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ON CONSTRUCTION OF QUANTUM MECHANICS ON CUBIC FORMS

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Two quadratic forms represent the basis of the theoretical constructions of contemporary geometry and physics. The main geometrical notion - an interval - is a quadratic form. The Hamilton operator is a function of square of momentum; relativistic relationship between mass, energy, momentum is represented by a quadratic form. Finally, in Mechanics and in Geometry it is these groups of motion which are of great importance that remain an unchanged invariant - a certain quadratic form.

The fact that in the fundamental theoretical derivations a quadratic form has a dominant role results from the human manner of thought operating mainly by the binary system: "yes-no", "right-left", "positive-negative", etc. Accordingly, positive and negative charged particles and antiparticles, two-projection (half) spin, two-dimensional phase space, distance between two particles, 2-particle interaction, parity and so on appear in theory.

The interconnection between relativistic relationship of energy-momentum-mass and such notions as particle-antiparticle, left and right, half spin operator was discovered in the framework of the Dirac theory [1].

Let us assume that there exists 3-charged state of particle. The quarks [2] possessing 3-colour charge try primarily to play role of such particles. Further, it is naturally believed by analogy that the particles of one tricharged group are related to each other similar to particles and antiparticles in the Dirac theory.

Due to the idea of the Dirac theory for a description of these hypothetical particles the forms one degree higher than in conventional theory, namely, cubic forms, should be applied. The idea of composing a physical theory based on polylinear forms has been suggested in [3,4,5,6]. In this paper we represent one of the ways of constructing such a theory. The hypothetical world being described by this theory, might be called the triad world. For brevity sake, we avoid strict determinations and proofs and present the main properties of space triad world and describe trends of constructing Quantum Mechanics of particles moving in this space.

1. Geometry of triad world (Trigeometry) possesses very interesting properties. It contains analogy of notion of straight line, plane (surface), intercept (of a

a straight line), angle between two (or three) straight line, triangle, so on. But in trigeometry "straight line" passes through three dots, accordingly, only by means of three dots an interoept may be separated from a "straight line"; "triangle" has four sides and four angles, moreover, the angle is a complex value. In Euclidean geometry (that is naturally for dyad world) there exist only two opposite directions; in triad world we have three natural opposite directions. Each direction in dyad world has appropriate signs (+) or (-). Solutions of the equation $x^3 - 1 = 0$: $(1, \theta, \theta^2)$ play the role of a three-digit unit in triad case.

Let R be a set of positive real numbers including zero. Suppose that an element of R can be contrast to any point on trigeometry straight line. An intercept on this line gives relationship between three points $x_1, x_2, x_3 \subseteq R$, $x_1 > x_2 > x_3$. Determine the length of the intercept as an expression of the form

$$P[x_1, x_2, x_3] := x_1 + \theta x_2 + \theta^2 x_3. \qquad (1)$$

When $x_2 = x_3$, expression (1) turns to the ordinary form for intercept length bounded by points (x_1, x_2) on the ordinary straight line.

In trigeometry the figure to be analogous to a right triangle consists of three perpendiculars (A, B, D) and a hypothenuse (C). For such a figure a relationship is valid (analogy of Pyphagore theorem):

$$C^3 = A^3 + B^3 + D^3 - 3ABD. \qquad (2)$$

Introduce three functions:

$$G_0(\alpha, \beta) := 1/\gamma, \quad G_1(\alpha, \beta) := \alpha/\gamma, \quad G_2(\alpha, \beta) := \beta/\gamma, \qquad (3)$$

where $\alpha = B/A$, $\beta = D/A$, $\gamma = C/A$. These functions are analogous to functions $\mathrm{Cos}\,\phi$, $\mathrm{Sin}\,\phi$. It is clear that

$$G_0^3 + G_1^3 + G_2^3 - 3G_0G_1G_2 = 1.$$

As has been shown by Greav's[7], one can specify the exponent function

$$\mathrm{Exp}(\theta\ \phi) := G_0(\phi) + \theta G_1(\phi) + \theta^2 G_2(\phi),$$
$$(4)$$
$$\phi := \phi_1(\alpha, \beta) + \theta \phi_2(\alpha, \beta).$$

The summing formulas of parameters at exponent multiplication have the following form:

$$\mathrm{Exp}(\theta\varphi_1)\,\mathrm{Exp}(\theta\varphi_2) = \mathrm{Exp}(\theta\varphi_3),$$

$$\varphi_3(\alpha_3, \beta_3) = \varphi_1(\alpha_1, \beta_1) + \varphi_2(\alpha_2, \beta_2),$$

$$\alpha_3 = (\alpha_1 + \alpha_2 + \beta_1\beta_2)/\xi, \quad \beta_3 = (\beta_1 + \beta_2 + \alpha_1\alpha_2)/\xi,$$

$$\xi = 1 + \alpha_1\beta_2 + \beta_1\alpha_2 .$$

The plane in trigeometry triplane is tridimensional manifold. On the triplane the transformations such as rotation, remaining invariant cubic form, can be set:

$$I(3) := p_1^3 + p_2^3 + p_3^3 - 3p_1p_2p_3 . \tag{5}$$

These transformations might be set by using cyclic number representation:

$$p_1' + p_2'\theta + p_3'\theta^2 = (G_0 + \theta G_1 + \theta^2 G_2)(p_1 + p_2\theta + p_3\theta^2) . \tag{6}$$

The generalization of tridimensional cubic form (5) in the case of higher measurements can be efficiently used with the help of Dicson algebra basis [8,9], since Dicson algebra with respect to cubic form plays the role like Clifford algebra with respect to the quadratic forms [4].

2. Dicson algebra of cubic degree. The cyclic algebra of $N>2$ degree was discovered by Dicson [9]. That is why we call it Dicson algebra and denote by sign Dic(N). Elements Dic(N) are defined by relations:

$$\mathrm{Dic}(N) := \oplus u^n v^m F ,$$

$$uv = vu\theta , \quad v^N = e , \quad u^N = e ,$$

e is algebraic unit, F is a commutative field, the characteristic of the field is not a divisor of N, θ is a primitive root of equation $x^N - 1 = 0$. As is easily seen, Dic(N) is a generation of quaternion algebra. As is shown in [4] one can get matrix basis for linearization of the N-forms, using $u, v \in \mathrm{Dic}(N)$.

Let us observe in more detail the neighbour of quaternions - the algebra Dic(3).

The generatrix elements can be realized with the help of matrix representation [10]:

$$u := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad v := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta^2 \end{pmatrix}.$$

The matrix basis of Dic(3), including a unit matrix E, consists of 9 elements

$$\{\alpha_k\} := \{\alpha_0 = E,\ \alpha_1 = u,\ \alpha_2 = v,\ \alpha_3 = uv,\ \alpha_4 = u^2 v,$$

$$\alpha_5 = \alpha_1^2,\ \alpha_6 = \alpha_2^2,\ \alpha_7 = \alpha_3^2,\ \alpha_8 = \alpha_4^2\}.$$

The matrix system $\{\alpha_k\}$ is linear-independent, complete and $sp(\alpha_k) = 0$, $k = 1,2,\dots,8$. Any matrix A(3x3) given in field F, can be expended in basis $\{\alpha_k\}$:

$$A = \sum_{k=0}^{9} a_k \alpha_k, \qquad a_k = sp(\alpha_k^2 A)/3.$$

Let two matrices are given

$$A_1 := a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4,$$

$$A_2 := b_1\alpha_5 + b_2\alpha_6 + b_3\alpha_7 + b_4\alpha_8.$$

$$A_1^3 = f_1(a_1, a_2, a_3, a_4)\,E,$$

$$A_2^3 = f_2(b_1, b_2, b_3, b_4)\,E,$$

where $f_1(a)$, $f_2(b)$ are cubic forms and

$$\det A_1 = f_1(a), \qquad \det A_2 = f_2(b).$$

Basis (2.3) allows to obtain the linear transforms of cubic form

$$I(a,b) = I(A) := \left(\sum_{n=1}^{4} a_n \alpha_n + b_n \alpha_{n+4} \right)^3 ,$$

where

$$\sum_{n=1}^{4} a_n b_n = 0 .$$

Proposition

$$\sum_{k=1}^{4} \pi_k \pi_{k+4} = 0 .$$

Let:

$$\pi := \pi_0 + \sum_{k=1}^{8} \pi_k \alpha_k , \quad A(3) = A_1 + A_2 .$$

$$\bar{\pi} := \pi_0 + \theta \sum_{k=1}^{8} \pi_k \alpha_k , \quad \bar{\bar{\pi}} := \pi_0 + \theta^2 \sum_{k=1}^{8} \pi_k \alpha_k , \quad \pi \bar{\pi} \bar{\bar{\pi}} = E .$$

Then the cubic form I(a,b) is invariant relatively to transformations

$$A' = \pi A \bar{\pi} \bar{\bar{\pi}} ,$$
$$A'' = \pi \bar{\pi} A \bar{\bar{\pi}} . \tag{7}$$

The transform (7) consists of 7-parameters. As quality matrix of transformation one can obtain:

$$\hat{g} := G_0(\phi) + G_1(\phi)\,\alpha + G_2(\phi)\,\alpha^2 , \quad \alpha^3 = E .$$

The cubic form I(a,b) consists of coordinates of two 4-dimensional orthogonal (in a common sense) subspaces. Form (5) is generalized nothing more not less than in such a way at the further growth of space dimensionaling in trigeometry.

3. C u b i c q u a n t u m m e c h a n i c s c o n s t r u c t i o n p r i n c i p a l. The Cubic Quantum Mechanics is a Quantum Mechanics of particles moving in the space of trigeometry. This means that the invariant

values of the theory need a trilinear form. A wave functions of free state is described by an exponentional function (4). Accordingly, the notion of "unitary" is exchanged.

If in Quantum Mechanics the observable has a discrete spectrum of eigen values, then its linearity operator is a matrix Λ. In this case

$$\Lambda\psi_n = \lambda_n\psi_n , \quad U\Lambda = D(\lambda)\, U ,$$

where ψ_n is an eigen vector, U - unitary matrix, $D(\lambda)$ - diagonal matrix of the eigenvalue. Any unitary matrix has the exponential representation

$$U := \exp(i\phi) , \quad U^{-1} = \exp(-i\phi) , \quad UU^{-1} = E .$$

Analogy of unitary matrices in Cubic Quantum Mechanics (3 - unitary matrices) is

$$U_1 = \mathrm{Exp}(\theta\phi) , \quad U_2 = \mathrm{Exp}(\theta^2\phi) , \quad U_3 = \mathrm{Exp}(\phi) ,$$

$$U_1 U_2 U_3 = E .$$

Fundamental elements of Quantum Mechanics are angular moment, spin and creation and annihilation operators. Let us consider the analogies of those notions in Cubic Quantum Mechanics.

<u>Determinations</u>

1. 3-linearity anticommutator of 3 given operators: b_1, b_2, b_3 is

$$\{b_1 , b_2 , b_3\} := b_1 b_2 b_3 + b_2 b_3 b_1 + b_3 b_1 b_2 .$$

2. 3 - linearity commutator between 3 operators is defined as:

$$[b_1 , b_2 , b_3] := b_1 b_2 b_3 + \theta b_2 b_3 b_1 + \theta^2 b_3 b_1 b_2 .$$

Since we introduce the notion of a trilinear anticommutator, the Hamilton operator can be defined by analogy with the ordinary oscillator in a following way:

$$H(3) := \frac{1}{3}\{b_1 , b_2 , b_3\} .$$

The operators b_1, b_2, b_3 affected by the given state function may be defined by the following formulas:

$$b_1 \Phi_n = \sqrt[3]{n+1}\ \Phi_{n+1},$$

$$b_2 \Phi_n = \sqrt[3]{n+\theta+1}\ \Phi_{n+\theta},$$

$$b_3 \Phi_n = \sqrt[3]{n}\ \Phi_{n+\theta^2}.$$

Using the formulas, it is easy to find spectra of operators $\hat{N} := b_1 b_2 b_3$, $H(3)$

$$\hat{N} \Phi_n = n\,\Phi_n, \quad H(3)\,\Phi_n = \left(n + \frac{2+\theta^2}{3}\right) \Phi_n .$$

Here $b^+ := b_1$ plays the role of the creation operator; and $b^- := b_2 b_3$, the role of annihiligation operator. Note that

$$[\,b^+, b^-\,] = \mathbb{1} .$$

the operators of a trispin are elements of Dicson algebra:

$$\Sigma_k = \theta \hbar \alpha_k / 3 .$$

The corresponding analogues of the operators of angular moment can be obtained by using boson operators

$$b_k^+,\ b_i^-,\ (i,k = 1,2,3)\ ,\ [\,b_k^+, b_i^-\,] = \delta_{ki}\ ,\ [\,b_k^{\pm}, b_i^{\pm}\,] = 0 .$$

In this case a triangular moment has the form:

$$J_k = \theta \hbar\, \Sigma_k^{(np)} b_n^+ b_p^- .$$

As we suppose that the moving of particle occurs in the trigeometry space, the dependence of energy and momentum must be in the form invariant with respect to transformations (7). The direct analogue of the energy operator of Pauli equation is the following expression:

$$H = \frac{1}{3m^2} \left(\sum_{k=1}^{8} \alpha_k p_k \right)^3 , \qquad \sum_{i=1}^{4} p_i\, p_{i+4} = 0 . \tag{8}$$

The cubic form is the analog of relativistic relation between energy, momentum and mass:

$$(\mathcal{E}/c^2)^3 = (\sum_{k=1}^{8} \alpha_k p_k)^3 + (mc)^3 . \qquad (9)$$

In this case the expression $\mathcal{E}_0 = mc^3$ corresponds to rest energy. At $c \to \infty$ expression (9) transforms into (8). Accordingly, the analogs of the Klein-Gordon and Dirac equations can be written/5/.

In the frames of Quantum Field Theory electric charged particles are described by means of the complex wave functions. In the Cubic Quantum Mechanics tricharged states will be described by Greaves numbers. According to the Dirac theory the Charges and the rest of masses of different signs correspond to particles and antiparticles. In Cubic Quantum Mechanics the charges and rest of mass of particles interrelating to each other as antiparticles have factors 1, θ, θ^2.

One of the possible areas of application of Cubic Quantum Mechanics is hadron physics, whose fundamental objects (quarks) possess tricolour state.

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