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SCHEME DEPENDENCE IN THE RENORMALIZATION GROUP

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The problem of scheme dependence of the renormalization group calculations in QCD has been a subject of lively discussions since 1979 until recently (see, for example, Refs.[1-7]). Various recipes are proposed to eliminate the effect of choosing a concrete renormalization scheme in calculations of physical quantities. Such recipes as well as various versions of the "optimization" procedure can be viewed, quite phenomenologically, as actually different methods for describing experimental data within QCD. However, ideologically, all these recipes are very similar, since the problem itself does not allow much freedom. The main goal of the present article is to clarify the situation.

Let us begin with general relations between different renormalization schemes. Consider a massless theory with a single coupling constant g. A physical quantity R in this case depends on the two arguments:

$$R = R(t, g) \,, \tag{1}$$

where $t = p^2/\mu^2$, p and μ are the momentum and renormalization parameter, respectively. According to the renormalizability principle, a change in μ can be entirely compensated by the corresponding change in g:

$$R\left(\frac{p^2}{\mu'^2}, g'\right) = R\left(\frac{p^2}{\mu^2}, g\right), \tag{2}$$

where

$$g' = \bar{g}\left(\frac{\mu'^2}{\mu^2}, g\right). \tag{3}$$

The function $\bar{g}(x, g)$ is the effective (or running) coupling constant. It obeys the equation

$$x \frac{\partial}{\partial x} \bar{g}(x, g) = \beta(\bar{g}(x, g)) \tag{4}$$

and the normalization condition

$$\bar{g}(1, g) = g . \tag{5}$$

An explicit form of the functions R, $\bar{g}$ and β depends on the renormalization scheme employed. Let us compare the expressions R and $\tilde{R}$ for the same physical quantity in different schemes. Transition from one scheme to another is equivalent to introducing counterterms into the Lagrangian, which in its turn corresponds to an appropriate variation of the coupling constant, $g \to \tilde{g} \equiv q(g)$. We may use the same notation, μ , for the renormalization parameter in both the cases. The resulting relation reads

$$\tilde{R}(t, q(g)) = R(t, g) . \tag{6}$$

Now we use the well-known equality

$$R(t, g) = R(1, \bar{g}(t, g)) \tag{7}$$

to obtain $\tilde{\bar{g}}$ and $\tilde{\beta}$:

$$\tilde{R}(1, q(\bar{g}(t,g))) = \tilde{R}(t, q(g)) = \tilde{R}(1, \tilde{\bar{g}}(t, q(g))) \tag{8}$$

and, therefore,

$$q(\bar{g}(t,g)) = \tilde{\bar{g}}(t, q(g)) . \tag{9}$$

Differentiating (9) with respect to $\ln t$ results in

$$\tilde{\beta}(q(g)) = \beta(g) \frac{dq(g)}{dg} . \tag{10}$$

This formula relates the β -functions of different schemes.

Now we introduce another representation for R which is in a sense more physical. Equation (4) can be written as follows:

$$\ln t = \int_g^{\bar{g}(t,g)} \frac{dx}{\beta(x)} \tag{11}$$

or

$$\ln t = \Psi(\bar{g}(t,g)) - \Psi(g) , \tag{12}$$

where

$$\Psi(g) = \int^{g} \frac{dx}{\beta(x)} \tag{13}$$

The lower limit in integral (13) is not shown explicitly. This means that the function Ψ is defined up to an arbitrary constant. In other words, $\Psi(g)$ is an indefinite integral of $1/\beta(x)$ at the point $x = g$.

If, as it is in QCD,

$$\beta(g) = -\beta_0 g^2 \left(1 + \beta_1 g + \beta_2 g^2 + \mathcal{O}(g^3)\right), \tag{14}$$

one usually defines

$$\Psi(g) = \frac{1}{\beta_0}\left[\frac{1}{g} + \beta_1 \ln g + \left(\beta_2 - \beta_1^2\right) g + \mathcal{O}(g^2)\right]. \tag{15}$$

Denoting

$$\ln t + \Psi(g) \equiv L \equiv \ln P^2/\Lambda^2 \tag{16}$$

we arrive at

$$L = \Psi\left(\bar{g}(t, g)\right), \tag{17}$$

$$R(t, g) = \Phi(L), \tag{18}$$

where

$$\Phi(x) = R\left(1, \Psi^{-1}(x)\right). \tag{19}$$

Equation (3) and the group nature of the renormalization transformations $(\mu, g) \to (\mu', g')$ imply that $\bar{g}$ as well as L and Λ are invariant under these transformations in the framework of a given renormalization scheme. So, representation (18) for $R(t, g)$ involves the only ("physical") parameter $\Lambda^2 = \mu^2 \exp(-\Psi(g))$ instead of two "unphysical" ones, μ and g.

Representation (18) for R is explicitly renormalization-group invariant. And what about its scheme invariance? A question about the scheme dependence of L or Λ themselves is, in my opinion, meaningless because they can be treated as free parameters of the perturbative expansions. But the question about scheme dependence of numerical coefficients of these expansions (i.e. expansions in $1/L$ and $\ln L$) is lawful. From (6), (16), and (18) we get

$$R(t,g)=\Phi(\ln t+\Psi(g))=\tilde{R}(t,q(g))=\tilde{\Phi}(\ln t+\tilde{\Psi}(q(g))). \quad (20)$$

On the other side, (10) and (13) imply that

$$\Psi(g)=\int^{g}\frac{dx}{\beta(x)}=\int^{g}\frac{dq(x)}{\tilde{\beta}(q(x))}=\int^{q(g)}\frac{dy}{\tilde{\beta}(y)}+\Delta=\tilde{\Psi}(q(g))+\Delta \quad (21)$$

(appearance of a constant Δ is due to arbitrariness of the lower limit of integration). We arrive at the remarkable formula:

$$\tilde{\Phi}(L)=\Phi(L+\Delta). \quad (22)$$

Now we are in a position to say that representation (18) for R is not only renormalization-group invariant but also, in fact, scheme-invariant. The effect of changing the renormalization scheme reduces, in terms of the function $\Phi(L)$, to mere rescaling of Λ or to an additive shift $L \to L+\Delta$ in the argument of Φ.

Consider, for clarity, a typical example of the function $\Phi(L)$. Let $R(1,g)$ be of the form

$$R(1,g)=1+z_1 g+z_2 g^2+z_3 g^3+\mathcal{O}(g^4). \quad (23)$$

By inversion of eq.(17) we obtain

$$\bar{g}(t,g)=\frac{1}{\beta_0 L}\left[1-\frac{\beta_1 \ln\beta_0 L}{\beta_0 L}+\frac{1}{(\beta_0 L)^2}\left(\beta_1^2\ln^2\beta_0 L-\beta_1^2\ln\beta_0 L+\beta_2-\beta_1^2\right)+\ldots\right]. \quad (24)$$

Substituting (24) into (7) gives

$$\Phi(L)=1+\frac{z_1}{\beta_0 L}+\frac{1}{(\beta_0 L)^2}\left[z_2-z_1\beta_1\ln\beta_0 L\right]+\frac{1}{(\beta_0 L)^3}\Big[z_3- \quad (25)$$
$$-2z_2\beta_1\ln\beta_0 L+z_1\left(\beta_1^2\ln^2\beta_0 L-\beta_1^2\ln\beta_0 L+\beta_2-\beta_1^2\right)\Big]+\mathcal{O}\left(\frac{1}{L^4}\right).$$

In another renormalization scheme we shall deal with another function, $\tilde{\Phi}(L)$, having other numerical coefficients of $L^{-n}\ln^m L$. However, due to (22), a single shift $L \to L-\Delta$ will restore the original values of all these coefficients simultaneously. Moreover, since the constant Δ appears in (21) irrespective of a specific physical object R, the above-mentioned shift $L \to L-\Delta$ relates the expressions for any physical quantity obtained in different schemes. Performing this shift we simultaneously transform all the QCD perturbative calculations from one scheme to another.

A numerical value of Δ can be found, e.g. from eq. (21). Assuming

$$q(g) = g\left(1 + q_1 g + q_2 g^2 + O(g^3)\right), \tag{26}$$

we easily obtain

$$\Delta = q_1/\beta_0 \ . \tag{27}$$

Another way to evaluate Δ is a direct comparison of $\tilde{\Phi}$ with Φ. For instance, in (25) the scheme dependence appears for the first time in the $1/L^2$ -term because τ_2 does depend on the choice of a particular scheme (β_0 , β_1 and τ_1 are scheme-invariant). It is easy to deduce from (22) that

$$\Delta = \frac{\tau_2 - \tilde{\tau}_2}{\beta_0 \tau_1} \ . \tag{28}$$

So, an explicit evaluation of Δ requires only a lower-order information.

Up to now we used, in fact, the only conceivable strategy to reduce, as much as possible, the renormalization-scheme dependence of perturbative results. If it were manageable to get these results in a closed form, i.e. nonperturbatively, the remaining scheme dependence (22) would be next to nothing. Really, it does not matter in which form, $\Phi(L)$ or $\Phi(L+\Delta)$, has one to compare theoretical predictions with experiment: a needed shift will be prescribed by the very comparison. However, in practice, we are to truncate a series like (25) for $\Phi(L)$. And an important question arises: which quantity is to be expanded in a truncated series, $\Phi(L)$ or $\Phi(L+\Delta)$, or something else? The point is that $\Phi(L)$ and $\Phi(L+\Delta)$ in a truncated form are not at all related by a mere shift $L \rightarrow L+\Delta$. It is the unavoidable truncation of perturbative series that causes the well--known umbiguities in renormalization group calculations of physical quantities and, therefore, entails non-equivalent conditions for comparison with experiment of theoretical results obtained in different schemes.

It seems only natural now that some authors have made considerable efforts to work out the methods for obtaining scheme-invariant results in every finite order of perturbation theory, as well. These attempts have led to the so-called scheme-invariant perturbation theory. The following recipe seems to be its simplest form[3,7]. In a given renormalization scheme one defines the function

$$\Phi_*(L) \equiv \Phi\left(L + \frac{r_2}{\beta_0 r_1}\right). \tag{29}$$

From (22)and (28) it follows that

$$\tilde{\Phi}_*(L) = \Phi_*(L), \tag{30}$$

i.e. the function Φ_* is the same in any scheme. Numerical coefficients of $L^{-n}\ln^m L$ in the series expansion of $\Phi_*(L)$ prove to be scheme-invariant combinations of original (generally scheme-dependent) coefficients r_i and β_j. In our case,

$$\Phi_*(L) = 1 + \frac{r_1}{\beta_0 L} - \frac{r_1\beta_1 \ln\beta_0 L}{(\beta_0 L)^2} + \frac{1}{(\beta_0 L)^3}\Big[r_3 - \frac{r_2^2}{r_1} + r_1\beta_2 - \beta_1 r_2 - r_1\beta_1^2 + r_1\beta_1^2\ln^2\beta_0 L - r_1\beta_1^2\ln\beta_0 L\Big] + \ldots \tag{31}$$

Actually, this recipe is nothing but the choice of such a scheme where the coefficient of $1/(\beta_0 L)^2$ in (25) is equal to zero.

Consider now one more scheme-invariant representation proposed in [7]. An idea is to introduce the variable a instead of L :

$$\beta_0 L - \frac{r_2}{r_1} = \frac{1}{a} + \beta_1 \ln a . \tag{32}$$

According to (29)

$$R(t,g) = \Phi_*\left(L - \frac{r_2}{\beta_0 r_1}\right) = \Phi_*\left(\frac{1}{\beta_0 a} + \frac{\beta_1}{\beta_0}\ln a\right), \tag{33}$$

so, the expansion of R in a should have scheme-invariant coefficients. Moreover, the logarithm in the r.h.s. of definition (32) causes this expansion to be an ordinary power series (without logarithms):

$$\bar{g}(t,g) = a - \frac{r_2}{r_1}a^2 + \left(\frac{r_2^2}{r_1^2} - \frac{\beta_1 r_2}{r_1} + \beta_2 - \beta_1^2\right)a^3 + O(a^4), \tag{34}$$

$$R(t,g)=1+z_1 a+a^3\left(z_3-\beta_1 z_2-\frac{z_2^2}{z_1}+\beta_2 z_1-2_1\beta_1^2\right)+\mathcal{O}(a^4). \quad (35)$$

To employ formulas of that type in practice requires solution of eq. (32) for $a(P^2/\Lambda^2)$. The Λ -parameter involved is to be fixed by comparing (35) with experiment at a certain momentum p_o.

In conclusion we discuss a version of the scheme-invariant perturbation theory proposed in /4/ for situation (23) and in /5/ for the case

$$R(1,g)=g\left(1+z_1 g+z_2 g^2+\dots\right). \quad (23')$$

Consider only the former case. Introduce a new variable ρ via

$$R(t,g)=1+z_1\rho. \quad (36)$$

Representing $\bar{g}$ in terms of ρ,

$$\bar{g}=\rho-\frac{z_2}{z_1}\rho^2+\left(2\frac{z_2^2}{z_1^2}-\frac{z_3}{z_1}\right)\rho^3+\mathcal{O}(\rho^4), \quad (37)$$

we can further rewrite the logarithmic derivative of R in a similar form:

$$t\frac{\partial}{\partial t}R(t,g)=\varphi(\rho), \quad (38)$$

$$\varphi(\rho)=-\beta_0\rho^2\left[z_1+z_1\beta_1\rho+\left(z_3+z_1\beta_2-2z_2\beta_1-\frac{z_2^2}{z_1}\right)\rho^2+\mathcal{O}(\rho^3)\right]. \quad (39)$$

It follows from definition (36) that the coefficients in (39) are scheme-invariant, i.e. $\tilde{\varphi}=\varphi$. These coefficients can be calculated immediately from the regularized (nonrenormalized!) expression for R /5/. Using

$$\varphi(\rho)=\frac{d\Phi(L)}{dL}=z_1\frac{d\rho}{dL}, \quad (40)$$

one can write down the following integral relation which is equivalent to (17):

$$L+const=z_1\int^{\rho}\frac{dx}{\varphi(x)}. \quad (41)$$

Its r.h.s. is an explicitly scheme-invariant function of ρ (or of R). Eventually, we obtain

$$f(R) = L + const , \tag{42}$$

where

$$f(R) = \tau_1 \int^{\rho} \frac{dx}{\varphi(x)} , \quad \tilde{f} = f . \tag{43}$$

One evaluates $f(R)$ on the basis of a certain approximation for φ. Then eq. (42) is used to compare R with experiment.

With this example we conclude our consideration of various versions of the scheme-invariant perturbation theory available in the literature. All these versions are absolutely equivalent in full theory whereas for truncated perturbative expansions this equivalence is lost. Phenomenologically, these are different variants of the renormalization-group calculational procedure in QCD. However, from the theoretical point of view, all these versions differ only in the way one truncates a perturbation series of type (25).

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