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S.A.Gogilidze*, V.V.Sanadze*,
Yu.S.Surovtsev, F.G.Tkebuchava*

ON HAMILTON CONSTRAINTS OF A RELATIVISTIC MEMBRANE

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*Tbilisi State University, Tbilisi, USSR

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To obtain the relativistic quantum field theory according to Dirac [1], space-time coordinates are considered on equal term with field functions and the initial state of a system is given on an arbitrary space-like hypersurface in the Minkowski space. The corresponding Lagrangian is reparametrization-invariant, and consequently is singular. The Hamilton formalism contains four constraints $H_\perp, H_i$ $(i=1,2,3)$; they are constraints of the first class, that is the Poisson bracket (PB), for any pair of the constraints is a linear combination of the constraints and, which is more remarkable, the coefficients in these linear combinations are universal. (They do not depend on the Lagrangian). The PB for $H_\perp, H_i$ have the form [1]:

$$\{H_r(y),H_s(y')\}=H_s\delta'_r(y,y')+H_r(y')\delta'_s(y,y'), \qquad (1)$$

$$\{H_r(y),H_\perp(y')\}=H_\perp(y)\,\delta'_r(y,y'), \qquad (2)$$

$$\{H_\perp(y),H_\perp(y')\}=-[H^r(y)+H^r(y')]\,\delta'_r(y,y'), \qquad (3)$$

$$\int d^3y\,\delta(y,y')=1, \quad \delta'_r(y,y')=\frac{\partial}{\partial y^r}\delta(y,y').$$

Dirac's procedure depends crucially on the fact that variables conjugate to surface variables enter into $H_\perp(y)$ and $H_i(y)$ in a linear form.

Teitelboim [2] has shown that the universal structure of (1)-(3) can just be derived from two assumptions, namely: (a) the constraints form a closed PB algebra,

and (b) a change in the canonical variables during the evolution from an initial surface to a final surface is independent of the particular sequence of intermediate surfaces used in the actual evolution of this change. This property is called the "path independence of dynamical evolution" [3].

To verify that such reparametrization-invariant theories as string or membrane theories, in which the constraints are not linear functions of momenta, satisfy assumption (b), we must show that the constraint algebra of these theories has a form like (1)-(3).

This question for the Nambu-Goto string was analysed in [4]. In the present paper we construct a generator which changes the shape of a membrane in the direction perpendicular to it and show that the algebra of membrane constraints can be transformed to form (1)-(3).

Consider a p-dimensional surface N, whose action is proportional to the volume swept by its motion in the D-dimensional space-time, with coordinates x^μ [5,6]

$$S=-\frac{1}{\Omega}\int d\xi^0 d\xi^1\cdots d\xi^p\,\sqrt{(-1)^p\det g_{ij}}\,, \qquad (4)$$

where $(\xi^0,\xi^1,\dots,\xi^p)\equiv(\tau,\sigma^1,\dots,\sigma^p)$ are coordinates in the p+1-dimensional subspace and g_{ij} is the metric on this subspace connected with the metric of the D-dimensional Minkowski space $\eta_{\mu\nu}$ by the following relation

$$g_{ij}=\frac{\partial x^\nu}{\partial\xi^i}\frac{\partial x^\mu}{\partial\xi^j}\eta_{\nu\mu}\,; \quad \nu,\mu=0,1,\dots,D-1\,, \qquad (5)$$

$$i,j=0,1,\dots,p\,; \quad \eta_{\nu\mu}=(+1,-1,\dots,-1).$$

To pass to the Hamiltonian form, the canonical momenta

$$P_\mu(\tau,\sigma)=\frac{\delta S}{\delta\left(\partial_\tau x^\mu(\tau,\sigma)\right)} \tag{6}$$

are constructed and p+1 primary constraints

$$\Phi_0(\sigma)=p^2-\frac{(-1)^p}{\Omega^2}G\,, \tag{7}$$

$$\Phi_\alpha(\sigma)=p^\mu\partial_\alpha x_\mu\equiv p\cdot\partial_\alpha x\,;\ \alpha=1,\dots,p \tag{8}$$

are obtained, where the $G=\det g_{\alpha\beta}\ (\alpha,\beta=1,2,\dots,p)$ is a cofactor of g_{00}, and $g_{\alpha\beta}$ is the metric of the p-dimensional space-like surface N with coordinates $\sigma^1,\dots,\sigma^p$.

In the theory with action (4) there are no other constraints, the canonical Hamiltonian is equal to zero and constraints (7)-(8) are the first-class constraints. The latter follows from the expressions [7] :

$$\{\Phi_\alpha(\sigma),\Phi_\beta(\sigma')\}=\Phi_\beta(\sigma)\delta'_\alpha(\sigma,\sigma')+\Phi_\alpha(\sigma')\delta'_\beta(\sigma,\sigma')\,, \tag{9}$$

$$\{\Phi_0(\sigma),\Phi_\beta(\sigma')\}=\left[\Phi_0(\sigma)+\Phi_0(\sigma')\right]\delta'_\beta(\sigma,\sigma')\,, \tag{10}$$

$$\{\Phi_0(\sigma),\Phi_0(\sigma')\}=\frac{(-1)^{p+1}}{\Omega^2}\left[\frac{\partial G(\sigma)}{\partial g_{\alpha\beta}}\Phi_\alpha(\sigma)+\frac{\partial G(\sigma')}{\partial g_{\alpha\beta}}\Phi_\alpha(\sigma')\right]\delta'_\beta(\sigma,\sigma'). \tag{11}$$

Here $\delta(\sigma,\sigma')=\prod_{\alpha=1}^{p}\delta(\sigma_\alpha-\sigma'_\alpha)$, $\delta'_\beta(\sigma,\sigma')=\frac{\partial}{\partial\sigma^\beta}\delta(\sigma,\sigma')$.

It is then clear that PB (9) will completely correspond to PB (1) if we introduce the notation $\Phi_\alpha\to H_\alpha$ and H_α is considered as a contravariant vector on surface N. The covariant components H^α are expressed in a standard way:

$$H^\alpha=g^{\alpha\beta}H_\beta=g^{\alpha\beta}\Phi_\beta\,, \tag{12}$$

$$g_{\alpha\beta'}\,g^{\beta'\beta}=\delta^\beta_\alpha\,. \tag{13}$$

Note that vector fields connected with the function H_α generate in the phase space coordinate changes tangent with respect to the p-dimensional surface N. We can take, as a constraint that will generate the surface motion in the direction perpendicular towards it, the following function

$$H_\perp(\sigma)=\frac{1}{2}\,\frac{1}{\Omega}\,G(\sigma)\Phi_0(\sigma)\,. \tag{14}$$

Calculating PB for $H_\perp(\sigma)$ and $H_\alpha(\sigma)$ at $\tau=\tau'$ we get

$$\{H_\alpha(\sigma),H_\perp(\sigma')\}=H_\perp(\sigma)\delta'_\alpha(\sigma,\sigma')\,, \tag{15}$$

$$\{H_\alpha(\sigma),H_\beta(\sigma')\}=H_\beta(\sigma)\delta'_\alpha(\sigma,\sigma')+H_\alpha(\sigma')\delta'_\beta(\sigma,\sigma')\,, \tag{16}$$

$$\{H_\perp(\sigma),H_\perp(\sigma')\}=(-1)^{p+1}g^{\alpha\beta}\left[H_\alpha(\sigma)+H_\alpha(\sigma')\right]\delta'_\beta(\sigma,\sigma')+R(\sigma,\sigma')\,, \tag{17}$$

where

$$R(\sigma,\sigma')=-\Omega\left[\sqrt{(-1)^p G(\sigma)}\,H_\perp(\sigma)H^\alpha(\sigma)+\sqrt{(-1)^p G(\sigma')}\,H_\perp(\sigma')H^\alpha(\sigma')\right]\delta'_\alpha(\sigma,\sigma')\,.$$

Expressions obtained for PB (15)-(17) completely correspond to expressions (1)-(3), with the exception of the term $R(\sigma,\sigma')$. But $R(\sigma,\sigma')$ is a function quadratic in constraints and therefore the Poisson bracket $\{R,B\}$, where B is an arbitrary function of the coordinates and momenta will reduce to zero. In other words, $R(\sigma,\sigma')$ is assumed to be zero in a strong sense.

As it is seen from expressions (1)-(3) and (15),(17) the PB for membrane constraints can be written in the same form, as Dirac constraints algebra.

The quantity $H_{\perp}(\sigma,\tau)$, constructed by formula (14), can therefore be interpreted as a constraint, the vector field of which generates the dynamical evolution of system (4) deforming the surface in the direction normal to it.

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