \exp(-S[\varphi]). \tag{11}$$

Let us consider in detail the "ergodicity" problem (11) /11,13/. Equation (9) may be viewed as the Schrödinger equation in imaginary "time" with non-Hermitian Hamiltonian:

$$\left(\frac{\partial}{\partial \tau} + \int dx \hat{\mathcal{H}}\right) P(\varphi, \tau) = 0, \tag{12}$$

where

$$\hat{\mathcal{H}} = \hat{p}^2 - i\hat{p}\frac{\delta S}{\delta \varphi} \quad , \quad \hat{p} = -i\frac{\delta}{\delta \varphi}. \tag{13}$$

If we set:

$$\rho(\varphi, \tau) = \Psi(\varphi, \tau) \exp\left(-\frac{1}{2} S[\varphi]\right) \tag{14}$$

a simple calculation leads to Schrödinger equation in imaginary "time" with the Hermitian Hamiltonian:

$$\left(\frac{\partial}{\partial \tau} + \int dx \hat{H}\right) \Psi(\varphi, \tau) = 0, \tag{15}$$

where

$$\begin{gathered} \hat{H} = \hat{p}^2 + \mathcal{U}(\varphi) \\ \mathcal{U}(\varphi) = \frac{1}{4}\left(\frac{\delta S}{\delta \varphi}\right)^2 - \frac{1}{2}\frac{\delta^2 S}{\delta \varphi^2} \end{gathered} \tag{16}$$

by "gauge" transformation (14) we have associated Hermitian Hamiltonian (16) with the known energetical band with non-Hermitian Hamiltonian (13). More detailed analysis of the correlation between "gauge" transformation (14) and energetical bands of operators (13,16) is given in works /11,13/.

If we denote the complete set of eigenstates of $\hat{H}$ by Ψ_n :

$$\hat{H}\Psi_n(\varphi) = E_n\Psi_n(\varphi)$$
$$\Psi_n(\varphi\tau) = \Psi_n(\varphi)\exp(-\tau E_n) \tag{17}$$
$$\Psi_0(\varphi) = \exp\left(-\tfrac{1}{2}S[\varphi]\right)$$

we may write for the solution of (15):

$$W(\varphi,\tau) = a_0 \exp\left(-\tfrac{1}{2}S[\varphi]\right) + \sum_{n=1}^{\infty} a_n \Psi_n(\varphi)\exp(-\tau E_n). \tag{18}$$

If we suppose that $E_n > 0$ for $n > 1$ then for large "time" limits of $W(\varphi,\tau)$ and $P(\varphi,\tau)$ we get:

$$\lim_{\tau\to\infty} W(\varphi,\tau) = a_0 \exp\left(-\tfrac{1}{2}S[\varphi]\right) \tag{19}$$

$$\lim_{\tau\to\infty} P(\varphi,\tau) = a_0 \exp\left(-S[\varphi]\right). \tag{20}$$

We fix the normalization by:

$$\lim_{\tau\to\infty} \int d[\varphi] P(\varphi,\tau) = 1 \tag{21}$$

and arrive finally at:

$$\lim_{\tau\to\infty} P(\varphi,\tau) = \frac{\exp(-S[\varphi])}{\int d[\varphi]\exp(-S[\varphi])} \tag{22}$$

as has been shown in [2]. In this section we have intentionally considered the simplest case of application of SQ method - scalar field theory with nondegenerate vacuum to show the leading part of PDD in solving the "ergodicity" problem in the SQ method. The essence of our approach (Section 4) consists in isolating single " time" PDD from the dynamical generating functional expression. But, first, let us consider the Gozzi [8] approach in constructing this expression.

3. Dynamical generating functional

In this section we briefly review the works[8] from the point of view of constructing the generating functional for the many "time" noise averages. To evaluate the latter let us consider, following[8], the generating functional:

$$\mathcal{Z}[J] = \int \mathcal{D}[\varphi]\mathcal{D}[\eta] P(\varphi,0) \Delta[\varphi - \varphi_\eta] \exp(J\varphi), \tag{23}$$

where $P(\varphi,0)$ is some initial probability, $\Delta[\varphi-\varphi_\eta]$ and $\mathcal{D}[\varphi]$ imply a lattice approximation of space supplemented by "time" axis:

$$\mathcal{D}[\varphi] = \prod_\tau \prod_x d\varphi(x\tau) \tag{24}$$

$$\Delta[\varphi - \varphi_\eta] = \prod_\tau \prod_x \delta(\varphi(x\tau) - \varphi_\eta(x\tau)). \tag{25}$$

After formal transformation /10/

$$\Delta[\varphi-\varphi_\eta] = \Delta\left[\dot{\varphi} + \frac{\delta S}{\delta\varphi} - \eta\right] \left\| \frac{\delta\eta}{\delta\varphi} \right\| \tag{26}$$

and the η integration we have the following expression:

$$\mathcal{Z}[J] = \int \mathcal{D}[\varphi] P(\varphi,0) \exp\left\{ -\int_0^\tau dt \left(-J\varphi + \frac{1}{4}\left(\dot{\varphi} + \frac{\delta S}{\delta\varphi}\right)^2 - \frac{1}{2}\frac{\delta^2 S}{\delta\varphi^2} \right) \right\}. \tag{27}$$

Performing the integration of the term

$$\int_0^\tau \dot{\varphi} \frac{\delta S}{\delta\varphi} dt = S[\varphi(\tau)] - S[\varphi(0)]$$

we get for $\mathcal{Z}[J]$:

$$\mathcal{Z}[J] = \int d[\varphi(0)]\, d[\varphi(\tau)]\, \mathcal{D}''[\varphi] P(\varphi,0) \exp\Big\{ \frac{1}{2}\big(S[\varphi(0)] - S[\varphi(\tau)]\big) + \tag{28}$$
$$+ \int_0^\tau dt \left(J\varphi - \mathcal{L}^{FP}(t)\right) \Big\} .$$

To understand the physical meaning of (28) we consider a simplest one-dimensional quantum mechanical system /3/. The kernel of the evolution operator is given by a path integral over all trajectories $x(t)$ from the initial state $x_0 = x(t_0)$ to the final state $x_f = x(t_f)$ of the exponential $S[x]$ where S is the classical action functional:

$$\langle x_f t_f | x_0 t_0 \rangle = \int d[x] \exp\left\{ -\frac{S[x]}{\hbar} \right\}. \tag{29}$$

If we consider vacuum expectation values of the position operators in the Heisenberg picture

$$\hat{Q}(t) = \exp\left\{-\frac{t\hat{H}}{\hbar}\right\} \hat{Q} \exp\left\{\frac{t\hat{H}}{\hbar}\right\},$$

where $\hat{H}$ is the Hamiltonian of a given quantum system, it is possible to obtain the following expression:

$$\langle 0|T\{\hat{Q}(t_1)\dots Q(t_n)\}|0\rangle = \int dx_0\, dx_f\, \Psi_0^*(x_f t_f)\, \Psi_0(x_0 t_0)\, \times \tag{30}$$

$$\times \int d''[x] \exp\left(-\frac{S[x]}{\hbar}\right) x(t_1)\dots x(t_n),$$

where

$$\begin{aligned} \Psi_0(x_0 t_0) &= \langle \Psi_0 | x_0 t_0 \rangle \\ \Psi_0^*(x_f t_f) &= \langle x_f t_f | \Psi_0 \rangle . \end{aligned} \tag{31}$$

Comparing (30) with (28) and taking into account (17) we see an analogy between the Fokker-Plank dynamical ensemble and the quantum system. We can also derive the second conclusion: to get the Euclidean measure $\Psi_0(\varphi)$ we have choose the initial probability density in the form:

$$P(\varphi, 0) = N \exp(-S[\varphi]). \tag{32}$$

The physical meaning of (32) is clear: we put the system from the beginning in the stationary (equilibrium) state and the presence of the Langevin dynamics does not modify it /8/. Implying that vacuum of $\hat{H}$ (16) is $\Psi_0(\varphi) = \exp\left(-\frac{1}{2}S[\varphi]\right)$ Gozzi named (28) the vacuum--vacuum generating functional.

Though this approach is elegant, it seems for us that the factor $P(\varphi, 0)$ is introduced "by hand" in expression (23) for the generating functional; therefore, in the following section we propose a probabilistic approach to construct it.

4. Dynamical generating functional. Probabilistic approach

It is the aim of this section to build the generating functional of the SQ method from the assumption that large "time" limit of noise average is equivalent to the quantum one without of additional "by hand" introduction of the factor $P(\varphi, 0)$ /8/. Due to the leading role of the Markov property we would like to do a brief survey of the

Markov process theory /14-16/. By definition a SP is Markovian under the condition that if a present state is known, any additional information about the history of a process is nonessential for future prognostication of the process behaviour. Thus, if the present is known, than past and future are relatively independent /16/. In particular, for a process governed by the Langevin equation to be Morkovian the correlations of noise η have to be δ -correlating at "time" τ. The Markov process is described by only two probability functionals: single "time" PDD and two "time" conditional PDD. By definition:

$$P(x_1;\dots;x_k|x_{k+1};\dots;x_n)=P(x_1;\dots;x_n)/P(x_{k+1};\dots;x_n). \tag{33}$$

For the Markov SP we have

$$P(x_1|x_2;\dots;x_n) = P(x_1|x_2) \tag{34}$$

$$P(x_1|x_3) = \int dx_2\, P(x_1|x_2)P(x_2|x_3) \tag{35}$$

$$P(x_1) = \int dx_2\, P(x_1|x_2)P(x_2). \tag{36}$$

This implies that $\tau_1 > \tau_2 > \dots > \tau_n$

Equations (35-36) are named the Chapman-Kolmogorov equations, respectively for the two "time" conditional PDD and single "time" PDD. We have to note that (36) is not the consequence of the Markov property but is a general property of the joint PDD:

$$\int dx_2\, P(x_1;x_2;x_3) = P(x_1;x_3). \tag{37}$$

Let us go back to building the generating functional. A starting point will be the following expression for the generating functional:

$$Z[J] = \int \mathcal{D}[\varphi]\, P[\varphi,\tau]\, exp(J\varphi), \tag{38}$$

where probability density functional:

$$P[\varphi,\tau] = \langle \Delta[\varphi-\varphi_\eta] \rangle_\eta \tag{39}$$

can be written in the following form /14/

$$P[\varphi,\tau] = \lim_{N\to\infty} P_N(\varphi_1\tau_1;\dots;\varphi_N\tau_N) \tag{40}$$

$$\varphi_i = \varphi(x_i), \quad \tau_0 \le \tau_i \le \tau$$

and $P_N(\ldots)$ is the N-dimensional joint PDD. Using the definition (33-37) for (40) we have

$$P_N(\varphi_1\tau_1;\ldots;\varphi_N\tau_N)=P(\varphi_1\tau_1|\varphi_2\tau_2)\ldots P(\varphi_{N-1}\tau_{N-1}|\varphi_N\tau_N)\,P(\varphi_N,\tau_N), \tag{41}$$

where

$$P(\varphi,\tau)=\Big\langle\prod_x\delta(\varphi(x\tau)-\varphi_\eta(x\tau))\Big\rangle_\eta \tag{42}$$

$$P(\varphi\tau|\varphi_1\tau_1)=\frac{\Big\langle\prod_{x,x_1}\delta(\varphi(x\tau)-\varphi_\eta(x\tau))\,\delta(\varphi(x_1\tau_1)-\varphi_\eta(x_1\tau_1))\Big\rangle_\eta}{P(\varphi_1,\tau_1)}. \tag{43}$$

For the functional (43) we have an obvious boundary condition:

$$P(\varphi\tau|\varphi_1\tau)=\delta[\varphi-\varphi_1]. \tag{44}$$

The boundary condition for the joint PDD (41) is the following:

$$P_N(\varphi_1\tau_1;\ldots;\varphi_N\tau_1)=\delta[\varphi_1-\varphi_2]\ldots\delta[\varphi_{N-1}-\varphi_N]\,P(\varphi_N,\tau_1). \tag{45}$$

The SQ prescription implies two operations over the many "time" stochastical averages $\langle\varphi_\eta(x_1\tau_1)\ldots\varphi_\eta(x_n\tau_n)\rangle_\eta$; namely

1) $\lim\limits_{\tau_i=\tau}\langle\ldots\rangle_\eta$

2) $\lim\limits_{\tau\to\infty}\langle\ldots\rangle_\eta$.

By means of (45) the first operation automatically isolates the single "time" PDD while the second one is responsible for the relaxation to equilibrium state. To express the conditional PDD (43) through the path integral we have to note that the "H amiltonian" operator $\hat{\mathcal{H}}$ (12) associated with $P(\varphi\tau|\varphi_1\tau_1)$ is non-Hermitian. After the "gauge" transformation

$$P(\varphi\tau|\varphi_1\tau_1)=W(\varphi\tau|\varphi_1\tau_1)\exp\Big\{-\tfrac{1}{2}\big(S[\varphi(\tau)]-S[\varphi(0)]\big)\Big\} \tag{46}$$

we get the Fokker-Plank equation with the Hermitian "Hamiltonian" (16). The boundary condition for Ψ is the same as for P (44)

$$\Psi(\varphi\tau|\varphi_1\tau) = \delta[\varphi-\varphi_1]. \tag{47}$$

Associating with the Hermitian "Hamiltonian" a Lagrangian $\mathcal{L}^{FP}$ according to the well-known prescription [11]

$$-\mathcal{L}^{FP} = p\dot{\varphi} - H \;, i\dot{\varphi} = \frac{\partial H}{\partial p} = -2p, \tag{48}$$

where H is the classical Hamiltonian function, we get for the Ψ following path integral expression

$$\Psi(\varphi\tau|\varphi_1\tau_1) = \int d[\varphi] \exp\left\{-\int_{\tau_1}^{\tau} dt\, \mathcal{L}^{FP}(t)\right\}, \tag{49}$$

where

$$\mathcal{L}^{FP}(t) = \frac{1}{4}(\dot{\varphi})^2 - \frac{1}{2}\frac{\delta^2 S}{\delta\varphi^2} + \frac{1}{4}\left(\frac{\delta S}{\delta\varphi}\right)^2. \tag{50}$$

Using (49-50), we arrive finally at the following path integral expression for the generating functional of the SQ method which coincides with the [8] vacuum-vacuum form of the generating functional

$$Z[J] = \int d[\varphi(0)]\, d[\varphi(\tau)]\, \mathcal{D}''[\varphi]\, P(\varphi,0) \exp\left\{-\frac{1}{2}S[\varphi(\tau)] + \frac{1}{2}S[\varphi(0)]\right\} \times \tag{51}$$

$$\times \exp\left\{-\int_0^{\tau} dt \left[\mathcal{L}^{FP}(t) - J\varphi\right]\right\}.$$

5. Conclusion

We have considered the simplest case, scalar field theory SQ, to build the generating functional. We think that the essential moment of the procedure is isolating of single " time" probability. This is true also in Zwanzinger's scheme [17]; hence this approach may be regarded also as an alternative to Alvarez-Estrada and Munoz-Sudupe [9] functional approach to the stochastic quantization of the Euclidean Yang-Mills field theory.

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