объединенный
институт
ядерных
исследований
дубна

E2-89-123

A.A.Osipov

EXTENSIONS OF THE KRICHEVER-NOVIKOV SUPERALGEBRAS IN THE RAMOND AND NEVEU-SCHWARZ CLOSED SUPERSTRING THEORY

Submitted to "Physics Letters B"

1989

Papers [1,2] considered the algebra of meromorphic vector fields $\mathrm{Vect}(\Sigma_g,P_\pm)$, which are holomorphic outside two arbitrary points $P_\pm$ and are globally specified on the whole compact Riemann surface Σ_g of genus g. In the space $\mathrm{Vect}(\Sigma_g,P_\pm)$ one can distinguish a count basis e_i , which makes up a subalgebra in $\mathrm{Vect}(\Sigma_g,P_\pm)$, to be called the Krichever-Novikov (KN) algebra. If g=0, it coincides with the Virasoro algebra, in this sense being a generalisation of the latter to the compact Riemann surface of an arbitrary genus.

Supersymmetric versions of the KN algebra were constructed in Ref.[3]. They correspond to the Ramond and Neveu-Schwarz superalgebras, known in the free string theory, and extend them to the case of interacting closed superstrings.

An extension of the KN algebra is possible in the closed string theory [4,5]. It describes a fixed-gauge quantum system and at g=0 turns into the known algebra of constraints and subsidiary conditions [6,7]. In this paper we shall show that there is a similar extension of the algebra of constraints for the Ramond and Neveu-Schwarz closed superstrings. In a particular case of g=0 the algebras, obtained here, coincide with the result of Ref.[8].

Let $\mathcal{F}_\lambda(\Sigma_g,P_\pm)$ be the tensor bundle on the compact Riemann surface Σ_g with two punctures $P_\pm$. Consider the meromorphic sections (λ-differentials), which are holomorphic outside the punctures and, possibly, a cut σ, connecting these points (λ is the conformal weight). Following Ref.[1], we can construct count bases $f_j^{(\lambda,\varkappa)}$ in the spaces of the sections. The cut σ occurs if $P_\pm$ are the branching points for $f_j^{(\lambda,\varkappa)}$. In this case for $f_j^{(\lambda,\varkappa)}$ there are continuous limits both on the upper and lower sides of the cut σ, related to each other as

$$f_j^{(\lambda,x)+} = \exp(2\pi i x)\ f_j^{(\lambda,x)-}. \tag{1}$$

In local complex coordinates $z_\pm$, chosen in the neighbourhood of points $P_\pm(z_\pm=0)$, $f_j^{(\lambda,x)}$ are of the form [1]

$$f_j^{(\lambda,x)} = a_j^{(\lambda,x)\pm}\ z_\pm^{\pm j\pm x-S(\lambda)}\ [1+O(z_\pm)]\ (dz_\pm)^\lambda. \tag{2}$$

In virtue of the Riemann-Roch theorem expansion (2) unambiguously determines $f_j^{(\lambda,x)}$ up to a constant. One can fix constants $a_j^{(\lambda,x)\pm}$, taking $a_j^{(\lambda,x)+} = 1$, $S(\lambda)=g/2-\lambda(g-1)$. Note that at $x=0$, $\lambda=0,1$ and $|j|\leq g/2$ the definition (2) is modified [1].

The following duality relation is valid:

$$\frac{1}{2\pi i}\oint f_i^{(\lambda,x)} f_{-j}^{(1-\lambda,-x)} = \delta(i-j). \tag{3}$$

The integration contour divides Σ_g into two parts $\Sigma_g^\pm$, so that $P_\pm\subset\Sigma_g^\pm$, and has no self-intersection points. Since the integrand is holomorphic outside $P_\pm$, the contour belongs to the homology class of C_+ , which is a circumference in the neighbourhood of P_+.

Below we shall use the following special symbols:

$$e_i=f_i^{(-1,0)},\quad A_i=f_i^{(0,0)},\quad \omega^i=f_{-i}^{(1,0)},\quad \Omega^i=f_{-i}^{(2,0)}. \tag{4}$$

If g is even, $i\in\mathbb{Z}$. If g is odd, $i\in\mathbb{Z}+1/2$. Let us also consider the objects with the half-integer conformal weight:

$$\varphi_\alpha=\begin{cases} f_j^{(-1/2,0)} \\ f_j^{(-1/2,1/2)} \end{cases},\qquad \Lambda_\alpha=\begin{cases} f_j^{(1/2,0)} \\ f_j^{(1/2,1/2)} \end{cases},$$

$$\Lambda^\alpha=\begin{cases} f_{-j}^{(1/2,0)} \\ f_{-j}^{(1/2,-1/2)} \end{cases},\qquad A^\alpha=\begin{cases} f_{-j}^{(3/2,0)} \\ f_{-j}^{(3/2,-1/2)} \end{cases}. \tag{5}$$

Two variants are possible here. The upper lines of the above expressions correspond to the first $\alpha=j\in\mathbb{Z}$, and the lower ones to the second $\alpha=j+1/2\in\mathbb{Z}+1/2$. It is easy to see that, unlike the case in (4), selection of the values of α does not depend on the parity of g.

Depending on the values taken by the index α, the basis elements e_i and φ_α make up the Ramond ($\alpha\in\mathbb{Z}$) or Neveu-Schwarz ($\alpha\in\mathbb{Z}+1/2$) type superalgebras on Σ_g. These algebras are of the form [3].

$$[e_i, e_j]=U^k_{ij}\, e_k\ , \qquad [e_i, \varphi_\alpha]=U^\beta_{i\alpha}\, \varphi_\beta\ , \qquad \{\varphi_\alpha, \varphi_\beta\}=U^i_{\alpha\beta}\, e_i\ . \tag{6}$$

where the brackets are the corresponding Lie derivatives, and the anticommutator is $\{\varphi_\alpha, \varphi_\beta\} \equiv \varphi_\alpha\varphi_\beta+\varphi_\beta\varphi_\alpha$. If one uses property (3), one can obtain expressions for the structure constants

$$U^k_{ij} = \frac{1}{2\pi i} \oint \Omega^k\ [e_i, e_j], \qquad U^i_{\alpha\beta} = \frac{1}{2\pi i} \oint \Omega^i\ \{\varphi_\alpha, \varphi_\beta\},$$

$$U^\beta_{i\alpha} = \frac{1}{2\pi i} \oint \kappa^\beta\ [e_i, \varphi_\alpha]. \tag{7}$$

Let us find possible extensions of these superalgebras in a way similar to that used for the KN algebra [4,5]. For this purpose we take the bases ω^i and κ^α and find the expressions in brackets

$$
\begin{aligned}
[e_i, \omega^j] &= \mathcal{L}_{e_i} \omega^j = - T^j_{ik}\ \omega^k,\\
[e_i, \kappa^\alpha] &= \mathcal{L}_{e_i} \kappa^\alpha = - T^\alpha_{i\beta}\ \kappa^\beta,\\
\{\varphi_\alpha, \omega^j\} &\equiv \varphi_\alpha\ \omega^j = - T^j_{\alpha\beta}\ \kappa^\beta,\\
\{\varphi_\alpha, \kappa^\beta\} &\equiv d(\varphi_\alpha \kappa^\beta) = - T^\beta_{\alpha i}\ \omega^i.
\end{aligned} \tag{8}
$$

Then, using (3), one can easily calculate the constants:

$$T^k_{ij} = \frac{1}{2\pi i} \oint \omega^k\ e_i dA_j\ , \qquad T^i_{\alpha\beta} = \frac{-1}{2\pi i} \oint \omega^i\ \varphi_\alpha \kappa_\beta\ ,$$

$$T^\beta_{i\alpha} = \frac{-1}{2\pi i} \oint \kappa_\alpha\ [e_i, \kappa^\beta], \qquad T^\beta_{\alpha i} = \frac{1}{2\pi i} \oint \varphi_\alpha\ \kappa^\beta dA_i\ . \tag{9}$$

Now we have all that is necessary for construction of the sought-for extensions of the superalgebras (6). We shall try to find them in the form

$$
\begin{aligned}
[\Phi_a, \Phi_b\rangle &= U^c_{ab}\, \Phi_c \quad , \\
[\Phi_a, \Psi^b\rangle &= B^b_a - T^b_{ac}\, \Psi^c \quad , \\
[\Psi^a, \Psi^b\rangle &= 0 \quad .
\end{aligned}
\tag{10}
$$

To shorten the formulae, we use a multi-index $a=(i,\alpha)$, where $i(\alpha)$ is the index of the Bose (Fermi) component of the supermultiplet. According to this we have $\Phi_a\sim(L_i, G_\alpha)$, $\Psi^a\sim(\psi^i, \chi^\alpha)$. The Virasoro-type generators L_i and the supercurrent G_α are expressed through the dynamic variable of the strings, propagating on Σ_g [3].

$$
L_i = \frac{1}{2}\, \ell^{mn}_i\, \alpha^\mu_m\, \alpha^\mu_n + \frac{1}{4}\, \ell^{\alpha\beta}_i\, b^\mu_\alpha\, b^\mu_\beta \; , \qquad G_\alpha = \ell^{\beta i}_\alpha\, b^\mu_\beta\, \alpha^\mu_i \; . \tag{11}
$$

As before, we assume that summation is made over the repeated indices. The Greek indices μ take the values 0,1,2,...D-1, where D is the dimensionality of the Minkowski space with metric $\eta^{\mu\nu}$, in which the string is immersed. The constants have the form

$$
\ell^{mn}_i = \frac{1}{2\pi i} \oint e_i\, \omega^m \omega^n , \qquad \ell^{\beta i}_\alpha = \frac{1}{2\pi i} \oint g_\alpha \lambda^\beta \omega^i \; ,
$$

$$
\ell^{\alpha\beta}_i = \frac{1}{2\pi i} \oint e_i (\lambda^\alpha d\lambda^\beta - \lambda^\beta d\lambda^\alpha) \; . \tag{12}
$$

The brackets in (10) are the graded Poisson brackets. Note in this connection that in the quantum case we shall deal with an algebra different from (10), since the first of the commutators will have a central term. It is not important for our considerations, so we confine ourselves to the classical case. The basic Poisson brackets are equal to

$$
[\alpha^\mu_i, \alpha^\nu_j] = \eta^{\mu\nu} \gamma_{ij} \; , \qquad \langle b^\mu_\alpha, b^\nu_\beta\rangle = \eta^{\mu\nu} \delta(\alpha+\beta) , \tag{13}
$$

where $\gamma_{ij} = \frac{1}{2\pi i} \oint A_i dA_j$, $\delta(\alpha+\beta) = \frac{1}{2\pi i} \oint h_\alpha h_\beta = \frac{1}{2\pi i} \oint h^\alpha h^\beta$.

The structure constants U^c_{ab} and T^c_{ab} are defined by formulae (7) and (9). The main requirement they must meet is validity of the graded Jacobi identities resulting from (10). These identities are satisfied if superalgebra (10) is confined to the system of contours C_τ , introduced in Ref.[1]. Contours C_τ on the surface Σ_g correspond to the string position at the moment of "time" τ and are defined as the level lines of the function $\tau = \mathrm{Re}\ p(Q)$, where $p(Q) = \int_{Q_0}^{Q} \omega$, Q_0 is an arbitrary initial point, and ω is the only meromorphic differential of the third kind that has simple poles with residues ± 1 at points $P_\pm$ and purely imaginary periods over all cycles. On the contour C_τ the basis $f_j^{(\lambda,x)}$ is complete, which is manifested in the existence of "delta functions"

$$\Delta_\tau^{(\lambda)}(Q,Q') = \sum_i f_i^{(\lambda,x)}(Q) f_{-i}^{(1-\lambda,-x)}(Q'), \tag{14}$$

with the main properties of the standard delta function [2]. Using them, one can easily establish validity of the Jacobi identities in question.

The central term B_a^δ is

$$B_i^j = \frac{1}{2\pi i} \oint \omega^j \, e_i \sigma \quad , \quad B_\alpha^\beta = \frac{1}{2\pi i} \oint g_\alpha \, h^\beta \sigma \quad , \tag{15}$$

where σ is the non-exact form on the considered system of closed contours of the Riemann surface Σ_g . If the third kind differential ω is taken to serve as σ, then at g=0 $B_a^\delta \big|_{g=0,\sigma=\omega} = \delta_a^\delta$, and algebra (10) completely coincides with the known extended algebra of constraints and subsidiary conditions in the free superstring theory [8].

To find the form of the generators Ψ^a, one must solve the second equation in (10). This is the way the general expression for ψ^i was obtained by in Ref.[4].

$$\psi(Q)=\psi^i A_i(Q)=X(Q_o)+\int_{Q_o}^{Q}(\pi+\sigma), \qquad Q_o, Q \in C_\tau . \tag{16}$$

Here $X(Q)=X_\mu(Q)\xi_\mu = x^i_\mu A_i(Q)\xi^\mu$ determines the string configuration in D-dimensional space-time, $\pi(Q)=\frac{1}{\sqrt{2}}\alpha^\mu_i\omega^i(Q)\xi_\mu$, $\xi_\mu=-2k_\mu/(kp)$, p_μ is the momentum of the string centre-of-mass. The light-like vector k_μ breaks the explicit Lorentz invariance of the theory, so it is only an auxiliary quantity ($k^2=0$ ensures validity of the last equation in (10)). The procedure of its removal from the physical results was discussed in the literature [9,10].

To determine the fermion component χ^α one must solve the equation

$$[G_\alpha,\psi^i\} = -T^i_{\alpha\beta}\,\chi^\beta. \tag{17}$$

Using (3) and properties of "delta function" (14), one can transform it into

$$[G_\alpha(Q'),\psi(Q)\} = \kappa^\alpha(Q')g_\alpha(Q)h_\beta(Q)\,\chi^\beta , \tag{18}$$

where $G(Q)=G_\alpha\kappa^\alpha(Q)$. Let us calculate the external differential of this expression. Then we have

$$[G_\alpha(Q'),d\psi(Q)\} = [G_\alpha(Q'),\pi(Q)\} = -\frac{1}{\sqrt{2}}\,b^\mu_\beta\,\xi_\mu\,\ell^{\beta i}_\alpha dA_i\,\kappa^\alpha(Q'),$$

on the left-hand side and

$$\kappa^\alpha(Q')\;d[g_\alpha(Q)h_\beta(Q)]\;\chi^\beta = -\kappa^\alpha(Q')\;T^i_{\alpha\beta}\;dA_i(Q)\;\chi^\beta .$$

on the right-hand side. Considering $\ell^{\beta i}_\alpha = -T^i_{\alpha\sigma}\,\delta(\sigma+\beta)$, we obtain

$$\chi^\alpha = -\frac{1}{\sqrt{2}}\,b^\mu_{-\alpha}\,\xi_\mu . \tag{19}$$

As established in Refs.[11,12], there is a relation between the Virasoro and KN algebras. For the free and interacting string this relation between algebras is through the corresponding linear transformation. One can show that there is a similar relation between the superalgebras obtained in Ref.[8] and considered in this paper. It will be discussed elsewhere.

Finally we note that all the results can be trivially extended to the conjugated sector of the closed superstring.

References

[1] I.M.Krichever and S.P.Novikov,Funk.Anal.i Pril.21 No.2(1987)46

[2] I.M.Krichever and S.P.Novikov,Funk.Anal.i Pril.21 No.4(1987)47

[3] L.Bonora,M.Martellini,M.Rinaldi, and J.Russo,Phys.Lett.B 206 (1988) 444.

[4] R.M.Kashaev and A.A.Osipov,Preprint JINR P2-88-862, Dubna 1988.

[5] R.M.Kashaev and A.A.Osipov,Preprint JINR E2-88-931, Dubna 1988.

[6] H.Aratyn,R.Ingermanson and A.J.Niemi,Preprint DOE/ER/01545-391 (1987).

[7] H.Aratyn,R.Ingermanson and A.J.Niemi,Nucl.Phys.B 299 (1988)507

[8] H.Aratyn,R.Ingermanson and A.J.Niemi,Phys.Lett.B 194 (1987)506

[9] L.Brink,M.Henneaux and C.Teitelboim,Nucl.Phys.B 293 (1987) 505

[10] H.Aratyn and R.Ingermanson,Preprint DOE/ER/01545-399 (1987).

[11] J.Alberty,A.Taormina and P.van Baal,Comm.Math.Phys.120 (1988) 249.

[12] L.Bonora,M.Matone and M.Rinaldi,Phys.Lett.B 216 (1989) 313.

Received by Publishing Department
on February 27, 1989.