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GREEN FUNCTIONS OF SCALAR PARTICLES IN STOCHASTIC FIELDS

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1. INTRODUCTION

At the present time it is very popular to represent different physical characteristics in the form of functional integrals (see, for example,[1,2]). However, calculations of functional integrals with the exception of the Gaussian integrals and a few integrals of a special form are of a serious difficulty. The main computing methods of functional integrals are, first, the quasi-classical approach or the method of stationary phase when it is considered that the main contribution to the integral comes from a function which minimized an integrand action (see, for example,[2]) and, second, the variational calculations (see[3]).

In this paper we proposed a variational method improving the Feynman method[3] and apply it for investigation of the asymptotic behaviour of the Green functions in stochastic fields. The idea of this method was formulated in[4].

We think that in such a difficult problem as calculations of functional integrals variational estimations help at least to understand and make someone feel the character of behaviour of a functional integral although they do not give the exact value of this integral.

2. VARIATIONAL METHOD

Here we formulate our variational method which will be used in what follows. Let the functional integral be given

$$I(g) = \int d\sigma_{\phi}\, e^{-gW[\phi]}, \tag{2.1}$$

$$d\sigma_{\phi} = \frac{1}{N_{\phi}} \delta\phi \exp\{-\frac{1}{2} \iint_{V} dx_1 dx_2\, \phi(x_1)\, D^{-1}(x_1,x_2)\, \phi(x_2)\}. \tag{2.2}$$

The notation is the following: $D^{-1}(x_1,x_2)$ is the distribution or the differential operator. The Green function $D(x_1,x_2)$ is defined by the equation

$$\int_V dy D^{-1}(x_1,y)D(y,x_2) = \delta(x_1-x_2)$$

and it satisfies some given boundary conditions.

The volume $V \subset \mathbf{R}^d$ over which the integration is performed in (2.2) can be both finite and infinite.

The functional differential in a lattice approximation is defined as

$$\delta\phi = \prod_{x\in V} d\phi(x).$$

The normalization constant N_ϕ is determined from the condition

$$\int d\sigma_\phi = 1.$$

$W[\phi]$ is a real functional, and g is a "coupling constant".

It is assumed that the functional integral (2.1) is defined on the Gaussian measure (2.2), i.e., at least there exists a perturbation series in the coupling constant g.

Let us formulate our variational method. The succession of our actions is the following. First of all, let us diagonalize the quadratic form in (2.2). We introduce the function $\Delta(x_1,x_2)$ satisfying the condition

$$\int_V dy\Delta(x_1,y)\Delta(y,x_2) = D(x_1,x_2). \tag{2.3}$$

In the cases under consideration this function can easily be found but it is enough for us to suppose its existence. Let us introduce the functional variable

$$\phi(x) = \int_V dy\Delta(x,y)\Phi(y) = (\Delta,\Phi)(x). \tag{2.4}$$

The functional integral (2.1.) can be written

$$I(g) = \frac{1}{N_\phi}\int \delta\Phi\exp\{-\frac{1}{2}\int_V dx\,\Phi^2(x) - gW[(\Delta,\Phi)]\}, \tag{2.5}$$

where the new constant N_ϕ is defined by the condition $I(0)=1$.

Let us choose in the volume $V \subset \mathbf{R}^d$ some orthonormal system of functions $\{g_{\{n\}}(x)\}$, where

$$\{n\} = (n_1,\ldots,n_d), \quad n_j = 0,1,2,\ldots \; (j=1,\ldots,d)$$

satisfies the conditions

$$\int_V d^dx\, g_{\{n\}}(x)\, g_{\{n'\}}(x) = \delta_{\{n,n'\}} = \delta_{n_1 n_1'} \cdot \ldots \cdot \delta_{n_d n_d'},$$

$$\sum_{\{n\}} g_{\{n\}}(x)\, g_{\{n\}}(x') = \delta^{(d)}(x - x') = \delta(x - x'). \tag{2.6}$$

The choice of the system (2.6) is arbitrary enough. The unique condition imposed on this system is that the functions $D(x_1, x_2)$ and $\Delta(x_1, x_2)$ can be developed over the functions of this system.

Let us represent the function $\Phi(x)$ over which the integration is performed in (2.5) in the form

$$\Phi(x) = \sum_{\{n\}} u_{\{n\}} g_{\{n\}}(x), \tag{2.7}$$

where the coefficients $U_{\{n\}}$ are independent variables. Then,

$$\int_V dx\, \Phi^2(x) = \sum_{\{n\}} u^2_{\{n\}},$$

$$(\Delta, \Phi)(x) = \sum_{\{n\}} \Delta_{\{n\}}(x)\, u_{\{n\}}, \quad \Delta_{\{n\}}(x) = \int_V dy\, \Delta(x, y)\, g_{\{n\}}(y). \tag{2.8}$$

The functional integral (2.5) can be written in the form of the infinitely multiple integral

$$I(g) = \int d\sigma_n \exp\{-gW[(\Delta, \Phi)]\},$$

$$d\sigma_n = \prod_{\{n\}} \frac{du_{\{n\}}}{(2\pi)^{d/2}} e^{-\frac{1}{2}\sum_{\{n\}} u^2_{\{n\}}}, \tag{2.9}$$

where the normalization constant is written in the explicit form.

We want to stress that the representation (2.9) is equivalent to (2.1).

Let us proceed to the variational estimation of the integral (2.9). We introduce the new variables in (2.9)

$$u_{\{n\}} = \frac{u'_{\{n\}}}{\sqrt{1 + q_{\{n\}}}} + s_{\{n\}}, \tag{2.10}$$

where the quantities $q_{\{n\}}$ and $s_{\{n\}}$ will be variational parameters. They satisfy the conditions

$$|\sum_{\{n\}} q_{\{n\}}| < \infty, \quad |\sum_{\{n\}} s_{\{n\}}| < \infty .$$

We would like to make the following remarks. Instead of (2.10) it is possible to do the substitution

$$u_{\{n\}} = \sum_{\{\ell\}} U_{\{n,\ell\}} \frac{u'_{\{\ell\}}}{\sqrt{1+q_{\{\ell\}}}} + s_{\{n\}} , \tag{2.11}$$

where U is an orthogonal real matrix: $\det U = 1$, $UU^T = I$. This matrix defines some rotation in the space of variables $\{u_{\{n\}}\}$. However, according to (2.7) it means the transition to another orthonormal basis (2.6). In other words, the basis enters into the set of our variational parameters.

Let us substitute (2.10) into (2.9). One can get

$$I(g) = \prod_{\{n\}} \frac{1}{\sqrt{1+q_{\{n\}}}} \cdot \int d\sigma_n \exp\{\frac{1}{2} \sum_{\{n\}} \frac{q_{\{n\}}}{1+q_{\{n\}}} u^2_{\{n\}} - \\ - \sum_{\{n\}} \frac{s_{\{n\}}}{\sqrt{1+q_{\{n\}}}} u_{\{n\}} - \frac{1}{2} \sum_{\{n\}} s^2_{\{n\}} - gW[(\Delta_q, \Phi) + (\Delta, s)]\}, \tag{2.12}$$

where

$$(\Delta_q, \Phi) = \sum_{\{n\}} \frac{\Delta_{\{n\}}(x)}{\sqrt{1+q_{\{n\}}}} u_{\{n\}} ; \quad (\Delta, s)(x) = \sum_{\{n\}} \Delta_{\{n\}}(x)\, s_{\{n\}} .$$

The measure $d\sigma_n$ is the same as in (2.9).

Let us use the inequality

$$\int d\sigma e^{-W} \geq \exp\{-\int d\sigma W\}$$

which is valid for any positive definite measures and any real functionals W. We obtain

$$I(g) \geq \exp\{-L[q] - \frac{1}{2}(s,s) - \int d\sigma_n W[(\Delta_q, \Phi) + (\Delta, s)]\},$$

$$L[q] = \frac{1}{2} \sum_{\{n\}} [\ln(1 + q_{\{n\}}) - \frac{q_{\{n\}}}{1 + q_{\{n\}}}].$$

$$(s, s) = \sum_{\{n\}} s^2_{\{n\}}. \qquad (2.13)$$

Representing our integral $I(g)$ in the form

$$I(g) = \exp\{-E(g)\} \qquad (2.14)$$

one can obtain from (2.13) for $E(g)$ the upper estimation

$$E(g) \le E_+(g),$$

$$E_+(g) = \min_{\{q,s\}} \{L[q] + \frac{1}{2}(s,s) + \qquad (2.15)$$

$$+ \int d\sigma_n W[(\Delta_q, \Phi) + (\Delta, s)]\}.$$

This formula is the desired unequality.

Thus, the variational parameters are, first, the orthonormal system (2.6) and, second, the parameters $\{q_{\{n\}}, s_{\{n\}}\}$ over which we have to compute the minimum in (2.15).

It should be noted that this variational estimation (2.15) gives the exact result for the quadratic functionals $W[\phi]$.

In conclusion, we want to remark that this variational method differs from the Feynman method [3] in that the additional parameters $s_{\{n\}}$ are introduced and the parameters $q_{\{n\}}$ are connected with the pure Gaussian measure just as the specific properties of the differential operator $D^{-1}(x_1, x_2)$ enter into the interaction functional W. Therefore, the variational equations obtained from (2.15) connect directly the parameters $q_{\{n\}}$ and $s_{\{n\}}$ with the behaviour of the Green function $D(x_1, x_2)$ so that a more precise estimation can be achieved.

3. GREEN FUNCTIONS IN THE FORM OF A FUNCTIONAL INTEGRAL

Let us consider the Green function satisfying the following equation:

$$[(i\frac{\partial}{\partial x_\mu} + V_\mu(x))^2 + W(x) + m^2] G(x, y | V, W) = \delta(x - y), \qquad (3.1)$$

where $W(x) \geq 0$. This equation is defined in the Euclidean space $\mathbf{R}^d$.

$$x^2 = x_1^2 + \dots + x_d^2\,, \quad \Box = \left(i\frac{\partial}{\partial x_\mu}\right)^2 = -\left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2}\right).$$

The solution of equation (3.1) can be written in the form according to the Feynman functional integral representation (see [5]):

$$G(x, y \mid V, W) = \int_0^\infty d\alpha e^{-\alpha m^2} T_\beta \exp\{-\alpha \int_0^1 d\beta \left(i\frac{\partial}{\partial x_\mu(\beta)} + V_\mu(x(\beta))\right)^2 -$$

$$- \alpha \int_0^1 d\beta W(x(\beta))\}\delta(x - y) =$$

$$= \int_0^\infty d\alpha e^{-\alpha m^2} \int \delta\Phi(\beta) \exp\{-\int_0^1 d\beta\, \Phi_\mu^2(\beta) + \tag{3.2}$$

$$+ 2i\sqrt{\alpha}\int_0^1 d\beta\, \Phi_\mu(\beta)\, V_\mu(x - 2\sqrt{\alpha}\int_\beta^1 d\beta'\,\Phi(\beta')) -$$

$$-\alpha \int_0^1 d\beta W(x - 2\sqrt{\alpha}\int_\beta^1 d\beta'\,\Phi(\beta'))\}\delta(x - y - 2\sqrt{\alpha}\int_0^1 d\beta\,\Phi(\beta)).$$

Here T_β is a symbol of a "chronological" ordering in the parameter β.

The normalization of the functional integral in (3.2) is chosen in the following way:

$$\int \delta\Phi \exp\{-\int_0^1 d\beta\,\Phi^2(\beta)\}\delta(x - 2\sqrt{\alpha}\int_0^1 d\beta\,\Phi(\beta)) =$$

$$= \int \left(\frac{dk}{2\pi}\right)^d e^{-ikx - \alpha k^2} = \frac{1}{(4\pi\alpha)^{d/2}} e^{-\frac{x^2}{4\alpha}}\,. \tag{3.3}$$

Let us perform the following transformations in (3.2). We will calculate the functional integral in the representation of basic vectors. The orthonormal basis in the interval $0 < \beta < 1$

will be taken in the form

$$g_n(\beta) = \int_{}^{1} \begin{cases} \sqrt{2}\cos 2\pi n\beta. \\ \sqrt{2}\sin 2\pi n\beta. \end{cases} \quad (n = 1, 2, \ldots). \tag{3.4}$$

We introduce the new variables of integration in (3.2)

$$\Phi_\mu(\beta) = \Phi_{o\mu} + a_\mu(\beta).$$

$$a_\mu(\beta) = \sum_{n=1}^{\infty} (u_{n\mu}\cos 2\pi n\beta + v_{n\mu}\sin 2\pi n\beta). \tag{3.5}$$

We have

$$\int_o^1 d\beta\, \Phi_\mu^2(\beta) = \Phi_{o\mu}^2 + \frac{1}{2}\sum_{n=1}^{\infty}(u_{n\mu}^2 + v_{n\mu}^2),$$

$$\int_o^1 d\beta\, \Phi_\mu(\beta) = \Phi_{o\mu},$$

$$\int_o^\beta d\beta' \Phi_\mu(\beta') = \beta\Phi_{o\mu} + A_\mu(\beta), \tag{3.6}$$

$$A_\mu(\beta) = \int_o^\beta d\beta' a_\mu(\beta') = \sum_{n=1}^{\infty} \frac{1}{2\pi n}(u_{a\mu}\sin 2\pi n\beta + v_{n\mu}(1 - \cos 2\pi n\beta)).$$

After introducing the new variables (3.5), using formulas (3.6) and performing the integration over $\Phi_{o\mu}$ with the condition (3.3), one can obtain for the functional integral (3.2)

$$G(x,y|V,W) = \int_o^\infty \frac{da}{(4\pi a)^{d/2}} e^{-am^2 - \frac{(x-y)^2}{4a}} \cdot R(x,y|V,W),$$

$$R(x,y|V,W) = \int d\sigma_a I_v(x,y|V) I_s(x,y|W), \tag{3.7}$$

$$I_v(x,y|V) = \exp\{i\int_o^1 d\beta\,(x - y - 2\sqrt{a}\,a(\beta))_\mu V_\mu(x\beta + y(1-\beta) + 2\sqrt{a}A(\beta))\}, \tag{3.8}$$

$$I_s(x,y|W) = \exp\{-a\int_0^1 d\beta W(x\beta + y(1-\beta) + 2\sqrt{a}A(\beta))\},$$

$$d\sigma_a = \prod_{n=1}^{\infty} \left(\frac{du\,dv}{2\pi}\right)^d \exp\{-\frac{1}{2}\sum_{n=1}^{\infty}(u^2_{n\mu} + v^2_{n\mu})\}. \qquad (3.9)$$

The representation (3.7) is a basis of our further calculations.

4. SCALAR PARTICLES IN A STOCHASTIC FIELD

As the first example of the application of our variational method we consider the problem of arising of a mass for scalar particles which is in a stochastic field. Our results can be formulated in the form of the following statement.

Statement. Let the equation

$$(\Box + g\phi^2(x))\,G(x,y|\phi) = \delta(x-y) \qquad (4.1)$$

be given in the Euclidean space R^4. The field $\phi(x)$ is a random Gaussian field with the correlation function

$$<\phi(x_1)\,\phi(x_2)>_\phi = D(x_1 - x_2) = \int\left(\frac{dk}{2\pi}\right)^4 \tilde{D}(k^2)\, e^{-ik(x_1 - x_2)} . \qquad (4.2)$$

The function $\tilde{D}(k^2)$ decreases rapidly enough so that

$$D_n = \int\left(\frac{dk}{2\pi}\right)^4 \tilde{D}(k^2)(k^2)^n < \infty, \ (n = 0, 1). \qquad (4.3)$$

Then, the following inequality is valid for the Green function averaging over the random field as

$$G(x-y) = <G(x,y|\phi)>_\phi \ \geq \frac{\text{const}}{\sqrt{(x-y)^2}}\, e^{-M_+\sqrt{(x-y)^2}} . \qquad (4.4)$$

Here

$$M_+ = \min_{\xi>0,\,\sigma>0,\,\lambda>0} \{\frac{1}{4\xi} + 2\sigma + \frac{\lambda}{2} +$$

$$+ g\xi \int ds\, e^{-s} \int_0^\infty \left(\frac{dk}{2\pi}\right)^4 \tilde{D}(k^2) [1 - \exp\{i(kn)\frac{s}{2\lambda} - \frac{\xi}{\sigma} k^2 (1 - e^{-\frac{\sigma}{\lambda}s})\}]\}, \quad (4.5)$$

where n is an Euclidean vector with $n^2 = 1$.

For the weak and strong coupling we have

$$M_+ = \begin{cases} \sqrt{gD_0}, & (g \ll 1), \\ 1.09\sqrt[4]{gD_1}, & (g \gg 1). \end{cases} \quad (4.6)$$

Now we proceed to prove this statement. According to the representation (3.7) the Green function in a random field $\phi(x)$ is written

$$G(x|\phi) = \frac{1}{(4\pi)^2} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-\frac{x^2}{4\alpha}} R(x|\phi),$$

$$R(x|\phi) = \int d\sigma_a \exp\{-g\alpha \int_0^1 d\beta\, \phi^2(x\beta + 2\sqrt{\alpha}A(\beta))\}, \quad (4.7)$$

where we put $y = 0$ for convenience.

The averaging of (4.7) over the Gaussian field $\phi(x)$ can be performed in the following way. The representation is valid

$$\exp\{-g\alpha \int_0^1 d\beta \phi^2(x\beta + 2\sqrt{\alpha}A(\beta))\} =$$

$$= \int d\sigma_b \exp\{-2i\sqrt{g\alpha} \int_0^1 d\beta\, b(\beta)\phi(x\beta + 2\sqrt{\alpha}A(\beta))\}, \quad (4.8)$$

where

$$d\sigma_b = \frac{1}{N_b} \delta b \exp\{-\int_0^1 d\beta\, b^2(\beta)\} =$$

$$= \frac{db_0}{\sqrt{\pi}} \prod_{n=1}^\infty \frac{dt_n ds_n}{2\pi} \exp\{-b_0^2 - \frac{1}{2}\sum_{n=1}^\infty (t_n^2 + s_n^2)\},$$

$$b(\beta) = b_0 + \sum_{n=1}^\infty (t_n \cos 2\pi n\beta + s_n \sin 2\pi n\beta).$$

Introducing the representation (4.8) in (4.7) and performing the averaging over the Gaussinan random field $\phi(x)$, one can get

$$G(x) = \langle G(x \mid \phi) \rangle_{\phi} = \frac{1}{(4\pi)^2} \int_0^{\infty} \frac{d\alpha}{\alpha^2} e^{-\frac{x^2}{4\alpha}} R(x, \alpha),$$

$$R(x, \alpha) = e^{-E(x,\alpha)} = \tag{4.9}$$

$$= \int d\sigma_b \int d\sigma_a \exp\{-2g\alpha \iint_0^1 d\beta_1 d\beta_2 b(\beta_1) D(x(\beta_1 - \beta_2) + 2\sqrt{\alpha} \int_{\beta_2}^{\beta_1} d\beta a(\beta) b(\beta_2)\}.$$

Now let us apply our variational method to (4.9). We introduce the variational parameters $\{p_n\}$ for the measure $d\sigma_b$ and $\{q_n\}$ for the measure $d\sigma_a$. The parameters s_n in (2.11) are put to be equal to zero for both the measures. The additional investigation omitted here shows that these parameters equal zero in the limit $x^2 \to \infty$. Using (2.13) one can get

$$R(x, \alpha) = e^{-E(x,\alpha)} \geq e^{-E_+(x,\alpha)},$$

$$E_+(x, \alpha) = \min_{\{q_n, p_n\}} \{4L[q] + L[p] + \tag{4.10}$$

$$+ 2g\alpha \iint_0^1 d\beta_1 d\beta_2 B_p(\beta_1 - \beta_2) \int \left(\frac{du}{\sqrt{2\pi}}\right)^4 e^{-\frac{u^2}{2}} D(x(\beta_1 - \beta_2) + 2u\sqrt{\alpha A_q(\beta_1 - \beta_2)})\}.$$

Here, the following formulas are introduced:

$$B_p(\beta_1 - \beta_2) = \int d\sigma_b\, b_p(\beta_1) b_p(\beta_2) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos 2\pi n(\beta_1 - \beta_2)}{1 + p_n} =$$

$$= \frac{1}{2}\delta(\beta_1 - \beta_2) - \sum_{n=1}^{\infty} \frac{p_n}{1 + p_n} \cos 2\pi n(\beta_1 - \beta_2), \tag{4.11}$$

$$\int d\sigma_a R\left(\int_{\beta_1}^{\beta_2} d\beta a_q(\beta)\right) = \int \left(\frac{du}{\sqrt{2\pi}}\right)^4 e^{-\frac{u^2}{2}} R(u\sqrt{A_q(\beta_1 - \beta_2)}),$$

$$A_q(\beta_1-\beta_2) = \sum_{n=1}^{\infty} \frac{2(1-\cos 2\pi n(\beta_1-\beta_2))}{(2\pi n)^2(1+q_n)},$$

$$L[r] = \sum_{n=1}^{\infty} [\ln(1+r_n) - \frac{r_n}{1+r_n}], \quad (r_n = q_n, p_n).$$

The behaviour of the Green function as $x^2 \to \infty$ is interesting for us. Let us proceed to calculation of this asymptotic behaviour in this limit. For this aim we put in the integral (4.9)

$$a = |x|\xi, \quad (|x| = \sqrt{x^2})$$

and

$$q_n = (\frac{|x|\sigma}{\pi n})^2, \quad p_n = (\frac{|x|\lambda}{\pi n})^2, \tag{4.12}$$

where σ and λ are variational parameters. Then, the following estimation for the Green function is valid

$$G(x) \geq \frac{1}{(4\pi)^2|x|} \int_0^{\infty} \frac{d\xi}{\xi^2} \exp\{-\frac{|x|}{4\xi} - E_+(|x|,\xi)\}. \tag{4.13}$$

Here $E_+(|x|,\xi)$ is defined by formula (4.10). In the case of the parameters (4.12) we obtain for formulas (4.11)

$$L[q] = \ln\frac{\mathrm{sh}\,|x|\sigma}{|x|\sigma} - \frac{|x|\sigma}{2\,\mathrm{cth}|x|\sigma} + \frac{1}{2} \xrightarrow[|x|\sigma\to\infty]{} \frac{1}{2}|x|\sigma,$$

$$B_p(\beta) = \frac{1}{2}[\delta(\beta) - |x|\lambda\frac{\mathrm{ch}|x|\lambda(1-2|\beta|)}{\mathrm{sh}|x|\lambda} + 1] \xrightarrow[|x|\lambda\to\infty]{}$$

$$\longrightarrow \frac{1}{2}[\delta(\beta) - |x|\lambda e^{-2|x|\lambda|\beta|} + 1], \tag{4.14}$$

$$A_q(\beta) = \frac{1}{4}\frac{\mathrm{ch}|x|\sigma - \mathrm{ch}|x|\sigma(1-2|\beta|)}{|x|\sigma\,\mathrm{sh}|x|\sigma} \xrightarrow[|x|\sigma\to\infty]{}$$

$$\longrightarrow \frac{1}{4|x|\sigma}(1-e^{-2|x|\sigma|\beta|}).$$

Substituting (4.14) into (4.10) and introducing new variables

$\beta_j \to \dfrac{\beta_j}{\lambda|x|}$ $(j = 1,2)$ one can get after some transformations as $|x| \to \infty$

$$G(x) \geq \frac{\text{const}}{|x|} e^{-M_+|x|}, \tag{4.15}$$

where M_+ is defined by formula (4.5).

The asymptotic behaviour of M_+ for the small and large g can be obtained in the following way. Introducing the variables

$$\sigma = g^{\rho}\sigma', \quad \lambda = g^{\rho}\lambda', \quad \xi = g^{-\rho}\xi',$$

where ρ is an independent parameter one can get

$$M_+ = g^{\rho} \min_{\xi,\sigma,\lambda} \{\frac{1}{4\xi} + 2\sigma + \frac{\lambda}{2} +$$

$$+ g^{1-2\rho}\xi \int_0^{\infty} ds e^{-s} \int (\frac{du}{2\pi})^4 \tilde{D}(u^2) [1 - \tag{4.16}$$

$$- \exp\{i(un) \frac{s}{g^{\rho}2\lambda} - \frac{\xi u^2}{g^{2\rho}\sigma}(1 - e^{-\frac{\sigma}{\lambda}s})\}]\}.$$

As $g \to 0$ we have $\rho = \dfrac{1}{2}$ and

$$M_+ = \sqrt{g} \min_{\xi,\sigma,\lambda} \{\frac{1}{4\xi} + 2\sigma + \frac{\lambda}{2} + \xi D_0\} = \sqrt{gD_0}.$$

For $g \to \infty$ we have $\rho = \dfrac{1}{4}$ and

$$M_+ = g^{1/4} \min_{\xi,\sigma,\lambda} \{\frac{1}{4\xi} + 2\sigma + \frac{\lambda}{2} + \frac{1}{8}\xi D_1(\frac{1}{2\lambda^2} + \frac{4\xi}{\sigma + \lambda})\} \leq$$

$$\leq 1{,}09\,(gD_1)^{1/4}.$$

5. SCALAR PARTICLES IN A STOCHASTIC VECTOR GAUGE FIELD

The next example is relevant to the paper[6] where the author claims that stochastic gauge fields (electromagnetic fields, for example) can lead to the confinement of particles which are in these fields. The confinement is considered to be reached if the Green function decreases at large distances more rapidly than any linear exponent, i.e.,

$$\lim_{|x|\to\infty} |G(x)|\exp\{N\sqrt{x^2}\} = 0,$$

for any $N > 0$. This condition means that these Green functions cannot describe asymptotically free states of particles.

Here, we show that it is not true. Our conclusion is based on the following statement.

Statement. Let the equation

$$[(i\frac{\partial}{\partial x_\mu} + V_\mu(x,y))^2 + m^2]G(x,y|F) = \delta(x-y), \tag{5.1}$$

be given in the Euclidean space $\boldsymbol{R}^4$. The vector field V_μ is defined

$$V_\mu(x,y) = \int_0^1 ds s F_{\mu\nu}(xs + y(1-s))(x-y)_\nu . \tag{5.2}$$

Here $F_{\mu\nu}(x)$ is a random Gaussian field with the correlation function

$$<F_{\mu\nu}(x)F_{\rho\sigma}(y)>_F = (\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})D(x-y), \tag{5.3}$$

$$D(x) = \int(\frac{dk}{2\pi})^4\tilde{D}(k^2)e^{-ikx},$$

where the function $\tilde{D}(k^2)$ decreases rapidly enough.

Then, the following inequality is valid for the Green function averaging over the random field $F_{\mu\nu}$ in the limit $(x-y)^2\to\infty$

$$G(x-y) = <G(x,y|F)>_F \geq \frac{\text{const}}{\sqrt{(x-y)^2}}e^{-m\sqrt{(x-y)^2}} . \tag{5.4}$$

In other words, a vector gauge random field does not give even a positive contribution to the mass of the particle.

We proceed to prove this statement. For the solution of equation (5.1) the representation (3.7) gives

$$G(x,y|F) = \int_0^\infty \frac{d\alpha}{(4\pi\alpha)^2} e^{-\alpha m^2 - \frac{(x-y)^2}{4\alpha}} R(x,y|V)$$

$$R(x,y|V) = \int d\sigma_a \exp\{i\int_0^1 d\beta (x-y+2\sqrt{\alpha}\,a(\beta))_\mu V_\mu (x\beta + y(1-\beta) + 2\sqrt{\alpha}\,A(\beta))\}, \tag{5.5}$$

where V_μ is defined by formula (5.2). Averaging (5.5) over the random field $F_{\mu\nu}$ and putting $y = 0$, one obtains

$$R(x) = \langle R(x,0|V)\rangle_F = \int d\sigma_a \exp\{-W[X]\}, \tag{5.6}$$

$$W[X] = \frac{1}{2}\iint_0^1 d\beta_1 d\beta_2 \iint_0^1 ds_1 ds_2 s_1 s_2 D(s_1 X(\beta_1) - s_2 X(\beta_2))Y,$$

$$Y = X_\mu(\beta_1) X'_\nu(\beta_1)\,\delta_{[\mu\nu,\rho\sigma]} X_\rho(\beta_2) X'_\sigma(\beta_2) = \tag{5.7}$$

$$= 4\alpha[x_\mu(A_\nu(\beta_1) - \beta_1 a_\nu(\beta_1)) - 2\sqrt{\alpha}\,A_\mu(\beta_1) a_\nu(\beta_1)]\,\delta_{[\mu\nu,\rho\sigma]}\times$$

$$\times[x_\rho(A_\sigma(\beta_2) - \beta_2 a_\sigma(\beta_2)) - 2\sqrt{\alpha}\,A_\rho(\beta_2) a_\sigma(\beta_2)],$$

where

$$\delta_{[\mu\nu,\rho\sigma]} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho},$$

$$X_\mu(\beta) = x_\mu\beta + 2\sqrt{\alpha}\,A_\mu(\beta),$$

$$X'_\mu(\beta) = \frac{\partial}{\partial\beta} X_\mu(\beta) = x_\mu + 2\sqrt{\alpha}\,a_\mu(\beta).$$

The variational estimation (2.13) gives for (5.6)

$$R(x,\alpha) = e^{-E(x,\alpha)} \geq e^{-E_+(x,\alpha)} \tag{5.8}$$

$$E_+(x,\alpha) = \min_{\{q_n\}} [4L[q] + W[q]],$$

$$W[q] = \int d\sigma_a W[X_q], \tag{5.9}$$

where

$$X_q(\beta) = x\beta - 2\sqrt{\alpha}A_q(\beta).$$

$$A_q(\beta) = \int_0^{\beta} d\beta' a_q(\beta'), \quad a_q(\beta) = \sum_{n=1}^{\infty} \frac{u_n \cos 2\pi n\beta + v_n \sin 2\pi n\beta}{\sqrt{1+q_n}}.$$

The variational parameters q_n are chosen in the form

$$q_n = \left(\frac{|x|\sigma}{\pi n}\right)^2, \quad (|x| = \sqrt{x^2}), \tag{5.10}$$

where σ is a variational parameter. Then, $L[q]$ is defined by formula (4.14). The convolutions of the fields $A_q(\beta)$ and $a_q(\beta)$ which arise when calculating the functional integral (5.9) are

$$\langle A_{q\mu}(\beta_1)A_{q\nu}(\beta_2)\rangle = \int d\sigma_a A_{q\mu}(\beta_1)A_{q\nu}(\beta_2) =$$

$$= \frac{\delta_{\mu\nu}}{4} \sum_{n=1}^{\infty} \frac{1 - \cos 2\pi n\beta_1 - \cos 2\pi n\beta_2 + \cos 2\pi n(\beta_1 - \beta_2)}{(\pi n)^2 + (\sigma|x|)^2} = \tag{5.11}$$

$$= \frac{\delta_{\mu\nu}}{8\sigma|x|\,\mathrm{sh}\,\sigma|x|}[\mathrm{ch}\sigma|x| - \mathrm{ch}\sigma|x|(1-2\beta_1) - \mathrm{ch}\sigma|x|(1-2\beta_2) + \mathrm{ch}\sigma|x|(1-2(\beta_1-\beta_2))] \to$$

$$\xrightarrow[|x|\sigma\to\infty]{} \frac{\delta_{\mu\nu}}{8\sigma|x|}[1 - e^{-2\sigma|x|\beta_1} - e^{-2\sigma|x|\beta_2} + e^{-2\sigma|x||\beta_1-\beta_2|}],$$

$$\langle A_{q\mu}(\beta_1)a_{q\nu}(\beta_2)\rangle = \frac{\partial}{\partial\beta_2}\langle A_{q\mu}(\beta_1)A_{q\nu}(\beta_2)\rangle \xrightarrow[\sigma|x|\to\infty]{}$$

$$\to \frac{\delta_{\mu\nu}}{4}[e^{-2\sigma|x|\beta_2} + \epsilon(\beta_1-\beta_2)e^{-2\sigma|x||\beta_1-\beta_2|}],$$

$$\langle a_{q\mu}(\beta_1)a_{q\nu}(\beta_2)\rangle = \frac{\partial^2}{\partial\beta_1\partial\beta_2}\langle A_{q\mu}(\beta_1)A_{q\nu}(\beta_2)\rangle \xrightarrow[\sigma|x|\to\infty]{}$$

$$\longrightarrow \frac{1}{2}\delta_{\mu\nu}[\delta(\beta_1-\beta_2) - \sigma|x|e^{-2\sigma|x||\beta_1-\beta_2|}].$$

The function (5.9) can be written

$$W(\sigma,|x|) = \frac{1}{2}\iint_0^1 d\beta_1 d\beta_2 \iint_0^1 ds_1 ds_2 s_1 s_2 \int \left(\frac{dk}{2\pi}\right)^4 \tilde{D}(k^2) \times$$

$$\times e^{i|x|(kn)(\beta_1 s_1 - \beta_2 s_2)} J(|x|, \sigma; \beta_1, \beta_2, s_1, s_2), \tag{5.12}$$

$$J = \int d\sigma_a e^{-i2\sqrt{a}(k[s_1 A_q(\beta_1) - s_2 A_q(\beta_2)])} \cdot Y_q,$$

where Y_q is defined by (5.7) where the vector $X(\beta)$ is changed by $X_q(\beta)$.

The integral for J can easily be calculated. However, we do not write down here this cumbersome expression but pick out from it the leading terms in the limit $|x| \to \infty$. It should be noted that

$$a = |x| \xi, \tag{5.13}$$

where $\xi = 0(1)$ as $|x| \to \infty$ because the asymptotic behaviour of the Green function in (5.5) is defined by a saddle point of the integrand.

The convolutions (5.11) considered as distributions of the variables β_1 and β_2 have the following smallness order

$$\langle A_{q\mu}(\beta_1) A_{q\nu}(\beta_2) \rangle = \frac{\delta_{\mu\nu}}{8\sigma|x|}[1 + 0(\frac{1}{\sigma|x|})],$$

$$\langle A_{q\mu}(\beta_1) A_{q\nu}(\beta_1) \rangle = \frac{\delta_{\mu\nu}}{4\sigma|x|}[1 + 0(\frac{1}{\sigma|x|})],$$

$$\langle A_{q\mu}(\beta_1) a_{q\nu}(\beta_2) \rangle = 0(\frac{1}{\sigma|x|}), \tag{5.14}$$

$$\langle A_{q\mu}(\beta_1) \beta_2 a_{q\nu}(\beta_2) \rangle = 0(\frac{1}{(\sigma|x|)^2}),$$

$$\langle a_{q\mu}(\beta_1) a_{q\nu}(\beta_2) \rangle = 0(\frac{1}{(\sigma|x|)^2}).$$

The limiting relation takes place

$$\int (\frac{dk}{2\pi})^4 F(k^2) e^{i|x|(kn)(\beta_1 s_1 - \beta_2 s_2)} \xrightarrow[|x| \to \infty]{}$$

$$\to \frac{1}{|x|} \delta(\beta_1 s_1 - \beta_2 s_2) \int \frac{d\vec{u}}{(2\pi)^3} F(\vec{u}^2) + 0(\frac{1}{|x|^2}), \tag{5.15}$$

if the function $F(u)$ decreases rapidly enough.

Since the limiting expressions for the convolutions (5.14) do not depend on β_1 and β_2, the integral over β_1 and β_2 can be calculated

$$\iint_0^1 d\beta_1 d\beta_2 \delta(\beta_1 s_1 - \beta_2 s_2) = \frac{1}{s_1}\theta(s_1 - s_2) + \frac{1}{s_2}\theta(s_2 - s_1). \tag{5.16}$$

Taking into account (5.13-16) and introducing the new variables $s_1 = s$ and $s_2 = s(1+t)/2$ the expressions for (5.12) can be written after some calculations

$$W[q] = |x| F(\eta)[1 + 0(\frac{1}{|x|})],$$

$$F(\eta) = \frac{3}{2}\eta \int_0^1 ds s^2 \int_0^1 dt \int \frac{d\vec{u}}{(2\pi)^3} \tilde{D}(\vec{u}^2)\, e^{-\vec{u}^2 \eta s^2 \frac{3+t^2}{4}} \times \tag{5.17}$$

$$\times [1 + \frac{1}{3}\eta \vec{u}^2 s^2 t^2],$$

where $\eta = \frac{\xi}{2\sigma}$. It is easily seen that

$$F(\eta) = \begin{cases} 0(\eta), & \eta \to 0 \\ 0(\frac{\ln\eta}{\eta}), & \eta \to \infty. \end{cases} \tag{5.18}$$

Finally, for the Green function (5.4) we obtain as $|x| \to \infty$

$$G(x) \geq \frac{\text{const}}{|x|} e^{-M_+ |x|},$$

$$M_+ = \min_{\xi,\sigma} \{ m^2\xi + \frac{1}{4\xi} + 2\sigma + F(\frac{\xi}{2\sigma}) \}.$$

Introducing $\eta = \frac{\xi}{2\sigma}$ one gets

$$M_+ = \min_{\xi,\eta} \{ m^2\xi + \frac{1}{4\xi} + \frac{\xi}{\eta} + F(\eta) \} =$$

$$= \min_{\eta} \{ \sqrt{m^2 + \frac{1}{\eta}} + F(\eta) \} = m.$$

Thus, we obtain (5.4).

6. SCALAR PARTICLES IN A SPACE WITH A FLUCTUATIVE METRICS

In this section we calculate a correction to the mass of a scalar particle which is in an Euclidean space with a weak stochastic correction to the metric of a flat Euclidean space $\boldsymbol{R}^4$. Suppose that this metric can be written

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \epsilon_{\mu\nu}(x). \tag{6.1}$$

The Lagrangian of scalar particles in the space with this metric has the form

$$L = \frac{1}{2} \int d^4x\sqrt{g}[g_{\mu\nu}(x)\frac{\partial\phi(x)}{\partial x_\mu}\frac{\partial\phi(x)}{\partial x_\nu} - m^2\phi^2(x)].$$

The equation of motion is

$$g_{\mu\nu}(x)\frac{\partial^2}{\partial x_\mu \partial x_\nu}\phi(x) + \frac{\partial g_{\mu\nu}}{\partial x_\mu}\frac{\partial\phi(x)}{\partial x_\nu} + g_{\mu\nu}\frac{\partial \ln\sqrt{g}}{\partial x_\mu}\frac{\partial\phi(x)}{\partial x_\nu} - m^2\phi(x) = 0. \tag{6.2}$$

The weak stochastic field $\epsilon_{\mu\nu}(x)$ should be considered as a gravity-like field, i.e., a field with the spin two. In this case, $\epsilon_{\mu\nu}(x)$ satisfies the conditions

$$\epsilon_{\mu\nu}(x) = \epsilon_{\nu\mu}(x), \quad \mathrm{tr}\epsilon = \epsilon_{\mu\mu}(x) = 0, \quad \frac{\partial}{\partial x_\mu}\epsilon_{\mu\nu}(x) = 0. \tag{6.3}$$

Then, the second term in (6.2) equals zero. The third term in (6.2) is $O(\epsilon^3)$ because

$$\sqrt{g(x)} = 1 + \frac{1}{4}\mathrm{tr}\epsilon^2(x) + O(\epsilon^3)$$

and after averaging over $\epsilon_{\mu\nu}$ the second term in $\sqrt{g}$ leads to a constant. Therefore, $\frac{\partial}{\partial x_\mu}\ln\sqrt{g} = O(\epsilon^3)$ and this term does not give any contribution to corrections of the second order.

As a result, the equation in a weak stochastic field is

$$[-g_{\mu\nu}(x)\frac{\partial^2}{\partial x_\mu \partial x_\nu} + m^2]\phi(x) = 0.$$

The equation for the Green function of a scalar particle can be written

$$[-(\delta_{\mu\nu} + \epsilon_{\mu\nu}(x))\frac{\partial^2}{\partial x_\mu \partial x_\nu} + m^2]G(x,y|\epsilon) = \delta(x-y). \tag{6.4}$$

Let us consider the stochastic field $\epsilon_{\mu\nu}(x)$. This field satisfies the conditions (6.3) and is a random Gaussian field with the correlation function

$$\langle \epsilon_{\mu\nu}(x)\epsilon_{\rho\sigma}(y)\rangle_\epsilon = D_{\mu\nu,\rho\sigma}(x-y) =$$

$$= \int\left(\frac{du}{2\pi}\right)^4 \tilde{D}(u^2)\,\Delta_{\mu\nu,\rho\sigma}(u)\,e^{-iu(x-y)} \quad ,$$

$$\Delta_{\mu\nu,\rho\sigma}(u) = d_{\mu\rho}d_{\nu\sigma} + d_{\mu\sigma}d_{\nu\rho} - \frac{2}{3}d_{\mu\nu}\,d_{\rho\sigma} \,, \tag{6.5}$$

$$d_{\mu\nu} = \delta_{\mu\nu} - \frac{u_\mu u_\nu}{u^2}.$$

The function $\tilde{D}(u^2)$ is supposed to decrease rapidly enough. We choose it in the form

$$\tilde{D}(u^2) = \frac{G}{u^2}e^{-u^2/\Lambda^2} \,. \tag{6.6}$$

Here $1/\Lambda$ defines the correlation length. It is natural to suppose that it is of the order of the Planck length

$$1/\Lambda \sim L_{P\ell} = \sqrt{\frac{G\hbar}{c^3}} \simeq 1.62\cdot 10^{-33}\ \text{sm}.$$

Let us consider equation (6.4). The solution of this equation can be represented in the form of a functional integral (we put $y = 0$):

$$G(x|\epsilon) = \int_0^\infty d\alpha\, e^{-m^2\alpha}\, T_\beta N[\epsilon]\int \delta\Phi \times$$

$$\times \exp\left\{-\int_0^1 d\beta\,\Phi_\mu(\beta)\,g^{-1}_{\mu\nu}(x(\beta))\Phi_\nu(\beta) - 2\sqrt{\alpha}\int_0^1 d\beta\,\Phi_\mu(\beta)\frac{\partial}{\partial x_\mu(\beta)}\right\}\delta(x)\,, \tag{6.7}$$

where

$$N[\epsilon] = \frac{1}{\sqrt{\det g(x)}} = \int\delta\Psi\exp\left\{-\int_0^1 d\beta\,\Psi_\mu(\beta)\,g_{\mu\nu}(x(\beta))\,\Psi_\nu(\beta)\right\},$$

$$N[0] = 1.$$

After standard transformations one can get

$$G(x|\epsilon) = \int_0^\infty d\alpha e^{-\alpha m^2} \iint \delta\Psi\delta\Phi\delta(x - 2\sqrt{\alpha}\int_0^1 d\beta\Phi(\beta)) \times$$

$$\times \exp\{-\int_0^1 d\beta[\Psi_\mu(\beta) g_{\mu\nu}(X(\beta))\Psi_\nu(\beta) + \Phi_{\mu\nu}(\beta) g^{-1}_{\mu\nu}(X(\beta))\Phi_\nu(\beta)]\}, \qquad (6.8)$$

$$X_\mu(\beta) = 2\sqrt{\alpha}\int_0^\beta d\beta' \Phi_\mu(\beta').$$

We consider the case of a weak stochastic field. Restricting oneself to the second order in the field $\epsilon_{\mu\nu}(x)$ one can get

$$g^{-1}_{\mu\nu}(x) = \delta_{\mu\nu} - \epsilon_{\mu\nu}(x) + \epsilon_{\mu\rho}(x)\epsilon_{\rho\nu}(x) + 0(\epsilon^3),$$

and

$$\int \delta\Psi \exp\{-\int_0^1 d\beta \Psi_\mu(\beta) g_{\mu\nu}(X(\beta))\Psi_\nu(\beta)\} =$$

$$= \exp\{\frac{1}{4}\delta(0)\int_0^1 d\beta\, \epsilon_{\mu\nu}(X(\beta))\epsilon_{\nu\mu}(X(\beta)) + 0(\epsilon^3)\}.$$

In this approximation the Green function has the form

$$G(x|\epsilon) = \int_0^\infty \frac{d\alpha}{(4\pi\alpha)^2} e^{-\alpha m^2 - \frac{x^2}{4\alpha}} J(x,\alpha|\epsilon), \qquad (6.9)$$

$$J(x,\alpha|\epsilon) = \int d\sigma_a \exp\{\frac{1}{4}\delta(0)\int_0^1 d\beta\, \epsilon_{\mu\nu}(X(\beta))\epsilon_{\nu\mu}(X(\beta)) -$$

$$- \int_0^1 d\beta \Phi_\mu(\beta)\epsilon_{\mu\rho}(\dot{X}(\beta))\epsilon_{\rho\nu}(X(\beta))\Phi_\nu(\beta) +$$

$$+ \int_0^1 d\beta \Phi_\mu(\beta)\epsilon_{\mu\nu}(X(\beta))\Phi_\nu(\beta)\}, \qquad (6.10)$$

$$X(\beta) = x\beta + 2\sqrt{\alpha}A(\beta), \qquad \Phi(\beta) = \frac{1}{2\sqrt{\alpha}}X'(\beta).$$

The averaging of (6.10) over the weak stochastic field $\epsilon_{\mu\nu}$ gives

$$J(x, a) = \langle J(x, a|\epsilon) \rangle_\epsilon =$$

$$= \exp\{\frac{1}{4}\delta(0) D_{\mu\nu,\nu\mu}(0) - \int d\sigma_a \int_0^1 d\beta \Phi_\mu(\beta) \Phi_\nu(\beta) D_{\mu\rho,\rho\nu}(0) +$$

$$+ \int d\sigma_a \frac{1}{2} \iint_0^1 d\beta_1 d\beta_2 \Phi_\mu(\beta_1) \Phi_\nu(\beta_1) D_{\mu\nu,\rho\sigma}(X(\beta_1) - X(\beta_2)) \Phi_\rho(\beta_2) \Phi_\sigma(\beta_2)\}.$$

As $|x| \to \infty$ the integral (6.9) is defined by the saddle point $a = |x|/2m$. Then

$$J(x, \frac{|x|}{2m}) = e^{|x| \delta m},$$

$$\delta m = \frac{5}{4} m \int (\frac{du}{2\pi})^4 \tilde{D}(u^2) [1 + \frac{2}{15} \cdot \frac{u^2}{(\frac{2u^2}{m})^2 + (un)^2} (1 - \frac{(un)^2}{u^2})^2]$$

and

$$G(x) \simeq e^{-(m - \delta m)|x|} \qquad (6.11)$$

If $\tilde{D}(k^2)$ is (6.6) then for $m \ll \Lambda$ one can obtain

$$\delta m = m \frac{5}{(8\pi)^2} G \Lambda^2 [1 + \frac{1}{12} \cdot \frac{m^2}{4\Lambda^2} \ln \frac{4\Lambda^2}{m^2}]. \qquad (6.12)$$

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