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HARMONIC SUPERSPACE: HOW IT WORKS?

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Supersymmetric field theory models constitute now a perspective branch of the modern elementary particle physics. They suggest new avenues to unifying all the fundamental interactions, including the gravitational one. They are a basic ingredient of the superstring theory. They solve the hierarchy problem of Grand Unification Theories. In their framework one succeeds, for the first time, in constructing *finite local field theories*, where the ultraviolet divergences from bosonic and fermionic loops are cancelled mutually. There is a lot of predictions that have to be checked in the future high energy experimental events.

For the simplest N=1 supersymmetry adequate approaches were worked out approximately ten years ago. The situation with extended supersymmetries turned out much more involved. Even for the smallest number N=2 of spinor generators till 1984 nobody could succeed in getting manifestly invariant off-shell description of the theory in terms of unconstrained superfields. Such a description is needed both for an effective handling of the theory (on the classical and quantum levels) and for revealing its geometric structure.

The description of this kind (for N=2) became possible in 1984 due to the invention of harmonic superspace[1].To date, the unconstrained formulations of all the N=2 supersymmetric theories(of matter,gauge and gravity fields) are completed, an essential progress is achieved in describing N=3 theories.The newly introduced harmonic coordinates come out as a kind of twistors (in an isospace rather than in Minkowski space-time), thus demonstrating a deep affinity to the twistor approach by R.Penrose and E.T.Newman.

The present notes are a very brief account of our lectures given at this Warsaw meeting. Details can be found in our papers [1-9].

I.The standard superspace for N-extended supersymmetry

$$R^{4|4N} = \{x^m,\ \theta_i^{\mu}, \bar\theta^{j\dot\mu}\} \qquad (1)$$

contains spinor coordinates $\theta_i^{\mu}, \bar\theta^{i\dot\mu}$ ($i=1,\dots N,\quad \mu,\dot\mu=1,2$). This superspace is suited for constructing unconstrained formulations only of N=1 theories.For N>1 one needs a more refined type of superspaces, the harmonic ones. The N=2 harmonic superspace is defined as a product of (1) and a 2-sphere

$$H^{4+2|8} = R^{4|8} \times \frac{SU(2)}{U(1)} \tag{2}$$

where SU(2) is the automorphism group of N=2 Poincaré superalgebra. $S^2 = \frac{SU(2)}{U(1)}$ is described by additional (harmonic) coordinates $u^{\pm}_i$ constrained by $u^{+i}u^-_i=1$ and defined up to an U(1) phase $u^{\pm\prime}_i = e^{\pm i\alpha}\, u^{\pm}_i$. It is convenient to choose a coordinatization

$$H^{4+2|4} = \{\, x^m, \theta^{+\mu}, \bar\theta^{+\dot\mu}, u^{\pm}_i,\ \theta^{-\mu}, \bar\theta^{-\dot\mu} \,\} \tag{3}$$

where we have projected θ^{μ}_i onto $u^{\pm}_i$: $\theta^{\pm\mu} = \theta^{i\mu}\, u^{\pm}_i, \bar\theta^{\pm\dot\mu} = \bar\theta^{i\dot\mu} u^{\pm}{}_i$.

A very important fact:N=2 supersymmetry can be realized in an analytic subspace of (3) having half the original spinor coordinates

$$A^{4+2|8} = \{\, \zeta^M = (x^m,\ \theta^{+\mu},\ \bar\theta^{+\dot\mu}),\ u^{\pm}_i \,\} \tag{4}$$

All physical, auxilliary and gauge degrees of freedom of N=2 theories of interest are combined into analytic harmonic superfields defined on the superspace (4). E.g.,N=2 matter supermultiplet (an SU(2) doublet of scalar fields φ^i and a Dirac spinor ψ^α, $\bar\kappa^{\dot\alpha}$ on shell) is described by a U(1) charge one superfield

$$q^+(\zeta,u) = \varphi^+(x,u) + \theta^{\alpha+}\psi_\alpha(x,u) + \bar\theta^+_{\dot\alpha}\, \bar\kappa^{\dot\alpha}(x,u) + \ldots \tag{5}$$

where the component fields φ^+, ψ_α , $\bar\kappa^{\dot\alpha}$,...are represented by harmonic expansions on S^2

$$\varphi^+(x,u) = \varphi^i(x)u^+_i + \varphi^{(ijk)}(x)u^+_i u^+_j u^-_k + \qquad , \tag{6}$$

$$\psi_\alpha(x,u) = \psi_\alpha(x) + \psi^{(ij)}_\alpha(x)\, u^+_i u^-_j \quad \ldots \qquad ,..\ \text{etc.}$$

We see that q^+contains, along with the physical fields,an infinite tower of extra fields emerging in harmonic expansions. These are necessary for closing N=2 supersymmetry off shell. In fact, they are auxiliary and are eliminated on shell by the equations of motion.

To construct actions in harmonic superspace we have to introduce derivatives in harmonic coordinates. Most important of them is the analyticity- preserving derivative D^{++}

$$D^{++} = u^+_i \partial/\partial u^-_i - 2i\theta^+\sigma^a\bar\theta^+\partial/\partial x^a \tag{7}$$

With its help the free action of q^+ can be written as

$$S = \int d\zeta^{-4} du\, \overset{*}{\bar{q}}{}^+ D^{++} q^+ \tag{8}$$

where $d\zeta^{-4}du$ is the analytic superspace integration measure. This action gives rise to the equation of motion

$$D^{++} q^+ = 0 \tag{9}$$

which can be shown to eliminate all the auxiliary fields and to yield the correct free equations for physical fields. Note that the integration over harmonic variables appearing in (8) is defined by the following simple rules

$$\int du\, 1 = 1, \quad \int du\, u^{+(i}u^{+j}\ldots u^{-k}u^{-l)} = 0 \tag{10}$$

It is also worth mentioning that the conjugation $\overset{*}{-}$ combines usual complex conjugation with the operation $*$

$$(u_i^+)^* = u_i^-\ ,\ (u_i^-)^* = -u_i^+\ ,\ (\theta^+)^* = \theta^-,\ (\bar{\theta}^+)^* = \bar{\theta}^- \tag{11}$$

which can be interpreted as an antipodal map of the sphere S^2 (Weyl's reflection). The analytic superspace is real with respect to $\overset{*}{-}$.

One may easily generalize (8) to the interaction case

$$S = \int d\zeta^{-4} du\, \left(\overset{*}{\bar{q}}{}^+ D^{++} q^+ + L^{+4}(\, q^+,\ \bar{q}^+,\ u^+,\ u^-\,)\right) \tag{12}$$

Here $L^{+4}(\, q^+,\ \bar{q}^+,\ u^+,\ u^-\,)$ is an arbitrary four-fold charged function of N=2 matter superfields (their number may be arbitrary) and of explicit harmonics $u_i^\pm$ .We wish to stress that this is the most general self-interaction of N=2 matter multiplets off shell[6] and it is just the concept of the analytic harmonic superspace that allowed to write down the general matter action. Any N=2 matter action in conventional N=2 superspace corresponds to a restricted class of self-interactions.

As has been shown by L.Alvarez-Gaumé and D.Z.Freedman[10], any self-interaction of N=2 matter involves, for physical bosons, a sigma model with hyper-Kähler target manifold (i.e. some special 4n dimensional Riemann manifold with the holonomy group in Sp(n)). So, any function in (12) corresponds to one or another hyper-Kähler manifold and can be used to explicitly compute the relevant metric in the Lagrangian of physical bosons(by eliminating auxiliary fields by their equations of motion). In other words, there is a one-to-one correspondence between the variety of all the hyper-Kähler metrics on the one hand and that of functions L^{+4} (q, u) on the other.For this reason, the function L^{+4}can be called *Hyper-Kähler potential*[2,6] (by analogy with the N=1 case where the superfield matter Lagrangian appears as a *Kähler potential*[11]). Now the potentials for a variety of hyper-Kähler metrics are known[6,12].These are simpler than the corresponding metrics.Recently we have established a direct relation of hyper-Kähler potentials with the conventional formulation of hyper-Kähler geometry [9].

2.That is all we wish to say here concerning N=2 matter self-interactions in harmonic superspace. Now we pass to N=2 gauge theories. Let q^+ belong to a representation of a gauge group with the generators T^A. The analyticity-preserving local gauge transformation looks as

$$q^{+\prime} = e^{i\lambda^A T^A} q^+, \qquad \lambda^A = \lambda^A (\zeta, u) \tag{13}$$

To make the q action (12) gauge invariant one has to covariantize the harmonic derivative D^{++} by adding an analytic superfield gauge connection V^{++}

$$D^{++} \longrightarrow \mathcal{D}^{++} = D^{++} + iV^{++}(\zeta,u), \tag{14}$$

$$V^{++\prime} = e^{i\lambda}(V^{++} - iD^{++}) e^{-i\lambda} \tag{15}$$

Stress a remarkable analogy with the ordinary (N=0) Yang-Mills connection transformation law $A(x)_m' = e^{i\lambda(x)}(A_m(x) - i\partial_m)e^{-i\lambda(x)}$. The unconstrained analytic superfield V^{++}encodes all the information on the structure of N=2 gauge theory[1].In this language, the standard N=2 Yang-Mills constraints in the superspace $R^{4|8}$ by Grimm, Sohnius, Wess [13] come out as the statement that the harmonic analyticity is preserved in gauge theory.

Note that the N=2 Yang-Mills action written in terms of V^{++} turns out to be of Chern-Simons form [4].

An important remark. Just as q^+, the analytic superfield V^{++} initially contains an infinite number of fields appearing from its harmonic decomposition. While in q^+ such fields are auxiliary and are eliminated by the field equations, in V^{++}such fields are gauge degrees of freedom and are completely eliminated by fixing the gauge (off shell). In this Wess-Zumino like gauge one recovers the familiar off-shell N=2 gauge multiplet: the gauge vector field $A_m(x)$, the doublet of gauginos $\psi^i_\alpha(x)$, the isosinglet complex scalar field $\phi(x)$ and the triplet of auxiliary scalar fields $D^{(ij)}(x)$, all in the adjoint representation of gauge group.

The harmonic superspace quantization procedure for the N=2 interacting matter and gauge superfields has been worked out in [3]. There the distributions in harmonic variables were defined and the propagators and the Feynman rules for the above superfields were constructed. The miraculous cancellations of ultraviolet divergences in the extended supersymmetry theories become transparent in the quantum harmonic superspace approach. In particular, it allowed to get the first consistent proof of the finiteness of N=4, d=2 nonlinear sigma models simply from the dimensionality considerations. At the same time, N=2, d=4 sigma models proved to be non-renormalizable.

To end with the N=2 case let us discuss N=2 supergravities. A convenient general way to construct different off-shell versions of Einstein supergravity is to start with the relevant conformal supergravity and then to compensate unwanted invariances (e.g., dilatations or conformal supersymmetry) by introducing extra compensator multiplets. The underlying group of conformal N=2 supergravity is the group of coordinate transformations in N=2 harmonic superspace which preserve analyticity i.e. leave the analytic superspace (4) invariant [7]

$$\delta\, \zeta^M = \lambda^M(\zeta, u)\ ,$$
$$\delta\, u^+_i = \lambda^{++}(\zeta, u)\, u^-_i\ , \qquad \delta\, u^-_i = 0,$$
$$\delta\, \theta^{\mu -} = \lambda^{\mu -}(\zeta, u, \theta^-, \bar\theta^-), \qquad \delta\, \bar\theta^{\dot\mu -} = \lambda^{\dot\mu -}(\zeta, u, \theta^-, \bar\theta^-). \tag{16}$$

The unconstrained prepotentials of conformal supergravity are the components of the analytic vielbein covariantizing the harmonic derivative D^{++}:

$$\mathfrak{D}^{++} = u^+_i\partial/\partial u^-_i + H^{+4}u^-_i\partial/\partial u^+_i + H^{++M}\partial/\partial\zeta^M + (H^{++\mu -}\partial/\partial\theta^{\mu -} + \text{h.c.}). \tag{17}$$

Like in the case of N=2 Yang-Mills connection V^{++}, the analyticity of these prepotentials amounts to the standard constraints in the real N=2 superspace. Their irreducible field content is revealed in the Wess-Zumino like gauge (similarly to the Yang-Mills case) and it is just that of the off-shell N=2 conformal supergravity multiplet.

To pass to Einstein supergravities one should introduce appropriate compensator superfields. All the versions of N=2 Einstein supergravities known previously [14] correspond to choosing certain constrained harmonic superfields as compensators. It can be shown that all these versions do not allow general matter self-couplings because it is impossible to construct proper superfield densities from the constrained compensators. The harmonic superspace provides a way out. It suggests the unconstrained analytic superfield q^{+} as a most appropriate choice for the compensator [7]. In such a way we arrive at a new off-shell version of N=2 Einstein supergravity. It contains an infinite number of auxiliary fields (in contrast to the versions known previously) and admits most general matter couplings. The versions with constrained compensators are related, via a duality transformation, to some special classes of matter actions within this ultimate version. Just this new version allows one to visualize the theorem by J.Bagger and E.Witten[15] on the one-to-one correspondence between the matter coupling in N=2 supergravity and quaternionic manifolds(these are some distinguished class of Riemann manifolds of the dimensionality 4n having their holonomy group in Sp(n)xSp(1)): in this case the physical bosons are always described by a nonlinear sigma model with a quaternionic manifold as the target one[8].

Thus all the N=2 theories have an adequate description in the framework of harmonic superspace. This approach not only allows to represent the previously known theories in a compact convenient form but also gives a possibility to fill some gaps (to construct most general off-shell matter couplings, to find the ultimate version of N=2 supergravity, to achieve a considerable simplification of analysis of divergences, etc.).

3.It helps also in understanding higher N theories. However, in each concrete case one meets subtleties specific for it and some novelties become necessary. At present, we succeeded to the end for N=3 gauge theory[4]. There the harmonic superspace is a direct product of the standard real N=3 superspace and the coset space $\frac{SU(3)}{U(1)xU(1)}$ where SU(3) is the N=3 supersymmetry automorphism group. Correspondingly, there are two U(1) charges and one has to deal with the harmonics

$$u^{a}_{\iota} = (u^{1,1}_{\iota}, u^{-1,1}_{\iota}, u^{0,-2}_{\iota}); \qquad u^{a\iota} = \bar{u}_{\iota}{}^{-a} ; \qquad \iota=1,2,3 \tag{18}$$

and

$$u^j_a\, u^b_j = \delta^b_a \; ; \qquad u^i_a\, u^a_j = \delta^i_j \; ; \qquad \mathrm{Det}\; u = 1. \tag{19}$$

After projecting spinor variables $\theta_{i\alpha}$, $\bar\theta^i_{\dot\alpha}$ onto harmonics (18) they become $\theta^{-1,-1}_\alpha, \theta^{1,-1}_\alpha$, $\theta^{0,2}_\alpha$; $\bar\theta^{1,1}_{\dot\alpha}$, $\bar\theta^{-1,1}_{\dot\alpha}$, $\bar\theta^{0,-2}_{\dot\alpha}$. The analytic N=3 superspace contains only part of them, just the second, third, fourth and sixth ones. The analytic Yang-Mills prepotentials $V^{2,0}, V^{1,3}$, $V^{1,-3}$ come out as connections in the harmonic derivatives $D^{2,0}$, $D^{1,3}$, $D^{1,-3}$ and they have an evident transformation law $\delta\, V^{ab} = D^{ab}\,\lambda$ where λ is a chargeless analytic superfield parameter. The action is of pure Chern-Simons type

$$I = {}^{1}/_{g^2}\; \mathrm{tr} \int d\zeta^{-4,0} du \;\{\, V^{1,3}\, F^{3,-3} + V^{1,-3}\, F^{3,3} + V^{2,0}\, F^{2,0} - iV^{2,0}\, [V^{1,3}, V^{1,-3}]\} \tag{20}$$

where F are the field strengths, e.g $F^{3,3} = -i[\, D^{2,0}, D^{1,3}]$. Recently there appeared a paper devoted to the quantization of this theory [16] along the lines of its N=2 prototype. Note that the Chern-Simons type action (20) was proposed in 1985 and it describes a quite nontrivial dynamics. Now the theories with Chern-Simons actions gained a rather great popularity in connection with the string field theory and the topological field theory [17].

There remains a lot of important problems in supersymmetric theories that one can hope to solve using the harmonic superspace techniques. This approach was used recently to attack N=4 super Yang-Mills[18], ten-dimensional super Yang-Mills and supergravity in the context of superparticle and superstring models[19], etc.

4.Somewhat unexpectedly, harmonics have deep implications in ordinary bosonic (N=0) gauge theories. Their use turns out to be very fruitful for solving the self-duality equations in Yang-Mills theory and for getting an unconstrained formulation of hyper-Kähler geometry[9]. Actually, the formalism of harmonics in these theories is an another form of the Penrose-Newman-Ward twistor approach [20]. Nevertheless, it allows to clarify some subtleties of the twistor approach and ,that seems to be most important, to understand the above bosonic theories and the supersymmetric gauge ones from the same standpoint.

The common geometric basis of the bosonic problems just mentioned and of the supersymmetric gauge theories is the preservation of certain "flat" analyticities upon passing to the case with interaction, when the derivatives defining these analyticities are covariantized in a proper way. In the bosonic case these analyticities are those with respect to bosonic variables while in the supersymmetry case Grassmann analyticities are underlying. The constraints normally imposed to define these theories (e.g., the self-duality equations in the N=0 Yang-Mills case or well-known constraints on the superfield strengths in N=1,2,3 super Yang-Mills theories) turn out to express the fact of preserving above analyticities. Such an interpretation of constraints opens up a universal way of solving the latter by passing to the basis (in the manifolds of involved fields and coordinates) where the analyticity becomes manifest. In this basis, the constraints are automatically satisfied as they become the integrability conditions for the relevant flat analyticities. The whole information on the structure of the theory proves to be encoded in unconstrained prepotentials which naturally come out in the process of passing to the analytic basis.

In [9] we discussed, from such a point of view, a number of bosonic theories and indicated their supersymmetric counterparts. For instance, N=1 super Yang-Mills theory has as its direct bosonic analog Yang complex gauge theory[22].Similarly, Kähler geometry (Riemannian geometry of a 2n dimensional manifold with the holonomy group in U(n)) can be formulated in a tight analogy with N=1 supergravity in the Ogievetsky-Sokatchev approach[21]. In these cases, ordinary complex Cauchy-Riemann analyticity or Grassmann Cauchy-Riemann one[23] (in the supersymmetry case) play the fundamental role. A new type of bosonic analyticity, the harmonic one, becomes relevant when trying to understand on similar grounds self-dual Yang-Mills theory and hyper-Kähler geometry. The first theory turns out to admit a formulation quite parallel to that of N=2 Yang-Mills in the harmonic superspace described above. The unconstrained prepotential formulation of hyper-Kähler geometry has many features in common with the harmonic superspace formulation of N=2 supergravity. These similarities root, of course, in the common nature of analyticities underlying both types of theories, viz. the bosonic and Grassmann harmonic analyticities.

To see how the harmonic analyticity works in the bosonic case let us apply to self-dual Yang-Mills in $R^4 = \{ x^{\alpha\dot{\alpha}} \}$

$$\mathcal{D}_{\beta\dot\alpha} = \partial_{\beta\dot\alpha} + iA_{\beta\dot\alpha}(x) \tag{21}$$

$$[\, \mathcal{D}_{\beta\dot\alpha}, \mathcal{D}_{\gamma\dot\rho}] = i(\, \varepsilon_{\beta\gamma} F_{\dot\alpha\dot\rho} + \varepsilon_{\dot\alpha\dot\rho} F_{\beta\gamma}\,) \tag{22}$$

$$F_{\beta\gamma} = 0 \tag{23}$$

Here, the indices β,γ and $\dot\alpha,\dot\rho$ refer, respectively, to two SU(2)'s out of which the Lorentz group O(4) of R^4 is composed: O(4)=SU(2)xSU(2) (the anti-self duality condition would correspond to nullifying the field strength $F_{\dot\alpha\dot\rho}$ instead of $F_{\beta\gamma}$). Clearly, the self-duality equation (23) can be equivalently rewritten as a constraint on the commutator of covariant derivatives

$$[\, \mathcal{D}_{(\beta\dot\alpha}\, , \mathcal{D}_{\gamma)\dot\rho}] = 0 \tag{24}$$

Now, let us interpret (24) within the harmonic extension of R^4, that is $R^4 x \frac{SU(2)}{U(1)} = \{\, x^{\alpha\dot\alpha},\, u^+_\alpha, u^-_\beta\, \}$ where SU(2) is one of two SU(2)'s entering O(4) (namely, the one acting on undotted indices). The harmonics $u^\pm_\alpha$ possess all the properties of $u^\pm_i$ discussed earlier, the only difference is that they are associated now with the part of Lorentz group rather than with the internal symmetry automorphism SU(2) as in the previous case. Just as in the supersymmetry case, the main advantage of introducing harmonics is the possibility to single out in the extended space an analytic subspace, that time

$$\{\, x^{+\dot\alpha},\, u^+_\alpha\, ,\, u^-_\beta\, \} \tag{25}$$

where $x^{+\dot\alpha} = x^{\alpha\dot\alpha}u^+_\alpha$. This subspace is closed under the whole Poincaré group of R^4, so one may define the Poincaré-covariant fields on it, $f^q(x^{+\dot\alpha},\, u^\pm_\alpha\,)$, which are solutions of the harmonic Cauchy-Riemann condition

$$\partial^+_{\dot\alpha}\, f^q = \partial/\partial x^{-\dot\alpha}\, f^q = 0\ , \qquad (x^{-\dot\alpha} = x^{\alpha\dot\alpha}u^-_\alpha\,) \tag{26}$$

Let us come back to the self-duality constraint (24). Projecting the covariant derivatives onto harmonics u^+ one rewrites (24) as

$$[\, \mathcal{D}^+_{\dot\alpha}\, , \mathcal{D}^+_{\dot\rho}\,] = 0, \qquad \mathcal{D}^+_{\dot\alpha} = u^{+\alpha}\, \mathcal{D}_{\alpha\dot\alpha} \tag{27}$$

that is easily recognized as the integrability condition for the existence of covariantly analytic fields belonging to nontrivial representations of Yang-Mills group

$$\mathcal{D}^+_{\dot\alpha} f^q = 0$$

The constraints in the form (27) imply a "pure gauge" solution for the u^+ projection of the Yang-Mills connection

$$A^+_{\dot\alpha} = e^{iv}\, \partial^+_{\dot\alpha}\, e^{-iv} \tag{28}$$

where $v(x^{\pm\dot\alpha}, u^\pm_\beta)$ is some harmonic field, arbitrary for the moment. However, one has to take into account that, by definition, $A^+_{\dot\alpha}=u^{+\alpha}A_{\alpha\dot\alpha}$, so

$$D^{++}A^+_{\dot\alpha} = D^{++}(e^{iv}\, \partial^+_{\dot\alpha}\, e^{-iv}) = 0, \qquad D^{++} = u^{+\alpha}\partial/\partial u^{-\alpha}$$

and v is in fact constrained. This condition can be rewritten as

$$\partial^+_{\dot\alpha}\ (e^{-iv}\, D^{++}\, e^{iv}) = 0 \tag{29}$$

Thus, the field $V^{++}(x^{+\dot\alpha},u^+_\alpha,u^-_\beta) = i\, e^{-iv}\, D^{++}\, e^{iv}$ is analytic while otherwise arbitrary. V^{++}*provides the most general solution of the self-duality equation*: any analytic field with the U(1) charge +2 gives rise to a self-dual Yang-Mills connection and vice versa. To restore the connection one should solve the nonlinear equation

$$D^{++}\, e^{-iv} = -\, i\, V^{++}\, e^{-iv} \tag{30}$$

The problem is to select those V^{++} which correspond to the solutions with a finite action, that is to instantons. An explicit form of the SU(2) one-instanton V^{++} has been given in [7]

$$V^{++}{}_i{}^j = -i\rho^{-2}\, x^+_i\, x^{+j}, \qquad \rho=\text{const}$$

V^{++} for general n-instanton solution of Atiyah, Drinfeld, Hitchin, Manin has been obtained in [24].

The harmonic space description of instantons is, of course, an another form of familiar Ward's construction [20]. The analytic space (25) is nothing else than the twistor space as it is described, e.g.,in the book by Atiyah [25]. The equation (30) can be identified with the Sparling equation which is an important ingredient of the Newman's version of the twistor approach (see, e.g.,[20]). However, the harmonic space view seems useful in some respects. Apart from the fact that it establishes profound analogies with the supersymmetric gauge theories, it translates technical details of the twistor approach into the down-to-earth language of customary differential geometry. An important point is that it offers the opportunity to take advantage of the well-known techniques of harmonic expansions on the sphere S^2 when treating fields on the analytic (twistor) space. All this makes it tempting to apply the formalism of harmonics to more involved problems, such as the self-duality equations in higher-dimensional spaces, the monopole business, etc. An elegant treatment of the t'Hooft-Polyakov monopoles in this language has been given in [24].

5.An important application of N=0 harmonic analyticity is the unconstrained formulation of hyper-Kähler geometry[9]. As has been already mentioned, this geometry is defined as a subclass of real Riemannian geometry in 4n dimensions, with the holonomy group contained in Sp(n). The holonomy group is generated by Riemann tensor so this definition can be reformulated as a restriction on this tensor or, equivalently, on the form of the commutator of covariant derivatives.

Let $\mathfrak{D}_a$ be such a covariant derivative, the index a being the tangent space one (a = 1,...4n). One always may substitute a by a pair of Sp(n)xSp(1) indices as $a \Rightarrow (\mu \iota)$ (μ = 1...2n, ι = 1,2).Then the defining constraint of hyper-Kähler geometry can be written in the form very resembling the self-duality condition (24) [9]

$$[\, \mathfrak{D}_{\mu(\iota}\, ,\, \mathfrak{D}_{\nu j)}] = 0 \qquad (31)$$

This resemblance prompts how to solve (31). One has to introduce harmonics $u^{\pm}_{\iota}$ on Sp(1)≅ SU(2), to extend $R^{4n} = \{ x^{\mu \iota} \}$ to the harmonic space $\{x^{\mu +}, x^{\mu -}, u^{\pm}_{\iota} \}$, to project $\mathfrak{D}_{\mu \iota}$ onto harmonics and to represent (31) as the integrability conditions for the relevant harmonic analyticity

$$[\, \mathfrak{D}^{+}_{\mu}\, ,\, \mathfrak{D}^{+}_{\nu}] = 0 \qquad , \; \mathfrak{D}^{+}_{\mu} = u^{+\iota}\mathfrak{D}_{\mu \iota} \qquad (32)$$

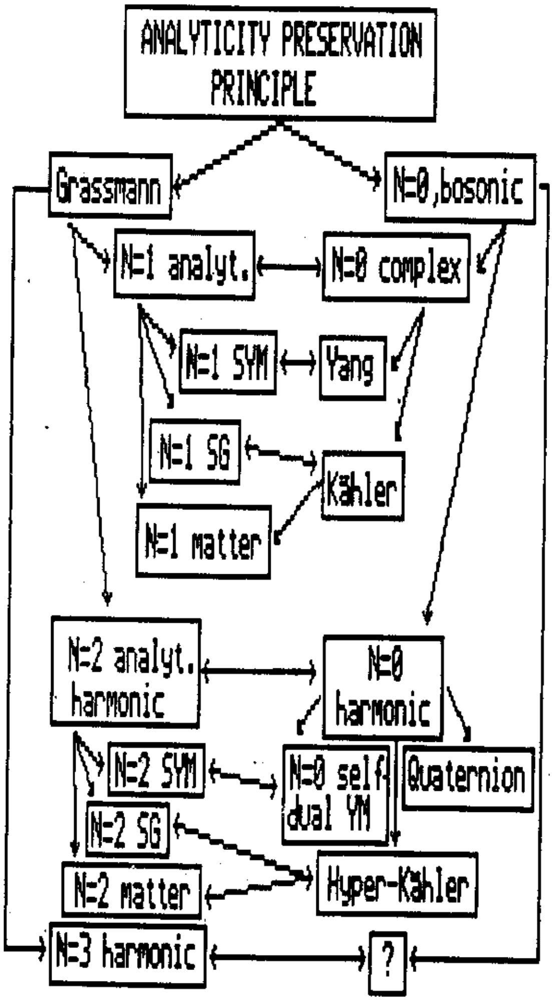
Without giving details, we quote the results. The constraint (32) can be solved by passing to the basis where the underlying analyticity becomes manifest and the derivative $\mathfrak{D}^{+}_{\mu}$ is reduced to the simple differentiation with respect to the coordinate $x^{-\mu}$. All the vielbeins and connections present originally in $\mathfrak{D}$ are eventually expressed via a single unconstrained object, the analytic potential $L^{+4}(x^{\mu+}, u^+_i, u^-_j)$. Any L^{+4} produces some hyper-Kähler metric and, vice versa, any hyper-Kähler manifold corresponds to some properly chosen L^{+4}. So, hyper-Kähler manifolds can be classified according to their potentials L^{+4}. Note that up to now there were no regular methods of computing hyper-Kähler metrics. Given a potential L^{+4}, one may compute the relevant metric by the algorithmic procedure worked out in our paper [9]. To know hyper-Kähler metrics is very important, e.g., in the four-dimensional case where they describe the gravitational instantons.

These results also allowed us to directly prove the aforementioned one-to-one correspondence between the matter couplings in N=2 supersymmetry and hyper-Kähler manifolds[10]. Recall that the N=1 matter superfield Lagrangian can be identified with a Kähler potential[11], with the N=1 matter chiral superfields being the coordinates of the relevant Kähler manifold. Quite similarly, the N=2 matter Lagrangian in the action (12) written via the unconstrained analytic superfields q^+ is nothing else than some hyper-Kähler potential and q^+'s are the coordinates of the corresponding analytic subspace (together with harmonics appearing explicitly in (12)). Thus we have proven, by the geometric reasoning, that the action (12) actually leads to the most general N=2 matter couplings.

The quaternionic geometry also can be considered within this framework and solved via an analytic potential, this time the quaternionic one. It also has the U(1) charge +4 but the formulas relating it to the metric differ from those in the hyper-Kähler case. Besides, there comes out a new geometric object, the complex coordinate of that Sp(1) which enters the holonomy group of quaternionic manifolds. The matter Lagrangian in N=2 supergravity can be identified with the quaternionic potential. This provides a manifestly geometric proof of the theorem by Bagger and Witten [8]. It is interesting that there is a natural geometric place for the analytic superfield compensator mentioned earlier: the latter is the holonomy Sp(1) coordinate appearing in the process of solving the quaternionic geometry constraints.

Finally, it seems to us remarkable that the need in the same objects,harmonics, comes from the areas so different at the first

Table
The Analyticity Preservation Principle reveals deep relationships between theories that are seemingly very different.



sight! Various analogies and relations between the constrained bosonic gauge theories and their supersymmetric counterparts following from the universal Analyticity Preservation Principle are summarized in the Table subjoined.It would be highly desirable to inquire to which more problems the concept of harmonic analyticity may be of relevance. In particular, it is an intriguing question what are the N=0 implications of SU(3) analyticity which underlies N=3 super Yang-Mills theory.

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