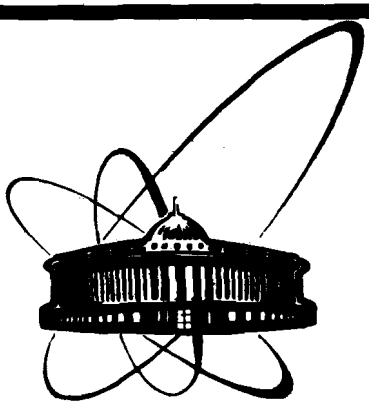

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V.Tapia*

A NOVEL METHOD TO SOLVE FUNCTIONAL DIFFERENTIAL EQUATIONS. APPLICATION TO QUANTUM FIELD THEORY

*Centro de Investigacion en Fisica, Universidad de Sonora, Apartado Postal A-088, Hermosillo, Sonora, Mexico

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I. Introduction

Recently Jackiw (1988) has revived the interest into the *Schrödinger representation* in quantum field theory. In this approach the dynamical quantities are expressed in terms of fixed-time canonical variables, $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$, which upon quantisation satisfy canonical commutation relations. The quantum field theory involves states $|\Psi>$ that, in the Schrödinger representation, are realised as functionals, $|\Psi> \rightarrow \Psi[\phi,t)$, of a time-independent, c-number field $\phi(\mathbf{x})$. The operator $\hat{\phi}(\mathbf{x})$ acts on these states by multiplication, $\Phi(\mathbf{x})|\Psi> \rightarrow \phi(\mathbf{x})\Psi[\phi,t)$, while the canonical momentum operator $\hat{\pi}(\mathbf{x})$ acts by variational differentiation, $\pi(\mathbf{x})|\Psi> \rightarrow -ih\delta\Psi[\phi,t)/\delta\phi(\mathbf{x})$.

In the Schrödinger representation the Schrödinger equation is given by

$$\hat{H}[\phi]\ \Psi[\phi,t) = ih\ \partial_t E\ \Psi[\phi,t) \ , \tag{I.1}$$

where the Hamiltonian is defined by

$$\hat{H}[\phi] = \int_\Sigma \hat{\mathcal{H}}(\phi(\mathbf{x}))\ d^3x \ , \tag{I.2}$$

where Σ is any space-like section of space-time. Since the Hamiltonian, and all other relevant quantities, are already integrated over space-like sections the use of functional methods is mandatory. The main drawback of this approach lies precisely on the use of functional methods. Only a few particular cases, e.g., those in which the starting Lagrangian is quadratic in the fields, can be solved exactly.

It must be, on the other hand, observed that $\Psi[\phi,t)$, being the total wave function, must be given as the product of the local wave functions $\psi(\phi(\mathbf{x}),t)$

$$\Psi[\phi,t) = \prod_{x\in\Sigma} \psi(\phi(\mathbf{x}),t) \ . \tag{I.3}$$

A remarkable property of the Schrödinger equation (I.1) is that it is solved by the local Schrödinger equation

$$\hat{\mathcal{H}}(\phi(\mathbf{x}))\ \psi(\phi(\mathbf{x}),t) = \tfrac{ih}{v}\ \partial_t \psi(\phi(\mathbf{x}),t) \ . \tag{I.4}$$

This Schrödinger equation is a differential equation in which the derivation operator is the variational one, which is iteratively defined at all orders by

$$\delta_A = \partial_A - d_a \delta_A{}^a \ , \tag{I.5a}$$

$$\delta_A{}^{a_1\cdots a_r} = \partial_A{}^{a_1\cdots a_r} - d_a \delta_A{}^{a a_1\cdots a_r} \ . \tag{I.5b}$$

$d_a = d/dx^a$ is the total or formal derivative with respect to x^a, the coordinates on the base space; $\partial_A^{a_1\cdots a_r} = \partial/\partial\phi^A_{a_1\cdots a_r}$, with ϕ^A the fields describing the system and $\phi^A_{a_1\cdots a_r} = d^r\phi^A/dx^{a_1}\cdots dx^{a_r}$. We will call them *variational differential equations*.

The solution to a variational differential equation will be a function containing the derivatives, with respect to the base space coordinates, of the fields up to a generic order s: a sth-order function. However, variational differential equations present a quite pathological behaviour. The problem lies on the fact that the variational derivation operator, in contrast with an ordinary derivation operator, doubles the order of the function on which acts. In fact, from the definition of the variational operator follows that, if $\psi(\phi(\mathbf{x}),t)$ is a function containing the derivatives of ϕ up to an order s, then $\delta\psi(\phi(\mathbf{x}),t)/\delta\phi$ will be a function containing the derivatives of $\phi(\mathbf{x})$ up to an order $2s$. Therefore, the different terms appearing in a variational differential equation will be of different orders. One way of solving this *impasse* is by making the solution of a variational differential equation to be a function containing the derivatives of the fields at all orders. But this takes us again back to functional methods.

If one insists into keep finite the order of the solution one sees that, in principle, unsolvable order incompatibilities appear between the different terms of a variational differential equation, unless one restrict the considerations to those functions for which their variational derivatives are of an order lesser than that naively implied by just doubling the order. In fact, the previous situation would drastically change if the order of $\delta\psi(\phi(\mathbf{x}),t)/\delta\phi$ would be lesser than that naively obtained just by doubling the order each time a variational derivation operator is acting. It is possible to characterise the functions for which this property holds. For sth-order functions we must just look at the kernel of sth-order functions mapping to rth-order, r<2s, functions under variational derivation. These functions, which we call *non-standard*, can be almost completely characterised in the sense that its dependence on the highest-order derivatives is completely determined. These functions turn to be polynomials in the highest-order derivatives of the fields with arbitrary functions of the lower-order derivatives as coefficients. The order of these polynomials strongly depends on the dimension of the base space. Then, functional differential equations reduce to a system of coupled partial differential equations for the coefficients we mentioned above.

Up to our knowledge, no general techniques for solving variational differential equations is actually known. We must emphasize furthermore that we do not know of any explicit reference to the facts quoted previously in the literature.

Let us see how the previous ideas apply to field theory. For a field theory in a (1+p)-dimensional space-time the Schrödinger equation becomes a functional differential equation in a p-dimensional base space. The Schrödinger equation can be solved by a variational Schrödinger equation on a p-dimensional base space. Therefore we need to characterise non-standard functions over p-dimensional base spaces. The Schrödinger equation reduces to a system of coupled partial differential equations. This provide therefore a non-perturbative scheme for quantum field theory.

In this article we will consider field theories in (1+1)-dimensions. In this case the Schrödinger equation is a variational differential equation in a one- dimensional base space. Therefore we need to characterise non-standard functions over one-dimensional base spaces. We do this for first- and second-order functions. The first-order case is quite trivial while the second one can be considered as a rereading of the results by Tapia (1985), therefore only the main results are presented.

Since most of our reasoning will be based on variational derivation we start by introducing functions over jets of fields over which the variational derivation operator acts. We introduce some elements of what can be called "Lagrangian analysis," i.e., the differential calculus for variational derivatives. Then we consider field theory, the canonical formalism and the canonical quantisation in which variational Schrödinger equations play a central role. We turn then to the massless scalar field and show in detail the order incompatibility we mentioned above.

Next we turn to characterise non-standard functions over one-dimensional base spaces. We then apply these results to the massless and massive, Klein-Gordon, scalar fields, in (1+1)-dimensions; in this two cases the solutions must be second- order non-standard functions. The Schrödinger equation reduces to two coupled partial differential equations. The massles case can be solved exactly. For the massive case we just write down the equations. We have not been able to solve them, but being similar to the Schrödinger equation for the harmonic oscillator in ordinary quantum mechanics we conjecture that they should admit Hermite-like solutions. If this is the case the energy spectrum is given by $E_n = \hbar\omega(n+1/2)$. This agrees with the fact that the Klein-Gordon field is the natural generalisation to higher dimensions of the harmonic oscillator.

We arrive finally to the conclusions and draw the forthcoming work in which the higher-dimensional, p>1, case will be considered.

II. Mathematical Preliminaries: Fibered Manifolds, Jet Prolongations, Functions on Jets and Functional Derivation

Let M be a differentiable manifold, with dim(M)=m, and let $B=(\beta,M,\pi)$ be a fibered manifold over the manifold M, with $\dim(\beta)=n$. The sth-order jet prolongation of a fibered manifold B, where s is any positive integer, will be denoted by $J^s(B)=(J^s(\beta),M,\pi^s)$; we agree that the zeroth-order prolongation of B coincides with B itself. We now give some local coordinate notations. On the manifold M we consider local coordinates x^a, a=1,...,m; on the fiber β at x we consider local coordinates $\phi^A(x)$, A=1,...,n, such that on the fibered manifold B we have local coordinates $(x^a,\phi^A(x))$. On the sth-order jet prolongation $J^{(s)}(B)$ we consider local coordinates

$$(x^a,\ \phi^A(x),\ \phi^A{}_a(x),\ \ldots,\ \phi^A{}_{a_1\ldots a_s}(x))$$

where

$$\phi^A{}_{a_1\ldots a_r}(x) = d^r\phi^A(x)/dx^{a_1}\ldots dx^{a_r} .$$

Let $\mathcal{F}^{(s)}(x)$ be an ordinary function on $J^{(s)}(B)$

$$\mathcal{F}^{(s)}(x) = \mathcal{F}^{(s)}(\phi^A(x),\ \phi^A{}_a(x),\ \ldots,\ \phi^A{}_{a_1\ldots a_s}(x)) , \qquad (II.1)$$

and let $f^{(s)}$ be the space of functions $\mathcal{F}^{(s)}(x)$. A function depending on up to the sth-order derivatives of the fields is to be called a sth-order function, for example, a sth-order Lagrangian.

Let us now consider functionals on $J^{(s)}(B)$. A functional is an ordinary function from $J^{(s)}(B)$ to some $\mathbb{R}^p$, with p some integer number. Equivalently it can be considered as a map which to any function $\mathcal{F}^{(s)}(x)$ associates a number. Two main kinds of functionals will be of importance for what follows: the σ-functionals, defined as the sum over M of the given function

$$F^{(s)}[\phi] = \int_M \mathcal{F}^{(s)}(x)\ d^mx , \qquad (II.2a)$$

and the π-functionals defined as the product of the given function over all M

$$Z^{(s)}[\phi] = \prod_{x\in M} Z^{(s)}(x) . \qquad (II.2b)$$

Let us now consider the variation $\delta F^{(s)}[\phi]$ of a σ-functional $F^{(s)}[\phi]$ under variations $\delta\phi^A(x)$ of $\phi^A(x)$, $\phi^A\to\phi^A+\delta\phi^A$. The result is

$$\delta F^{(s)}[\phi] = \int_M \delta_A\mathcal{F}^{(s)}(x)\ \delta\phi^A(x)\ d^mx + \int_{\partial M} n_a(x)\ [\delta_A{}^a\mathcal{F}^{(s)}(x)\ \delta\phi^A(x) + \delta_A{}^{ab}\mathcal{F}^{(s)}(x)\ \delta\phi^A{}_b(x) + \ldots]\ d^{m-1}S , \qquad (II.3)$$

where

$$\delta\phi^A{}_{a_1\ldots a_r}(x) = (\delta\phi^A(x))_{a_1\ldots a_r} , \text{ etc} . \qquad (II.4)$$

We have furthermore defined, at all orders,

$$\delta_A = \partial_A - d_a\delta_A{}^a , \qquad (II.5a)$$

where $\delta_A{}^a$ is defined iteratively by

$$\delta_A{}^{a_1\ldots a_r} = \partial_A{}^{a_1\ldots a_r} - d_a\delta_A{}^{aa_1\ldots a_r} . \qquad (II.5b)$$

$d_a=d/dx^a$ is the total or formal derivative with respect to x^a; $\partial_A{}^{a_1\ldots a_r}=\partial/\partial\phi^A{}_{a_1\ldots a_r}$. ∂M is the boundary of M and n_a is the normal to ∂M; $d^{m-1}S$ is its (m-1)-dimensional volume element. Throughout all the work we will restrict our attention to functions satisfying

$$\delta_A{}^{a_1\ldots a_r}\mathcal{F}^{(s)}(x)\Big|_{\partial M} = 0 , \qquad (II.6)$$

Therefore, equation (II.3) reduces to

$$\delta F^{(s)}[\phi] = \int_M \delta_A\mathcal{F}^{(s)}(x)\ \delta\phi^A(x)\ d^mx . \qquad (II.7)$$

The variational derivative of $\mathcal{F}^{(s)}(x)$ is defined as the factor appearing in (II.7) multiplying the variation $\delta\phi^A$ under the integral sign. Therefore, according with (II.5), is given at all orders by

$$\delta_A\mathcal{F}^{(s)} = \partial_A\mathcal{F}^{(s)} - d_a\partial_A{}^a\mathcal{F}^{(s)} + \cdots + (-)^r\, d_{a_1}\cdots d_{a_r}\partial_A{}^{a_1\ldots a_r}\mathcal{F}^{(s)} + \cdots + (-)^s\, d_{a_1}\cdots d_{a_s}\partial_A{}^{a_1\ldots a_s}\mathcal{F}^{(s)} , \qquad (II.8)$$

and therefore coincides with the usual Euler-Lagrange derivative.

It can now be checked that

$$\delta_A : f^{(s)} \to f^{(2s)} . \qquad (II.9)$$

An important property of the variational derivative is

$$\delta_A(d_a\Lambda^a(\phi,\ldots)) \equiv 0 . \qquad (II.10)$$

Therefore we can write

$$\delta_A \delta_B = \delta_A \partial_B \ . \qquad \text{(II.11)}$$

A word concerning the definition of the total, or formal, derivative is in order here. As appearing in (II.5) it is defined in the following way

$$d_a = \partial_a + \phi^A{}_a \, \partial_A + \cdots + \phi^A{}_{a a_1 \cdots a_r} \, \partial_A{}^{a_1 \cdots a_r} + \cdots \ . \qquad \text{(II.12)}$$

Therefore, it is a map acting as

$$d_a : f^{(s)} \to f^{(s+1)} \ . \qquad \text{(II.13)}$$

Therefore, the formal derivative goes from the fiber $J^{(s)}(B)$ at x to the fiber $J^{(s+1)}(B)$ at x. We see that in this definition the concept of infinitesimally close fiber does not appear. In the usual definition the formal derivative is defined by taking resource to two infinitesimally close fibers $J^{(s)}(B)$ at x and $J^{(s)}(B)$ at $x+\Delta x$. But both definitions are completely equivalent.

III. Lagrangian Analysis

Here we introduce some elements of what can be called "Lagrangian analysis." The results of this section will be useful for developing quantum field theory. In this Lagrangian analysis the role of the derivation is played by the variational derivative. But δ_A directly cannot be the correct operator since it is not a derivation operator, in fact, it does not satisfy the Leibniz rule

$$\delta_A(FG) \neq G\delta_A F + F\delta_A G \ . \qquad \text{(III.1)}$$

This is due to the fact that the variational derivative is a on-linear oerator. Therefore, we must look for an alternative definition. The problem can be solved if we conveniently redefine the product of functions defined on different points. Next we define an operator $\mathcal{D}_A(x)$ satisfying all the requirements for being a differential operator. With the previous prescription for the product of functions this differential operator gives a convenient generalisation of the Leibniz rule for the variational derivative.

Next we consider integration. In order to do that we take resource to a remarkable identity derived quite recently in (Tapia *et al.*, 1989). This identity allows to define an operator playing the role of integration. In this section all functions $\mathcal{F}$ are assumed as defined on a jet $J^{(s)}(B)$ of a generic order s.

First of all we must give a prescription on how the product of

operators in different point is to be defined. We do it in the following way

$$\mathcal{F}_1(x_1) \cdots \mathcal{F}_r(x_r) = V^{r-1} \int_M \mathcal{F}_1(x_1;z) \cdots \mathcal{F}_r(x_r;z) \, d^m z \ , \qquad \text{(III.2)}$$

where

$$\mathcal{F}_1(x) = \int_M \mathcal{F}_1(x;z) \, d^m z \ , \qquad \text{(III.3)}$$

For ordinary functions it is enough to consider

$$\mathcal{F}(x;z) = \delta^{(m)}(x-z) \, \mathcal{F}(z) \ , \qquad \text{(III.4)}$$

where $\delta^{(m)}(\cdot)$ is a m-dimensional delta function. However, some operator valued functions have less trivial kernels. Let us consider for example the operator

$$\mathcal{D}_A(x) = \int_M [\delta^{(m)}(x-z) \, \partial_A(z) + d\delta^{(m)}(x-z)/dz^a \, \partial_A{}^a(z) + \cdots] \, d^m z \ . \qquad \text{(III.5)}$$

According to the rule (III.2) this operator acting on ordinary functions is defined by

$$\mathcal{D}_A(x) \, \mathcal{F}(y) = V \int_M \delta^{(m)}(y-z) \, [\delta^{(m)}(x-z) \, \partial_A(z) + d\delta^{(m)}(x-z)/dz^a \, \partial_A{}^a(z) + \cdots] \, \mathcal{F}(z) \, d^m z \ , \qquad \text{(III.6)}$$

such that when restricted to functions satisfying (II.6) we obtain

$$\mathcal{D}_A(x) \, \mathcal{F}(y) = V \int_M \delta^{(m)}(y-z) \, \delta^{(m)}(x-z) \, \delta_A \mathcal{F}(z) \, d^p z = V \, \delta^{(m)}(x-y) \, \delta_A \mathcal{F}(x) \ . \qquad \text{(III.7)}$$

The Leibniz rule takes the following form

$$\begin{aligned} \mathcal{D}_A(x) \, [\mathcal{F}(y) \, \mathcal{G}(z)] &= V^2 \int_M \delta^{(m)}(y-t) \, \delta^{(m)}(z-t) \, [\delta^{(m)}(x-t) \, \partial_A(t) + d\delta^{(m)}(x-t)/dx^a \, \partial_A{}^a(t) + \cdots] \, \mathcal{F}(t) \, \mathcal{G}(t) \, d^p t \\ &= V^2 \int_M \delta^{(m)}(y-t) \, \delta^{(m)}(z-t) \, \delta^{(m)}(x-t) \, [\mathcal{F}(t) \, \delta_A \mathcal{G}(t) + \mathcal{G}(t) \, \delta_A \mathcal{F}(t)] \, d^p t \\ &= V^2 \, \delta^{(m)}(y-x) \, \delta^{(m)}(z-x) \, x \, [\mathcal{F}(x) \, \delta_A \mathcal{G}(x) + \mathcal{G}(x) \, \delta_A \mathcal{F}(x)], \end{aligned} \qquad \text{(III.8)}$$

which is a correct generalisation of the Leibniz rule for the

variational derivative. For the fundamental variables we then obtain

$$\mathcal{D}_A(x)\ \phi^B(y) = V\ \delta^{(m)}(x-y)\ \delta^B_A\ . \qquad \text{(III.9)}$$

The same operator acting on functionals is given by

$$\mathcal{D}_A(x)\ F[\phi] = \mathcal{D}_A(x) \int_M \mathcal{F}(z)\ d^m z = \int_M \mathcal{D}_A(z)\ \mathcal{F}(z)\ d^m z$$

$$= \int_M V\ \delta^{(m)}(x-z)\ \delta_A \mathcal{F}(z)\ d^m z = V\ \delta_A \mathcal{F}(x)\ . \qquad \text{(III.10)}$$

It is then easy to check that

$$\mathcal{D}_A(x)\ (F[\phi]\ G[\phi]) = V^2\ [\mathcal{G}(x)\ \delta_A \mathcal{F}(x) + \mathcal{F}(x)\ \delta_A \mathcal{G}(x)]\ . \qquad \text{(III.11)}$$

The next step is to consider integration over jets, i.e., to look for an operator inverse to the derivation one. In order to do that we will make use of a remarkable identity for functions on jets derived recently in (Tapia *et al.*, 1989). Doing so we will be able to construct an integration operator S which is inverse to $\mathcal{D}$. Let us consider a function $\mathcal{F}$ and define $f^a(\mathcal{F})$ by

$$f^a(\mathcal{F}) = \int_0^1 [\phi^A\ (\delta_A{}^a(\mathcal{F}))\hat{}\ + \phi^A{}_b\ (\delta_A{}^{ba}(\mathcal{F}))\hat{}\ + \ldots]\ d\tau\ , \qquad \text{(III.12)}$$

where $(\cdot)\hat{}$ means that in the corresponding expression all variables, ϕ, $\partial\phi$, etc, have been scaled by a real factor $\tau\in[0,1]$ (i.e., $\hat{\phi}=\tau\phi$, etc). Then we have

$$d_a f^a(\mathcal{F}) = -\phi^A \int_0^1 (\delta_A(\mathcal{F}))\hat{}\ d\tau + \mathcal{F} - \mathcal{F}(0)\ , \qquad \text{(III.13)}$$

where $\mathcal{F}(0)=\mathcal{F}(\tau)\Big|_{\tau=0}$. Since $\mathcal{F}(0)$ is an irrelevant constant we can simply write

$$\mathcal{F} \equiv \phi^A \int_0^1 (\delta_A(\mathcal{F}))\hat{}\ d\tau + d_a f^a(\mathcal{F})\ . \qquad \text{(III.14)}$$

This is our announced identity.

In (Tapia *et al.*, 1989) the previous identity was used to prove, to any order, that the necessary and sufficient condition for obtaining identically vanishing field equations is that the Lagrangian is a divergence, as can be read inmediatly from (III.14).

It was, furthermore, used to remove from field theory the ambiguities related to the non-invariance of the energy-momentum tensor under the addition of a divergence to the Lagrangian. In this case one must select a representative for d-equivalent Lagrangians in which the divergence part has been removed. We must in that case look for a linear operator f acting on functions $\mathcal{F}$ by

$$\tilde{\mathcal{F}} \equiv f(\mathcal{F}) = \mathcal{F} + d_a g^a(\mathcal{F})\ , \qquad \text{(III.15)}$$

with $g^a(\mathcal{F})$ some function to determine. In (Tapia *et al.*, 1989) it was shown that the correct operator is given by

$$\tilde{\mathcal{F}} \equiv f(\mathcal{F}) \equiv \mathcal{F} - d_a f^a(\mathcal{F}) = \phi^A \int_0^1 (\delta_A(\mathcal{F}))\hat{}\ d\tau\ . \qquad \text{(III.16)}$$

A further property of $f(\cdot)$ is that of being a projector

$$f^2 = f\ . \qquad \text{(III.17)}$$

Therefore, when (III.17) is applied to $\tilde{\mathcal{F}}$ we obtain

$$\tilde{\mathcal{F}} = \phi^A \int_0^1 (\delta_A(\tilde{\mathcal{F}}))\hat{}\ d\tau\ . \qquad \text{(III.18)}$$

We will call the functions $\tilde{\mathcal{F}}$ satisfying (III.19) d-invariant functions.

Let us now consider ordinary functions on jets

$$\mathcal{F}(x) = \int_M \mathcal{F}(x;z)\ d^m z\ . \qquad \text{(III.19)}$$

The differential of this function is given by

$$d\mathcal{F}(x) = \int_M [\partial_A(z)\mathcal{F}(x;z)\ d\phi^A(z) + \partial_A{}^a(z)\mathcal{F}(x;z)\ d\phi^A{}_a(z) + \cdots]\ d^m z$$

$$= \int_M [\delta_A(z)\mathcal{F}(x;z)\ d\phi^A(z)$$

$$+ d_a(z)[\partial_A{}^a(z)\mathcal{F}(x;z)\ d\phi^A(z) + \cdots]]\ d^m z$$

$$= \int_M \delta_A(z)\mathcal{F}(x;z)\ d\phi^A(z)\ d^m z\ , \qquad \text{(III.20)}$$

since we assume that $\mathcal{F}(x;z)$ satisfies (II.6).

The integration operator must be such that

$$\mathsf{S}(d\mathcal{F}(x)) = d(\mathsf{S}(\mathcal{F}(x))) = \mathcal{F}(x)\ . \qquad \text{(III.21)}$$

We have found that the right definition for $\mathsf{S}(\cdot)$ doing the work is

$$\mathsf{S}(\cdot) = \int_0^1 (\cdot)\hat{}\ . \qquad \text{(III.22)}$$

In fact

$$\mathsf{S}(d\mathcal{F}(x)) = \int_0^1 \left[\int_M \delta_A(y)\mathcal{F}(x;z)\ d\phi^A(z)\ d^m z\right]\hat{}$$

$$= \int_M \phi^A(z) \int_0^1 [\delta_A(z)\mathcal{F}(x;z)]\hat{}\ d\tau\ d^m z$$

$$= \int_M \left[\mathcal{F}(x;z) - d_a(z) f^a(\mathcal{F}(x;z))\right] d^m z = \mathcal{F}(x)\ , \qquad \text{(III.23)}$$

where again we have used the fact we assumed that $\mathcal{F}(x;z)$ satisfies (II.6). This shows that (III.22) is the correct integration operator.

The previous corresponds to the line integral. The next step is to construct higher-dimensional integrals. The most important of them will be the volume integral over the jet. This volume integral is defined as the natural generalisation of definition (III.22) for the

line integral

$$S_B(d^s\mathcal{F}) = \int_0^1 \cdots \int_0^1 \mathcal{F}^\wedge \phi^1 \cdots \phi^n \, d\tau_1 \cdots d\tau_n \ , \qquad \text{(III.24)}$$

n times

The previous definitions correspond to undefined integrals. In order to obtain the defined one the previous expression must still be evaluated between two sets of values for the fields $(\phi,\partial\phi,\cdots)$, i.e., between two configurations of the system

$$S_d(d\mathcal{F}) = \Big|_{\phi_1}^{\phi_f} \int_0^1 (d\mathcal{F})^\wedge \ . \qquad \text{(III.25)}$$

In order to check that the previous definition works properly let us consider the simple example

$$\mathcal{F} = e^{-\phi^2} \ , \qquad \text{(III.26)}$$

with $\phi_1=-\infty$, $\phi_f=\infty$. Then

$$\begin{aligned} S_d(e^{-\phi^2}) &= \Big|_{\phi=-\infty}^{\phi=\infty} \int_0^1 (e^{-\phi^2})^\wedge \, \phi \, d\tau = \Big|_{\phi=-\infty}^{\phi=\infty} \int_0^1 e^{-\tau^2\phi^2} \, \phi \, d\tau \\ &= \Big|_{\phi=-\infty}^{\phi=\infty} \int_0^\phi e^{-u^2} \, du = \int_0^\infty e^{-u^2} \, du - \int_0^{-\infty} e^{-u^2} \, du \\ &= \int_{-\infty}^{\infty} e^{-u^2} \, du = \sqrt{\pi} \ . \end{aligned} \qquad \text{(III.27)}$$

The next natural step is to consider variational differential equations. In order to motivate their introduction we will consider field theory where, at the quantum mechanical level, they play a fundamental role.

IV. Field Theory

Field theory is the study of dynamical systems in which the dimension of the base space is greater than one. There a dynamical system is described by the fields $\phi^A(x^\mu)$, where A=1,...,n, and n is the number of fields; we assume they are sections of suitable vector bundles over the base space. x^μ, μ=0,...,m-1, are local coordinates in the m-dimensional base space.

IV.1. The Lagrangian Formalism

As in classical mechanics the dynamical information of a physical system is contained in the Lagrangian density

$$\mathcal{L} = \mathcal{L}(\phi^A, \phi^A{}_\mu) \ , \qquad \text{(IV.1)}$$

where $\phi^A{}_\mu = \partial\phi^A/\partial x^\mu$. For simplicity we restrict our considerations to first-order Lagrangians and to the case in which the Lagrangian density does not depend explicitly on the coordinates x^μ of the base space; the generalisation to the explicitly dependent case is straightforward.

The dynamics of the system is governed by field equations

$$\delta^{(m)}_A \mathcal{L} = 0 \ , \qquad \text{(IV.2)}$$

where

$$\delta^{(m)}_A = \partial_A - d_\mu \partial^\mu{}_A \ , \qquad \text{(IV.3)}$$

where m in $\delta^{(m)}_A$ means that the base space on which δ acts is m-dimensional.

In the classical mechanics of discrete systems the time plays a quite preferential role since it is the only coordinate of the base space. When going to field theory one must face an increase of the base space dimension from one to m. This problem is solved by considering field theory as classical mechanics with an infinite number of degrees of freedom, the canonical theory.

The first problem is to individuate an evolution parameter playing the role of the time. In order to achieve this we must assume that the base space is a space- time, i.e., a Riemannian space Ω with signature (1,p), m=1+p. We furthermore assume that Ω is simply connected. This allows to introduce a 1+p splitting of space-time by means of a system of simply connected space-like surfaces Σ and a time-like interval T, we then can locally write $\Omega=\Sigma\otimes T$. This is equivalent to the splitting $x^\mu=(t,x^i)$, i=1,...,p, has the usual meaning of a space-like index. We furthermore restrict our considerations to regions Ω of space-time of the form $\Omega=\Sigma\otimes[t_1,t_2]$, $[t_1,t_2]\in T$. Then Ω is limited in the time-like direction by the space-like surfaces Σ_1 and Σ_2 at times t_1 and t_2, respectively. In the space-like directions is limited by $\partial\Sigma$. We assume that Σ is an open space therefore S has the topology of a p-sphere, S^p. Σ can then be written as $\Sigma=I\otimes\partial\Sigma$, with $I=[0,\infty)$ a radial-like interval. The boundary is formally put at infinity.

The second step is to consider the system as a mechanical one but with an infinite number of degrees of freedom: the values of the field components at the various points of the space-like surfaces Σ for fixed t. Then the discrete label i in q^i (of classical mechanics) becomes a continuous label x^i plus additional discrete labels A; in

this way q^i is replaced by the fields $\phi^A(x^i)$. The sum over the discrete label i becomes an integration over the continuous label x^i over all Σ plus a sum over the discrete label A. The derivatives of the field components with respect to the time, $\dot{\phi}^A(x^i)=d\phi^A(x^i)/dt$, are defined as the velocities. Dots denote total time derivatives. In what follows we will suppress the continuous label x^i; this cannot give rise to any confusion.

The equations of motion can be considered as those from classsical mechanics but with an infinite number of degrees of freedom. In this case one must face an increase of the configuration space dimension from n to infinity. In this transition partial derivatives become partial variational (Lagrangian) derivatives. Therefore, in order to obtain canonical field theory we must just consider a 1+p splitting of the previous results.

The variational derivative can be rewritten as

$$\delta^{(m)}_A = \delta^{(p)}_A - d_t \dot{\partial}_A \ , \tag{IV.4}$$

where

$$\delta^{(p)}_A = \partial_A - d_1 \partial_A{}^1 \ , \tag{IV.5a}$$

$$\dot{\partial}_A = \partial/\partial\dot{\phi}^A \ , \tag{IV.5b}$$

such that

$$\delta^{(m)}_A \mathcal{L} = \delta^{(p)}_A \mathcal{L} - \dot{\pi}_A(\mathcal{L}) = 0 \ , \tag{IV.6}$$

where

$$\pi_A(\mathcal{L}) = \dot{\partial}_A \mathcal{L} \ , \tag{IV.7}$$

is the momenta canonically conjugated to ϕ^A.

IV.2. The Hamiltonian Formalism

A Hamiltonian mechanics for field theory may be set up still in parallel with classical mechanics. The canonical Hamiltonian is defined as

$$H_c[\mathcal{L}] = \int_\Sigma \mathcal{H}_c(\mathcal{L}) \ d\Sigma \ . \tag{IV.8}$$

where $\mathcal{H}_c(\mathcal{L})$ is the canonical Hamiltonian density

$$\mathcal{H}_c(\mathcal{L}) = \dot{\phi}_A \ \pi_A(\mathcal{L}) - \mathcal{L} \ . \tag{IV.9}$$

The variation of $\mathcal{H}_c(\mathcal{L})$ is

$$\delta\mathcal{H}_c(\mathcal{L}) = - \partial_A \mathcal{L} \ \delta\phi^A - \partial_A{}^1 \mathcal{L} \ \delta\phi^A{}_1 + \dot{\phi}^A \ \delta\pi_A(\mathcal{L}) \ , \tag{IV.10}$$

which shows that $\mathcal{H}_c(\mathcal{L})$ has the dependence

$$\mathcal{H}_c(\mathcal{L}) = \mathcal{H}_c(\phi^A, \ \phi^A{}_1, \ \pi_A(\mathcal{L})) \ . \tag{IV.11}$$

The variables $(\phi^A, \phi^A{}_1, \pi_A(\mathcal{L}))$ span the phase space.

The variation of the Hamiltonian is

$$\delta H_c[\mathcal{L}] = \int_\Sigma \delta\mathcal{H}_c(\mathcal{L}) \ d\Sigma = \int_\Sigma \left[\dot{\phi}^A \ \delta\pi_A(\mathcal{L}) - \delta^{(p)}_A \mathcal{L} \ \delta\phi^A\right] d\Sigma \ . \tag{IV.12}$$

From here one obtains

$$\frac{\delta^{(p)}\mathcal{H}_c(\mathcal{L})}{\delta\pi_A(\mathcal{L})} = \dot{\phi}^A \ , \tag{IV.13a}$$

$$\frac{\delta^{(p)}\mathcal{H}_c(\mathcal{L})}{\delta\phi^A} = - \delta^{(p)}_A \mathcal{L} \ . \tag{IV.13b'}$$

When the field equations (IV.2) hold eq.(IV.13b') reduces to

$$\frac{\delta^{(p)}\mathcal{H}_c(\mathcal{L})}{\delta\phi^A} = - \dot{\pi}_A(\mathcal{L}) \ . \tag{IV.13b}$$

The Lagrangian dynamics is reproduced through the Hamilton equations (IV.13).

Let us now consider ordinary functions on the phase space $\mathcal{F}=\mathcal{F}(\phi(\mathbf{x}),\pi(\mathbf{x}))$. Then we define σ-functionals

$$F[\phi,\pi] = \int_\Sigma \mathcal{F}(\phi,\pi) \ d^p z \ . \tag{IV.14}$$

A symplectic structure is induced on the phase space when considering the time derivative of a functional in the phase space which is given by

$$\dot{F} = \{F,H_c\} \ , \tag{IV.15}$$

where

$$\{F,G\} = \int_\Sigma \left(\frac{\delta^{(p)}\mathcal{F}(\mathbf{z})}{\delta\phi^A(\mathbf{z})} \frac{\delta^{(p)}\mathcal{G}(\mathbf{z})}{\delta\pi_A(\mathbf{z})} - (\mathcal{F}\leftrightarrow\mathcal{G})\right) d^p z \ , \tag{IV.16}$$

is the Poisson bracket. This Poisson bracket induces a symplectic structure on the phase space. This is true only for functionals restricted to satisfy (II.6) which has been assume in deriving (IV.16).

The canonical variables can be written as functionals with a delta function as kernel

$$\phi^A(\mathbf{x}) = \int_\Sigma \phi^A(\mathbf{z}) \ \delta^{(p)}(\mathbf{x}-\mathbf{z}) \ d^p z \ , \tag{IV.17a}$$

$$\phi^A{}_1(\mathbf{x}) = \int_\Sigma \phi^A{}_1(\mathbf{z}) \ \delta^{(p)}(\mathbf{x}-\mathbf{z}) \ d^p z \ , \tag{IV.17b}$$

$$\pi_A(\mathbf{x}) = \int_\Sigma \pi_A(\mathbf{z})\ \delta^{(p)}(\mathbf{x}-\mathbf{z})\ d^p z\ , \qquad \text{(IV.17c)}$$

where boldface letters stand for space-like. This allows to define the densities

$$\varphi^A(\mathbf{x};\mathbf{z}) = \phi^A(\mathbf{z})\ \delta^{(p)}(\mathbf{x}-\mathbf{z})\ , \qquad \text{(IV.18a)}$$

$$\varphi^A{}_1(\mathbf{x};\mathbf{z}) = \phi^A{}_1(\mathbf{z})\ \delta^{(p)}(\mathbf{x}-\mathbf{z})\ , \qquad \text{(IV.18b)}$$

$$\Pi_A(\mathbf{x};\mathbf{z}) = \pi_A(\mathbf{z})\ \delta^{(p)}(\mathbf{x}-\mathbf{z})\ . \qquad \text{(IV.18c)}$$

Then, one obtains the Lagrangian derivatives

$$\frac{\delta^{(p)}\varphi^A(\mathbf{x};\mathbf{z})}{\delta\phi^B(\mathbf{z})} = \delta^A_B\ \delta^{(p)}(\mathbf{x}-\mathbf{z})\ , \qquad \frac{\delta^{(p)}\varphi^A(\mathbf{x};\mathbf{z})}{\delta\pi_B(\mathbf{z})} = 0\ ,$$

$$\frac{\delta^{(p)}\varphi^A{}_1(\mathbf{x};\mathbf{z})}{\delta\phi^B(\mathbf{z})} = \delta^A_B\ d\delta^{(p)}(\mathbf{x}-\mathbf{z})/dx^1\ , \qquad \frac{\delta^{(p)}\varphi^A{}_1(\mathbf{x};\mathbf{z})}{\delta\pi_B(\mathbf{z})} = 0\ ,$$

$$\frac{\delta^{(p)}\Pi_A(\mathbf{x};\mathbf{z})}{\delta\phi^B(\mathbf{z})} = 0\ , \qquad \frac{\delta^{(p)}\Pi_A(\mathbf{x};\mathbf{z})}{\delta\pi_B(\mathbf{z})} = \delta^B_A\ \delta^{(p)}(\mathbf{x}-\mathbf{z})\ . \qquad \text{(IV.19)}$$

Therefore, for the canonical variables one obtains

$$\{\phi^A(\mathbf{x}),\pi_B(\mathbf{y})\} = \delta^A_B\ \delta^{(p)}(\mathbf{x}-\mathbf{y})\ . \qquad \text{(IV.20a)}$$

$$\{\phi^A{}_1(\mathbf{x}),\pi_B(\mathbf{y})\} = \delta^A_B\ d\delta^{(p)}(\mathbf{x}-\mathbf{y})/dx^1\ . \qquad \text{(IV.20b)}$$

The Hamilton equations are obtained by putting F equal to the canonical variables in (IV.15).

V. Quantum Field Theory

There exist several equivalent methods for going to quantum mechanics. Here we adopt the canonical quantisation method. We start by introducing the fundamentals of quantum field theory which can be considered as the direct extension of the canonical method for discrete systems with ordinary derivatives replaced by variational derivatives.

V.1. Fundamentals of Quantum Field Theory

A generic classical observable, represented by some function $\mathcal{F}=\mathcal{F}(\phi,\pi)$, is associated with the quantum mechanical observable

represented by the operator $\hat{\mathcal{F}}=\mathcal{F}(\hat{\phi},\hat{\pi})$, where careted quantities are the quantum mechanical counterparts of the classical ones. Summarising canonical quantisation is obtained through the map

$$\hat{}:\mathcal{F} \to \hat{\mathcal{F}} = \mathcal{F}(\hat{\phi},\hat{\pi})\ . \qquad \text{(V.1)}$$

This map must be a homomorphism, i.e., it must preserve the algebra of observables

$$[\hat{A},\hat{B}] = \tfrac{ih}{v}\ \widehat{\{A,B\}}\ . \qquad \text{(V.2)}$$

V is the p-dimensional volume element of Σ

$$V = \int_\Sigma d^p x\ . \qquad \text{(V.3)}$$

For an open space, which is the case, this quantity is infinity, but we will take the limit $V\to\infty$ at the end of the calculation since other V's will appear in the game.

For the canonical field variables one must have

$$[\hat{\phi}^A(\mathbf{x}),\hat{\Pi}_B(\mathbf{y})] = \tfrac{ih}{v}\ \delta^A_B\ \delta^{(p)}(\mathbf{x}-\mathbf{y})\ , \qquad \text{(V.4a)}$$

$$[\hat{\phi}^A{}_1(\mathbf{x}),\hat{\Pi}_B(\mathbf{y})] = \tfrac{ih}{v}\ \delta^A_B\ d\delta^{(p)}(\mathbf{x}-\mathbf{y})/dx^1\ , \qquad \text{(V.4b)}$$

and all other brackets equal to zero.

On the Hilbert space of state vectors there is furthermore defined an internal product $\langle\psi_1|\psi_2\rangle$. The physical state vectors are normalised in such a way that

$$\langle\psi|\psi\rangle = 1\ . \qquad \text{(V.5)}$$

The macroscopic, or observable value, of a quantum mechanical operator is given by

$$\langle F\rangle = \langle\psi|\hat{F}|\psi\rangle\ . \qquad \text{(V.6)}$$

Since the Lagrangian is classically connected with observable quantities it must be a real quantity. Quantum mechanically the equivalent condition is Hermiticity

$$\langle\psi_1|\hat{F}|\psi_2\rangle = \langle\psi_1|\hat{F}|\psi_2\rangle^* = \langle\psi_2|\hat{F}^\dagger|\psi_1\rangle = \langle\psi_2|\hat{F}|\psi_1\rangle\ , \qquad \text{(V.7)}$$

where * denotes complex conjugation.

The quantum mechanical behaviour of the system is described by the state vector $|\psi\rangle$. The physical states are those satisfying the Schrodinger equation

$$\hat{H}\ |\psi\rangle = \hat{E}\ |\psi\rangle\ , \qquad \text{(V.8)}$$

where $\hat{H}$ is the quantum mechanical Hamiltonian

$$\hat{H} = H_c[\hat{\phi},\hat{\pi}] \ ; \qquad (V.9)$$

and $\hat{E}$ is the energy operator.

V.2. Canonical Quantum Field Theory

Now we look for a concrete operator representation for $\hat{\phi}$ and $\hat{\pi}$ satisfying (V.4). The operators turn to be differential operators on jets. Canonical quantisation in field theory is obtained by generalising the usual replacement $q\to q$, $p\to -ih\partial$, of classical mechanics to $\phi\to\phi$, $\pi\to -(ih/V)\delta/\delta\phi$. For the Hilbert space one must consider the space of p-functionals $\Psi[\phi,t)$. In this representation the state vectors $|\psi>$ are represented by functionals $\psi[\phi,t)$, which we will call wave functions.

The internal product is defined through functional integration as

$$<\psi_1|\psi_2> = \int \bar{\psi}_1[\phi,t)\ \psi_2[\phi,t)\ \mu(\phi)\ \mathcal{D}^n\phi \ , \qquad (V.10)$$

where μ stands by a measure.

In the field representation the basic operators $\hat{\phi}$ and $\hat{\pi}$ are given by

$$\hat{\phi}^A(x) = \int_\Sigma \hat{\varphi}^A(x;z)\ d^pz \ , \qquad (V.11a)$$

$$\hat{\phi}^A{}_I(x) = \int_\Sigma \hat{\varphi}^A{}_I(x;z)\ d^pz \ , \qquad (V.11b)$$

$$\hat{\pi}_A(x) = -\tfrac{ih}{V}\,\mathcal{D}_A(x) = \int_\Sigma \hat{\Pi}_A(x;z)\ d^pz \ , \qquad (V.11c)$$

with

$$\hat{\varphi}^A(x;z) = \hat{\phi}^A(z)\ \delta^{(p)}(x-z) \ , \qquad (V.12a)$$

$$\hat{\varphi}^A{}_I(x;z) = \hat{\phi}^A{}_I(z)\ \delta^{(p)}(x-z) \ , \qquad (V.12b)$$

$$\hat{\Pi}_A(x;z) = -\tfrac{ih}{V}\,[\delta^{(p)}(x-z)\ \partial_A(z) + d\delta^{(p)}(x-z)/dz^I\ \partial_A{}^I(z) + \cdots] \ . \qquad (V.12c)$$

It must be observed that the space over which these operators act can be identified with $J^{(s)}(B)$ which can be considered as a subspace of the original phase space. This fact guarantees that one can obtain an irreducible representation for these operators; in mathematical language this is called a polarisation of the phase space. Due to the previous fact this is called the field representation. A momentum

representation can be obtained by selecting a convenient polarisation of the phase space.

This time [,] does not stand only by a commutation, as for classical field theory an integration is involved in between. The Poisson bracket is replaced by the commutator defined by

$$[\hat{A}(x),\hat{B}(y)] = \int_\Sigma [\hat{\mathcal{A}}(x;z)\ \hat{\mathcal{B}}(y;z) - \hat{\mathcal{B}}(y;z)\ \hat{\mathcal{A}}(x;z)]\ d^pz \ . \qquad (V.13)$$

where $\hat{\mathcal{A}}$ is such that

$$\hat{A}(x) = \int_\Sigma \hat{\mathcal{A}}(x;z)\ d^pz \ . \qquad (V.14)$$

The basic commutators are given by

$$[\hat{\phi}^A(x),\hat{\pi}_B(y)] = \int_\Sigma (\hat{\varphi}^A(x;z)\hat{\Pi}_B(y;z) - \hat{\Pi}_B(y;z)\hat{\varphi}^A(x;z))\ d^pz = \tfrac{ih}{V}\,\delta^A_B\,\delta^{(p)}(x-y) \ . \qquad (V.15a)$$

$$[\hat{\phi}^A{}_I(x),\hat{\pi}_B(y)] = \tfrac{ih}{V}\,\delta^A_B\,d\delta^{(p)}(x-y)/dx^I \ . \qquad (V.15b)$$

The canonical Hamiltonian is correspondingly changed to

$$\mathcal{H}_c(\phi^A,\ \phi^A{}_I,\ \pi_A(\mathcal{L})) \to \hat{\mathcal{H}}_c = \mathcal{H}_c(\phi^A,\ \phi^A{}_I,\ -\tfrac{ih}{V}\,\delta^{(p)}_A) \ , \qquad (V.16)$$

where $\delta^{(p)}=\delta^{(p)}/\delta\phi^A$, stands for the nariational derivative acting only on space-like directions

$$\frac{\delta^{(p)}}{\delta\phi^A} = \frac{\partial}{\partial\phi^A} - d_I\frac{\partial}{\partial\phi^A{}_I} + d_I d_J\frac{\partial}{\partial\phi^A{}_{IJ}} - \cdots \ . \qquad (V.17)$$

One furthermore defines

$$\hat{H}[\phi] = \int_\Sigma \hat{\mathcal{H}}_c(\phi^A(x))\ d^px \ . \qquad (V.18)$$

The quantity $\bar{\psi}[\phi,t)\psi[\phi,t)$ is interpreted as the probability of finding the system in a given classical configuration.

The energy operator is given by

$$\hat{E} = ih\,\partial_t \ , \qquad (V.19)$$

therefore the Schrödinger equation can be rewritten as

$$\hat{H}[\phi]\ \psi[\phi,\ t) = ih\,\partial_t\psi[\phi,\ t) \ . \qquad (V.20)$$

V.3 Local Quantum Field Theory

Since the total wave function $\Psi[\phi,t)$ is a p-functional it can be written as the product of the local wave functions $\psi(x,t)$ at any point

of the space-like section Σ

$$\Psi[\phi,t) = \Pi_{x\in\Sigma}\ \psi(\mathbf{x},t) \ . \tag{V.21}$$

Therefore, the left-hand side of eq.(V.20) can be rewritten as

$$\hat{H}[\phi]\ \psi[\phi,t) = \int_{\Sigma} \hat{\mathcal{H}}(\phi(\mathbf{y}))\ \Pi_{x\in\Sigma}\ \psi(\mathbf{x},t)\ d^{p}y \ . \tag{V.22}$$

The previous equation is obviously solved by the local Schrödinger equation

$$\hat{\mathcal{H}}_{c}(\phi(\mathbf{x}))\ \psi(\mathbf{x},t) = ih\ \partial_{t}\psi(\mathbf{x},t) \ , \tag{V.23}$$

since in that case

$$\hat{\mathcal{H}}(\phi(\mathbf{y}))\ \Pi_{x\in\Sigma}\ \psi(\mathbf{x},t) = ih\ \partial_{t}\Pi_{x\in\Sigma}\ \psi(\mathbf{x},t) \ , \tag{V.24}$$

and then

$$\begin{aligned}\hat{H}[\phi]\ \psi[\phi,t) &= \int_{\Sigma(y)} ih\ \partial_{t}\Pi_{x\in\Sigma}\ \psi(\phi(\mathbf{x}),t)\ d^{p}y \\ &= ih\ \partial_{t}\psi[\phi,t) \ .\end{aligned} \tag{V.25}$$

We will concentrate our efforts into the static case

$$\psi[\phi,t) = \psi[\phi]\ e^{-iEt/h} \ , \tag{V.26}$$

for which the Schrödinger equation reduces to

$$\hat{H}[\phi]\ \psi[\phi] = E\ \psi[\phi] \ . \tag{V.27}$$

In this case the local wave function reduces to

$$\psi(\mathbf{x},t) = e^{-iEt/h}\ \psi(\mathbf{x}) \ , \tag{V.28}$$

such that the local Schrödinger equation reduces to

$$\hat{\mathcal{H}}_{c}(\phi(\mathbf{x}))\ \psi(\phi(\mathbf{x})) = \tfrac{E}{v}\ \psi(\phi(\mathbf{x})) \ , \tag{V.29}$$

$\psi(\mathbf{x})$ is a function depending only on the space-like derivatives of ϕ up to an order s. The order up to which the space-like derivatives appear in $\psi(\mathbf{x})$ is intentionally left unspecified. The problem is therefore reduced to the resolution of eq.(V.29).

However, variational differential equations, to which equation (V.29) belongs, present a quite pathological behaviour. The problem lies on the fact that the variational derivation operator, in contrast with an ordinary derivation operator, doubles the order of the function on which acts. In fact, form the definition of the variational operator follows that, if $\psi(\phi(\mathbf{x}),t)$ is a function containing the derivatives of ϕ up to an order s, then $\delta\psi(\phi(\mathbf{x}),t)/\delta\phi$ will be a function containing the derivatives of $\phi(\mathbf{x})$ up to an order

$2s$. Therefore, the different terms appearing in a variational differential equation will be of different orders. This creates, in principle, unsolvable incompatibilities between the different terms appearing in eq.(V.29). In the next section we explicitly illustrate the situation with an example: the Klein-Gordon field.

Now two different questions can be addressed to the wave function $\psi(\phi(\mathbf{x}))$. One can define the probability of finding the system in the state corresponding to the "classical" space-like configuration ϕ:

$$\mathcal{P}[\phi] = \bar{\psi}[\phi]\ \psi[\phi] = \int_{\Sigma}\int_{\Sigma} \bar{\psi}(\phi(\mathbf{x}))\ \psi(\phi(\mathbf{y}))\ d^{p}x\ d^{p}y \ . \tag{V.30}$$

Secondly, the correlated probability:

$$\mathcal{P}(\mathbf{x},\mathbf{y}) = S_{B}(\bar{\psi}(\phi(\mathbf{x}))\ \psi(\phi(\mathbf{y})\ d\phi^{1}\cdots d\phi^{n}) \ . \tag{V.31}$$

Apart from being solutions of the Schrödinger equation (V.29), physical states must have a finite norm, i.e., they should be normalizable to one

$$S_{B}(\mathcal{P}[\phi]\ d\phi^{1}\cdots d\phi^{n}) = \int_{\Sigma(x)}\int_{\Sigma(y)} \mathcal{P}(\mathbf{x},\mathbf{y})\ d^{p}x\ d^{p}y = 1 \ . \tag{V.32}$$

This condition in fact selects the physically sensible solutions of the Schrödinger equation.

V.3. The Massless Scalar Field

The massless scalar field is the simplest field theory to start with. Apart from this fact is the natural generalisation to higher dimensions of the free particle of classical mechanics. This analogy is also useful when one wants to check and compare results. The massless scalar field is described by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\ \eta^{\mu\nu}\ \phi_{\mu}\ \phi_{\nu} \ . \tag{V.33}$$

In 1+p dimensions the previous Lagrangian is decomposed as

$$\mathcal{L} = \tfrac{1}{2}\ (\tfrac{1}{c^{2}}\ \dot{\phi}^{2} + \eta^{ij}\ \phi_{i}\ \phi_{j}) \ . \tag{V.34}$$

The momenta canonically conjugated to ϕ is

$$\pi = \tfrac{1}{c^{2}}\ \dot{\phi} \ , \tag{V.35}$$

such that the canonical Hamiltonian density is given by

$$\mathcal{H} = \tfrac{1}{2}\ (c^{2}\ \pi^{2} - \eta^{ij}\ \phi_{i}\ \phi_{j}) \ . \tag{V.36}$$

Quantum field theory is obtained with the usual replacements (V.13) and (V.14). For the quantum Hamiltonian we obtain

$$\hat{\mathcal{H}} = \frac{1}{2}\left(-\frac{h^2c^2}{V^2}\left(\frac{\delta^{(p)}}{\delta\phi}\right)^2 - \eta^{ij}\,\phi_i\,\phi_j\right) . \qquad (V.37)$$

The corresponding static Schrödinger equation is therefore

$$\frac{1}{2}\left(-\frac{h^2c^2}{V^2}\left(\frac{\delta^{(p)}}{\delta\phi}\right)^2 - \eta^{ij}\,\phi_i\,\phi_j\right)\psi = \frac{E}{V}\,\psi . \qquad (V.38)$$

The first step in order to solve this equation is to rewrite it as

$$\left[-\left(\frac{\delta^{(p)}}{\delta\phi}\right)^2 - \lambda^2\,\eta^{ij}\,\phi_i\,\phi_j\right]\psi = \varepsilon\,\psi , \qquad (V.39)$$

where

$$\lambda = \frac{V}{hc} , \qquad \varepsilon = \frac{2EV}{h^2c^2} . \qquad (V.40)$$

Now we carry out the point transformation

$$\phi = \alpha\,\bar{\phi} , \qquad x = \beta\,\bar{x} , \qquad (V.41)$$

with

$$\alpha^2 = h\,c\,V^{-(p-1)/p} , \qquad \beta^2 = V^{2/p} . \qquad (V.42)$$

Then, eq.(V.39) is reduced to

$$\left[-\left(\frac{\delta^{(p)}}{\delta\phi}\right)^2 - \eta^{ij}\,\phi_i\,\phi_j\right]\psi = \tilde{\varepsilon}\,\psi , \qquad (V.43)$$

where bars have been omitted. $\tilde{\varepsilon}$ is a dimensionless number. Then for the energy we obtain

$$E = \frac{hc}{2V^{1/p}}\,\tilde{\varepsilon} . \qquad (V.44)$$

In order to see more clearly how the problem of finding a solution to a equation of this kind goes on let us rewrite it as

$$-\left(\frac{\delta^{(p)}}{\delta\phi}\right)^2\psi = (\eta^{ij}\,\phi_i\,\phi_j + \tilde{\varepsilon})\,\psi . \qquad (V.45)$$

Let us first assume that ψ is a zeroth-order function, i.e., it depends only on ϕ. But this cannot be a solution since the lhs (left-hand side) is a zeroth-order function while the rhs (right-hand side) is a first-order function. The next possibility is ψ being a first-order function, $\psi=\psi(\phi,\phi_i)$. But in that case the lhs will, in principle, be a fourth- order function while the rhs is still a first-order function. This kind of argumentation can be repeated for ψ of any order. If $\psi\epsilon f^{(s)}$, then the lhs is in $f^{(4s)}$ while the rhs is in $f^{(s)}$. The final situation is always the same: an order incompatibility for each side of eq.(V.44). This is the order incompatibility we alluded above. We do not know of any explicit reference to this fact in the literature. Up to our knowledge, no general techniques for solving variational differential equations is actually known.

The previous situation would drastically change if the order of the lhs would be lesser than that naively implied by just doubling the order each time a variational differentiation operator is acting. This is in fact possible and we are going to explain how this happens. The functions having this particular behaviour are to be called non-standard functions.

Let us observe that for a field theory in (1+p) dimensions the local Schrödinger equation is defined on a p-dimensional base space. The simplest case is that of a field theory in (1+1) dimensions. In this case the base space for the local Schrödinger equation is one-dimensional. Then we need only to characterise non-standard functions in one dimension. This is the subject of the next section.

VI. Non-Standard Functions in One Dimension

Here we are going to characterise non-standard functions in one dimension. We will consider only the first- ($s=1$) and second-order ($s=2$) cases which are of direct application to most of the local Schrödinger equations in (1+1) dimensions. The first-order case is quite trivial. The second-order case can be considered as a rereading of the results by Tapia (1985) such that we present only the main results. The generalisation to s>2 is straightforward, but with an eye on physical applications we restrict our considerations to the second-order case. In this section, following the tradition of classical mechanics, we adopt the following notation: the coordinate of the base space is thought of as the time t, while the coordinates in the fiber by q^i, i=1,...,N, with N the dimension of the configuration space. Dots will denote total time derivatives.

VI.1. The Case s=1

We must analyse functions of the type $F^{(1,r)}$, with r=2,1,0. Let us start by rewriting $\delta_1 F^{(1)}$

$$\delta_1 F^{(1)} = \partial_1 F^{(1)} - (\dot{\partial}_1 F^{(1)})\cdot$$

$$= \partial_1 F^{(1)} - \partial_j\dot{\partial}_1 F^{(1)}\,\dot{q}^j - W_{ij}\,\ddot{q}^j , \qquad (VI.1)$$

were

$$W_{ij} = \dot{\partial}_i\dot{\partial}_j F^{(1)} , \qquad (VI.2)$$

is the Hessian matrix.

The case r=2. This is the standard case which is characterised by

$$\text{rank}(W_{ij}) \geq 1 \ . \tag{VI.3}$$

The case r=1. Now we want the maximum order of derivatives appearing in (VI.1) to be one. Due to the fact that the second-order derivatives of the coordinates appear only in the last term one must have that

$$W_{ij} = \dot{\partial}_i \dot{\partial}_j F^{(1,1)} \equiv 0 \ . \tag{VI.4}$$

The solution is

$$F^{(1,1)} = A_i(q)\ \dot{q}^i + A_0(q) \ . \tag{VI.5}$$

In this case (VI.1) reduces to

$$\delta_1 F^{(1,1)} = F_{ij}\ \dot{q}^j + E_i \ , \tag{VI.6}$$

where

$$F_{ij} = \partial_i A_j - \partial_j A_i \ , \tag{VI.7}$$

$$E_i = \partial_i A_0 \ . \tag{VI.8}$$

The case r=0.Now we want the maximum order of derivatives appearing in (VI.7) to be zero. Due to the fact that the first derivatives appear only in the last term, it must be

$$F_{ij} = \partial_i A_j - \partial_j A_i \equiv 0 \ , \tag{VI.9}$$

i.e.,

$$A_i = \partial_i A \ . \tag{VI.10}$$

In this case we will have

$$\delta_1 F^{(1,0)} = \partial_i (A_0 - \partial_t A) \ . \tag{VI.11}$$

VI.2. The Case s=2

We must now analyse functions of the type $F^{(2,r)}$, with r=4,3,2,1,0. Let us start by rewriting $\delta_1 F^{(2)}$

$$\delta_1 F^{(2)} = \partial_i F^{(2)} - (\dot{\partial}_i F^{(2)})^{\cdot} + (\ddot{\partial}_i F^{(2)})^{\cdot\cdot} \ . \tag{VI.12a}$$

It must be remarked that these equations are of fourth-order in the time derivatives and can be written in a more extended form that

displays the fourth-order time derivatives as

$$\delta_1 F^{(2)} = \partial_i F^{(2)} - \tilde{d}_0 \dot{\partial}_i F^{(2)} + \tilde{d}_0{}^2 \ddot{\partial}_i F^{(2)} - (C_{ij} - 2\ \tilde{d}_0 W_{ij})\ \ddot{q}^j$$

$$+ \ddot{\partial}_i \dot{\partial}_j \partial_k F^{(2)}\ \dot{q}^j\ \dot{q}^k + W_{ij}\ \ddot{q}^j \tag{VI.12b}$$

where

$$\tilde{d}_0 = \dot{q}^i\ \partial_i + \ddot{q}^i\ \dot{\partial}_i \ , \tag{VI.13}$$

$$W_{ij} = \ddot{\partial}_i \dot{\partial}_j F^{(2)} \ , \tag{VI.14}$$

is the generalised Hessian matrix of $F^{(2)}$ with respect to the accelerations, and

$$C_{ij} = \dot{\partial}_i \ddot{\partial}_j F^{(2)} - \dot{\partial}_j \ddot{\partial}_i F^{(2)} \ . \tag{VI.15}$$

The case r=4. As before, this is the standard case which is characterised by

$$\text{rank}(W_{ij}) \geq 1 \ . \tag{VI.16}$$

The case r=3. In this case we want the maximum order of derivatives appearing in (VI.13) to be three. Since the fourth-order derivatives appear only in the last term it must be

$$W_{ij} = \ddot{\partial}_i \ddot{\partial}_j F^{(2)} \equiv 0 \ . \tag{VI.17}$$

In this way $F^{(2,3)}$ is linear and non-homogeneous in the accelerations

$$F^{(2,3)} = f_i(q,\dot{q})\ \ddot{q}^i - V(q,\dot{q}) \ . \tag{VI.18}$$

Equation (VI.13) reduces to

$$\delta_1 F^{(2,3)} = F_{ij}(q,\ \dot{q},\ \ddot{q})\ \dot{q}^j + E_i(q,\ \dot{q},\ \ddot{q}) \ , \tag{VI.19}$$

where

$$F_{ij} = \dot{\partial}_i f_j - \dot{\partial}_j f_i \ , \tag{VI.20a}$$

$$E_i = \partial_i V - (\dot{\partial}_i V)^{\cdot} - (\partial_j f_i)^{\cdot}\ \dot{q}^j$$

$$- (\partial_i f_j + \partial_j f_i - \dot{C}_{ij})\ \ddot{q}^j \ , \tag{VI.20b}$$

The case r=2. In this case we want the maximum order of derivatives appearing in (VI.19) to be two. Since third-order derivatives appear only in the last term we must have

$$F_{ij} = \dot{\partial}_i f_j - \dot{\partial}_j f_i \equiv 0 \ , \tag{VI.21}$$

in this way f_i is given by

$$f_i = \dot{\partial}_i f \ , \qquad \text{(VI.22)}$$

and the $F^{(2,2)}$ is given by

$$F^{(2,2)} = \ddot{q}^j \ \dot{\partial}_j f(q, \dot{q}) - V(q, \dot{q}) \ . \qquad \text{(VI.23)}$$

VII. Applications

Now we show how the previous considerations apply to quantum field theory. We restrict our considerations to field theories in 1+1 dimensions. First we consider the massless scalar field. The exact solution is found. Next we consider the massive scalar field, Klein-Gordon. We write down the corresponding equations but we do not solve them. However, fue to the similarity with the harmonic oscillator of classical mechanics one can conjecture how the spectrum is done.

VII.1. The Massless Scalar Field

In this case the Schrödinger equation reduces to

$$[- (\frac{\delta^{(1)}}{\delta\phi})^2 + (\phi_1)^2] \ \psi = \varepsilon \ \psi \ . \qquad \text{(VII.1)}$$

The first trial is $\psi\epsilon f^{(0)}$, which, we already know, it does not work. The second possibility is $\psi\epsilon f^{(1)}$. Then $\partial_\phi\psi\epsilon f^{(1,1)}$, and the most general form for $\partial_\phi\psi$ is

$$\partial_\phi\psi = \tilde{\psi}_0(\phi) + \tilde{\psi}_1(\phi) \ \phi_1 \ . \qquad \text{(VII.2)}$$

For ψ we then obtain

$$\psi = \psi_0(\phi) + \psi_1(\phi) \ \phi_1 + \psi_2(\phi_1) \ , \qquad \text{(VII.3)}$$

such that we can rewrite $\partial_\phi\psi$ as

$$\partial_\phi\psi = \partial_\phi\psi_0 + \partial_\phi\psi_1 \ \phi_1 \ . \qquad \text{(VII.4)}$$

It can be easily verified, the proof is left to the reader, that (VII.3) cannot be a solution of (VII.1) due to the fact that ϕ_1 appears polynomically and it is impossible to reconcile the requirements of null coefficients for all the different orders.

The next possibility is to try with $\psi\epsilon f^{(2)}$. For this to be the case it must be $\partial_\phi\psi\epsilon f^{(2,2)}$. Therefore

$$\partial_\phi\psi = \tilde{f}(\phi,\phi_1) + \tilde{g}(\phi,\phi_1) \ \phi_{11} \ . \qquad \text{(VII.5)}$$

Therefore, for ψ we obtain

$$\psi = f(\phi,\phi_1) + g(\phi,\phi_1) \ \phi_{11} + h(\phi_1,\phi_{11}) \ , \qquad \text{(VII.6)}$$

such that $\partial_\phi\psi$ can be rewritten as

$$\partial_\phi\psi = \partial_\phi f + \partial_\phi g \ \phi_{11} \ . \qquad \text{(VII.7)}$$

For $\delta_\phi\partial_\phi\psi$ we obtain

$$\delta_\phi\partial_\phi\psi = [f - f_v \ v + g_u \ v^2]_{uu} + [2 \ g_u - f_{vv} + g_{uv} \ v]_u \ \phi_{11} \ , \qquad \text{(VII.8)}$$

where

$$u = \phi \ , \qquad v = \phi_1 \ . \qquad \text{(VII.9)}$$

Since this will be linear in ϕ_{11} we can put h=0. The KGS equation is now written as

$$- [f - f_v \ v + g_u \ v^2]_{uu} - [2 \ g_u - f_{vv} + g_{uv} \ v]_u \ \phi_{11} = - (v^2 - \varepsilon) \ (f + g \ \phi_{11}) \ . \qquad \text{(VII.10)}$$

We have therefore the following two equations

$$- [f - f_v \ v + g_u \ v^2]_{uu} + (v^2 - \varepsilon) \ f = 0 \ , \qquad \text{(VII.11a)}$$

$$- [2 \ g_u - f_{vv} + g_{uv} \ v]_u + (v^2 - \varepsilon) \ g = 0 \ . \qquad \text{(VII.11b)}$$

Let us redefine f and g by

$$f = F \ , \qquad \text{(VII.12a)}$$

$$g = G/v \ . \qquad \text{(VII.12b)}$$

Then, the differential equations acquire the most symmetric form

$$[F - v \ F_v + v \ G_u]_{uu} - (v^2 - \varepsilon) \ F = 0 \ , \qquad \text{(VII.13a)}$$

$$[G - v \ F_v + v \ G_u]_{uv} - (v^2 - \varepsilon) \ G = 0 \ . \qquad \text{(VII.13b)}$$

From here it follows that

$$F_v - G_u = - \frac{2vF}{(v^2-\varepsilon)} \ . \qquad \text{(VII.14)}$$

Then, equations (VII.13) can be rewritten as

$$H_{uu} - (v^2 - \varepsilon) \ F = 0 \ , \qquad \text{(VII.15a)}$$

$$H_{uv} - (v^2 - \varepsilon)\, G = 0 \ , \qquad \text{(VII.15b)}$$

where

$$H = \frac{(3v^2-\varepsilon)}{(v^2-\varepsilon)}\, F \ . \qquad \text{(VII.16)}$$

Then equations (VII.15a) can be equivalently rewritten as

$$(3v^2 - \varepsilon)\, H_{uu} - (v^2 - \varepsilon)^2\, H = 0 \ . \qquad \text{(VII.17)}$$

The solution to this equation is

$$H = a_1\, e^{h} + a_2\, e^{-h} \ , \qquad \text{(VII.18)}$$

where

$$h = u\, \frac{v^2-\varepsilon}{\sqrt{3v^2-\varepsilon}} \ . \qquad \text{(VII.19)}$$

F is easily obtained from (VII.15a), while G is obtained by using (VII.15b)

$$G = \frac{v(3v^2+\varepsilon)}{(v^2-\varepsilon)(3v^2-\varepsilon)^{3/2}}\, (a_1(1 + h)\, e^{h} - a_2(1 - h)\, e^{-h}) \ . \qquad \text{(VII.20)}$$

VII.2. The Massive Scalar Field

Now we consider the massive scalar field, Klein-Gordon. The Klein-Gordon field is the natural generalisation to higher dimensions of the harmonic oscillator of classical mechanics. As before, this analogy is also useful when one wants to check and compare results. The Klein-Gordon field is described by the Lagrangian

$$\mathcal{L}_{KG} = \tfrac{1}{2}\, (\eta^{\mu\nu}\, \phi_\mu\, \phi_\nu - m^2\, \phi^2) \ . \qquad \text{(VII.21)}$$

In 1+p dimensions the previous Lagrangian is decomposed as

$$\mathcal{L}_{KG} = \tfrac{1}{2}\, (\frac{1}{c^2}\, \dot{\phi}^2 + \eta^{ij}\, \phi_i\, \phi_j - m^2\, \phi^2) \ , \qquad \text{(VII.22)}$$

where m has dimensions [lenght]$^{-1}$. The momenta canonically conjugated to ϕ is

$$\pi = \frac{1}{c^2}\, \dot{\phi} \ , \qquad \text{(VII.23)}$$

such that the canonical Hamiltonian density is given by

$$\mathcal{H} = \tfrac{1}{2}\, (c^2\, \pi^2 - \eta^{ij}\, \phi_i\, \phi_j + m^2\, \phi^2) \ . \qquad \text{(VII.24)}$$

Quantum field theory is obtained with the usual replacements (V.13) and (V.14). For the quantum Hamiltonian we obtain

$$\hat{\mathcal{H}} = \tfrac{1}{2}\, (-\frac{\hbar^2 c^2}{v^2}\, (\frac{\delta^{(p)}}{\delta\phi})^2 - \eta^{ij}\, \phi_i\, \phi_j + m^2\, \phi^2) \ . \qquad \text{(VII.25)}$$

The corresponding static Schrödinger equation is therefore

$$\tfrac{1}{2}\, (-\frac{\hbar^2 c^2}{v^2}\, (\frac{\delta^{(p)}}{\delta\phi})^2 - \eta^{ij}\, \phi_i\, \phi_j + m^2\, \phi^2)\, \psi = \frac{E}{v}\, \psi \ . \qquad \text{(VII.26)}$$

The first step is to rewrite this equation as

$$[-(\frac{\delta^{(p)}}{\delta\phi})^2 - \lambda^2\, \eta^{ij}\, \phi_i\, \phi_j - \mu^2\, \phi^2]\, \psi = \varepsilon\, \psi \ , \qquad \text{(VII.27)}$$

where

$$\lambda = \frac{v}{\hbar c} \ , \qquad \mu = \frac{mv}{\hbar c} \ , \qquad \varepsilon = \frac{2Ev}{\hbar^2 c^2} \ . \qquad \text{(VII.28)}$$

If we carry out the point transformation

$$\phi = \bar{\phi}/\sqrt{\mu} \ , \qquad x = \frac{\lambda}{\mu}\, \bar{x} \ , \qquad \varepsilon = \mu\, \bar{\varepsilon} \ , \qquad \text{(VII.29)}$$

eq.(VII.27) is reduced to

$$[-(\frac{\delta^{(p)}}{\delta\phi})^2 - \eta^{ij}\, \phi_i\, \phi_j + \phi^2]\, \psi = \varepsilon\, \psi \ , \qquad \text{(VII.30)}$$

where bars have been omitted. This equation shows that ε is a dimensionless number. Then for the energy we will have

$$E = \frac{\hbar\omega}{2}\, \varepsilon \ , \qquad \text{(VII.31)}$$

where ω=mc, which, as must be, does not depends on V.

Now we restrict our considerations to the 1+1 dimensional case. In this case eq.(VII.30) reduces to

$$[-(\frac{\delta^{(1)}}{\delta\phi})^2 + (\phi_1)^2 + \phi^2]\, \psi = \varepsilon\, \psi \ . \qquad \text{(VII.32)}$$

The first trial is $\psi \epsilon f^{(0)}$, which, we already know, it does not work. The second possibility is $\psi \epsilon f^{(1)}$. Then $\partial_\phi \psi \epsilon f^{(1,1)}$, and the most general form for $\partial_\phi \psi$ is

$$\partial_\phi \psi = \bar{\psi}_0(\phi) + \bar{\psi}_1(\phi)\, \phi_1 \ . \qquad \text{(VII.33)}$$

For ψ we then obtain

$$\psi = \psi_0(\phi) + \psi_1(\phi)\, \phi_1 + \psi_2(\phi_1) \ , \qquad \text{(VII.34)}$$

such that we can rewrite $\partial_\phi \psi$ as

$$\partial_\phi \psi = \partial_\phi \psi_0 + \partial_\phi \psi_1\, \phi_1 \ . \qquad \text{(VII.35)}$$

It can be easily verified, the proof is left to the reader, that (VII.34) cannot be a solution of (VII.32) due to the fact that ϕ_1 appears polynomically and it is impossible to reconcile the requirements of null coefficients for all the different orders.

The next possibility is to try with $\psi \epsilon f^{(2)}$. For this to be the

case it must be $\partial_\phi \psi \in f^{(2,2)}$. Therefore

$$\partial_\phi \psi = \tilde{f}(\phi,\phi_1) + \tilde{g}(\phi,\phi_1)\ \phi_{11} \ . \qquad \text{(VII.36)}$$

Therefore, for ψ we obtain

$$\psi = f(\phi,\phi_1) + g(\phi,\phi_1)\ \phi_{11} + h(\phi_1,\phi_{11}) \ . \qquad \text{(VII.37)}$$

such that $\partial_\phi \psi$ can be rewritten as

$$\partial_\phi \psi = \partial_\phi f + \partial_\phi g\ \phi_{11} \ . \qquad \text{(VII.38)}$$

For $\delta_\phi \partial_\phi \psi$ we obtain

$$\delta_\phi \partial_\phi \psi = [f - f_v\ v + g_u\ v^2]_{uu} + [2\ g_u - f_{vv} + g_{uv}\ v]_u\ \phi_{11} \ , \qquad \text{(VII.39)}$$

where

$$u = \phi \ , \qquad v = \phi_1 \ . \qquad \text{(VII.40)}$$

Since this will be linear in ϕ_{11} we can put h=0. The KGS equation is now written as

$$- [f - f_v\ v + g_u\ v^2]_{uu} - [2\ g_u - f_{vv} + g_{uv}\ v]_u\ \phi_{11} = - (u^2 + v^2 - \varepsilon)\ (f + g\ \phi_{11}) \ . \qquad \text{(VII.41)}$$

We have therefore the following two equations

$$- [f - f_v\ v + g_u\ v^2]_{uu} + (u^2 + v^2 - \varepsilon)\ f = 0 \ , \qquad \text{(VII.42a)}$$

$$- [2\ g_u - f_{vv} + g_{uv}\ v]_u + (u^2 + v^2 - \varepsilon)\ g = 0 \ . \qquad \text{(VII.42b)}$$

Let us redefine f and g by

$$f = F \ , \qquad \text{(VII.43a)}$$

$$g = G/v \ . \qquad \text{(VII.43b)}$$

Then, the differential equations acquire the most symmetric form

$$[F - v\ F_v + v\ G_u]_{uu} - (u^2 + v^2 - \varepsilon)\ F = 0 \ , \qquad \text{(VII.44a)}$$

$$[G - v\ F_v + v\ G_u]_{uv} - (u^2 + v^2 - \varepsilon)\ G = 0 \ . \qquad \text{(VII.44b)}$$

From here it follows that

$$F_v - G_u = - \frac{2(vF-uG)}{(u^2+v^2-\varepsilon)} \ . \qquad \text{(VII.45)}$$

Then, equations (VII.44) can be rewritten as

$$H_{uu} - (u^2 + v^2 - \varepsilon)\ F = 0 \ , \qquad \text{(VII.46a)}$$

$$H_{uv} - (u^2 + v^2 - \varepsilon)\ G = 0 \ , \qquad \text{(VII.46b)}$$

where

$$H = \frac{1}{(u^2+v^2-\varepsilon)} [(u^2+3v^2-\varepsilon)\ F - 2uv\ G] \ . \qquad \text{(VII.47)}$$

Then equations (VII.46) can be equivalently rewritten as

$$(u^2 + 3v^2 - \varepsilon)\ H_{uu} - 2uv\ H_{uv} - (u^2 + v^2 - \varepsilon)^2\ H = 0 \ . \qquad \text{(VII.48)}$$

Now the problem is reduced to solve equation (VII.48) since then F and G can be found through (VII.46).

However, till now all different attempts we have made in order to integrate this equation have not lead to success. Up to now we have not been able to solve equation (VII.48). Comparison of eqs.(VII.44) or (VII.46) with the Hermite equation

$$H'' - (x^2 - \varepsilon))\ H = 0 \ , \qquad \text{(VII.49)}$$

leads us to conjecture that under the requirement of finiteness of the solutions to the Schrödinger equation the spectra for ε will be given by

$$\varepsilon_n = 2n + 1 \ . \qquad \text{(VII.50)}$$

Therefore, the energy eigenvalues are given by

$$E_n = \frac{\hbar\omega}{2}\ \varepsilon_n = \hbar\omega\ (n + \tfrac{1}{2}) \ . \qquad \text{(VII.51)}$$

Even when we have not been able to solve eq.(VII.48) the important result has been to show that functional differential equations can be reduced to a set of coupled partial differential equations.

VIII. Conclusions

We have introduced a method to solve variational differential equations, particularly those of the Schrödinger type appearing in quantum field theory. The method is based on the construction of the kernels of functions $f^{(s,r)}$. The generalisation to higher-orders, the natural continuation of this work, presents new problem absent for the first-order case. In particular we will prove that the functions $f^{(2,3)}$ are polynomials. Particularly interesting is the following case

$$\mathcal{F} = \phi \left(\phi_{00}\, \phi_{11} - (\phi_{01})^2\right) , \qquad \text{(VIII.1)}$$

For this Lagrangian one obtains

$$\phi\, \delta\mathcal{F} = \mathcal{F} . \qquad \text{(VIII.2)}$$

We can furthermore write

$$(\phi\delta)^2\mathcal{F} - \mathcal{F} = 0 . \qquad \text{(VIII.3)}$$

We already know of a similar, with a little bit of good will, eigenvalue equation: the Wheeler-DeWitt equation. This analogy has been the main motivation to develope the method introduced in this article. We hope these results will be of applicability to the Wheeler-DeWitt equation. This will be reported in a next paper.

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Тапиа В. E2-89-309

Новый метод решения функциональных дифференциальных уравнений. Применение к квантовой теории поля

Предлагается новый метод решения линейных функциональных дифференциальных уравнений с использованием локального линейного функционального дифференциального уравнения, которое содержит вариационные производные относительно соответствующих полей. Решение вариационного дифференциального уравнения будет содержать производные всех порядков от полей, но это вернет нас обратно к функциональным методам. Для того чтобы решение имело конечный порядок, необходимо ограничиться функциями, которые являются полиномами относительно производных высшего порядка с произвольными функциями низших порядков как коэффициентов. Тогда функциональные дифференциальные уравнения превращаются в систему связанных уравнений в частных производных для указанных коэффициентов. Мы применяем эти результаты к квантовой теории поля. Уравнение Шредингера превращается в систему связанных уравнений в частных производных. Это дает невозмущенную схему для квантовой теории поля. Мы учитываем как безмассовые, так и массовые /Клейна-Гордона/ скалярные поля.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1989

Tapia V. E2-89-309

A Novel Method to Solve Functional Differential Equations. Application to Quantum Field Theory

We present a method to solve functional linear differential equations based on the fact that a functional linear differential equation is solved by a local differential equation containing variational derivatives with respect to the concerned fields.The solution of a variational differential equation will be a function containing the derivatives of the fields at all orders. But this takes us again back to functional methods.In order to keep finite the order of the solution one must restrict the considerations to those functions which are polynomials in the highest-order derivatives of the fields with arbitrary functions of the lower-order derivatives as coefficients.Then, functional differential equations reduce to a system of coupled partial differential equations for the coefficients we mentioned above. We apply these results to quantum field theory. The Schrödinger equation reduces to a system of coupled partial differential equations. This provides therefore a non-perturbative scheme for quantum field theory. We consider the massless and the massive, Klein-Gordon, scalar fields.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1989