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CONVERGENT ELECTROMAGNETIC NUCLEON MASS DIFFERENCE, CORNWALL-NORTON MOMENTS AND LIGHT CONE EXPANSIONS

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CONVERGENT ELECTROMAGNETIC NUCLEON MASS DIFFERENCE, CORNWALL-NORTON MOMENTS AND LIGHT CONE EXPANSIONS

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Сходимость электромагнитной разности масс нуклонов, моменты Корнвэла-Нортона и разложения на световом конусе

Условия для сходимости электромагнитной разности масс нуклонов сформулированы при помощи моментов Корнвэла-Нортона.
Обсуждаются различные аспекты разложений на световом конусе. Доказано, что моменты однозначно связаны с обобщенными функциями - коэффициентами ряда Тэйлора для коммутатора.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Convergent Electromagnetic Nucleon Mass Difference, Cornwall-Norton Moments and Light Cone Expansions

Conditions for convergence of electromagnetic nucleon mass difference are formulated in terms of Cornwall-Norton moments.
Various aspects of light-cone expansions are discussed. It is proved that the momenta are uniquely related to the coefficient distributions of the Taylor series for the commutator.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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0. Introduction

In the past the problem of finite electromagnetic mass corrections for nucleons (of order e^2)[1] has been considered in connection with deep-inelastic scattering mostly in the case of scaling.

Providing that the structure functions describing deep-inelastic electron nucleon scattering behave dilatation-invariantly in the Bjorken region

$$\begin{aligned} W_1(\nu,\xi) &\sim F_1(\xi) \qquad & \nu = 2pq \to \infty \\ & & q^2 = -Q^2 \to -\infty \qquad (0.1)\\ W_2(\nu,\xi) &\sim \frac{1}{\nu} F_2(\xi) & \xi = -\frac{q^2}{\nu} \text{ fixed}, \end{aligned}$$

finiteness of electromagnetic mass corrections is expressed in the form of sum rules involving the scaling functions F_i. In dependence of the underlying parton structure one or two sum rules have been obtained[2].

In the present note we formulate this problem in terms of the moments of structure functions

$$\mu_{i,n}(Q^2) = \int_0^1 d\xi\, \xi^{n-1} W_i(\xi, Q^2) \quad , \quad \begin{array}{l} n = 0,2,4,\dots \\ q^2 = -Q^2 < \infty \, . \end{array} \qquad (0.2)$$

The consideration is thus extended to those cases where the individual moments show a different asymptotic behavior for $Q^2 \to \infty$ .So, especially, the well-known asymptotically free field theories (AFT)[3] are covered which predict a behavior

$$\begin{aligned} \mu_{1,n} &\sim [\log Q^2]^{\gamma_{1,n}} \\ & & (0.3) \\ \mu_{2,n} &\sim \frac{1}{Q^2} [\log Q^2]^{\gamma_{2,n}} \qquad \text{for } Q^2 \to \infty \end{aligned}$$

instead of the scaling law (0.1).

In Section 1. we express the amplitude of virtual Compton scattering by the $\mu_{i,n}(Q^2)$ in order to obtain convergence conditions for the Cottingham integral in terms of moments. Special attention is paid to the subtraction necessary for the definition of $\mu_{1,0}(Q^2)$ and to contributions from the so-called fixed pole. Finally we apply the convergence conditions found to a special field theoretic model [3].

In Section 2. we discuss various aspects of the light-cone expansion (LCE) [4] which up to now represents the only link between non-perturbative QFT (e.g., Callan-Symanzik equations) and observable quantities like the moments. The well-known LCE abstracted from operator product expansion will be considered in subsection 2.1. An inspection of the usual procedures to derive relations to the moments $\mu_{i,n}(Q^2)$ suggests that this LCE should be understood in a restricted sense only.

Subsection 2.2 is devoted to another possible definition for an asymptotic series at the light cone without ref ering to Lagrangian field theory. In general, however, such a series does not contain sufficient information to allow a term-by-term correspondence to characteristic objects defined in momentum space.

In subsection 2.3 we finally turn back to an investigation of the moments $\mu_n(Q^2)$ in the framework of general QFT.

It will be proved that the moments are uniquely related to the coefficients of the Taylor series for the matrix element of the even extended commutator $\bar{C}(x) = \varepsilon(x_0)\tilde{C}(x)$. Although this well-defined series is primary not an expansion on the light cone there are reasons to suppose that near the light cone it coincides with the LCE of 2.1 .

In Section 3. we summarize some conclusions.

1. Conditions for Finite Nucleon Mass Difference in Terms of Moments

From the beginning the problem of finite electromagnetic nucleon mass corrections was connected via the Cottingham formula [5]

$$\delta m = \delta m^p - \delta m^n \tag{1.1}$$

$$\delta m^{p,n} = \frac{\alpha\pi}{i}\int d^4q \frac{g_{\mu\nu}}{q^2+i\varepsilon} T^{p,n}_{\mu\nu}(p,q)$$

with the large q -behavior of the amplitude $T_{\mu\nu}(p,q)$ for virtual Compton scattering in forward direction (averaged over nucleon spins).

During the last years the scaling functions $F_i(\xi)$ of (0.1) have been used to determine the amplitude $T_{\mu\nu}(p,q)$ for high q. On this basis sum rules for the $F_i(\xi)$ have been obtained which guarantee the finiteness of the Feynman integral (1.1).

In this section we want to study the same problem of finite nucleon mass difference characterizing the virtual Compton amplitude by the moments $\mu_{i,n}(Q^2)$ of (0.2). Because of positivity $W_i(p,q) \geqslant 0$ these have to fulfill the inequalities

$$\mu_{i,n+2}(Q^2) \leq \mu_{i,n}(Q^2) \quad . \tag{1.2}$$

In the following, however, we allow a different behavior of the moments such that asymptotically

$$\begin{aligned} \mu_{1,n}(Q^2) &\lesssim (Q^2)^0 \\ \mu_{2,n}(Q^2) &\lesssim \frac{1}{Q^2} \qquad \text{for } Q^2 \to \infty \quad . \end{aligned} \tag{1.3}$$

The restrictions (1.3) are in accordance with dimensional analysis for the matrix element $\langle p | j_\mu(x) j_\nu(0) | p \rangle$ based on dim $j_\mu(x) = 3$.[6]

Let us now express the virtual Compton amplitude

$$T_{\mu\nu}(p,q) = \frac{i}{4\pi} \sum_\sigma \int dx\, e^{iqx} \langle p,\sigma | T\{ j_\mu(x) j_\nu(0) \} | p,\sigma \rangle \tag{1.4}$$

$$= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) T_1(pq,q^2) + \left(p_\mu - \frac{pq}{q^2} q_\mu\right)\left(p_\nu - \frac{pq}{q^2} q_\nu\right) T_2(pq,q^2) ,$$

$$p = (1,0,0,0)$$

in terms of the moments $\mu_{i,n}(Q^2)$ to insert into the Cottingham integral (1.1).

The structure functions $W_i(p,q)$ are connected with the invariant amplitudes by

$$W_i(p,q) = \mathrm{Im}\, T_i(p,q) \quad . \tag{1.5}$$

For spacelike q (i.e. $Q^2 = -q^2 > 0$) we apply dispersion relations to the $T_i(Q^2,\nu)$ at Q^2 fixed . Based on the Regge phenomenology T_1 needs one subtraction [7]

$$T_1(Q^2,\nu) = T_0(Q^2,\nu=0) + \frac{2\nu^2}{\pi} \int_{Q^2}^{\infty} \frac{d\nu'\, W_1(Q^2,\nu')}{\nu' \cdot (\nu'^2 - \nu^2)} \tag{1.6}$$

$$T_2(Q^2,\nu) = \frac{2}{\pi} \int_{Q^2}^{\infty} \frac{d\nu'\, W_2(Q^2,\nu') \cdot \nu'}{\nu'^2 - \nu^2} \tag{1.7}$$

where we have used $W_i(-\nu,Q^2) = -W_i(\nu,Q^2)$.

In the region of analyticity $|\nu| < Q^2$ the T_i can be represented by a convergent Taylor series

$$T_i(\nu,Q^2) = \sum_{n=0}^{\infty} \frac{\nu^n}{n!} \left(\frac{\partial}{\partial \nu}\right)^n T_i(\nu,Q^2) \Big|_{\substack{\nu=0 \\ Q^2 \text{ fixed}}} . \tag{1.8}$$

On the other hand from (1.6) and (1.7) we obtain for $|\nu| < Q^2$

$$\begin{aligned} T_2(\nu,Q^2) &= \frac{2}{\pi} \sum_{n=0}^{\infty} \nu^{2n} \int_{Q^2}^{\infty} d\nu' \frac{W_2(Q^2,\nu')}{\nu'^{2n+1}} \\ &= \frac{2}{\pi} \sum_{n=0}^{\infty} \left(\frac{\nu}{Q^2}\right)^{2n} \int_0^1 d\xi \cdot \xi^{2n-1} W_2(\xi,Q^2) , \end{aligned} \tag{1.9}$$

$$T_1(\nu,Q^2) = T_0(Q^2) + \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{\nu}{Q^2}\right)^{2n} \int_0^1 d\xi \cdot \xi^{2n-1} W_1(\xi,Q^2)$$

$$\text{with } \xi = \frac{Q^2}{\nu'} \quad . \tag{1.10}$$

Because of the uniqueness of the expansion (1.8) we establish

$$\mu_{i,n}(Q^2) = \frac{\pi (Q^2)^n}{2\, n!} \left(\frac{\partial}{\partial \nu}\right)^n T_i(\nu, Q^2) \Big|_{\nu=0} \qquad (1.11)$$

besides of $i=1,\ n=0$ .Therefore the amplitude T_2 is completely determined in terms of the moments $\mu_{2,n}(Q^2)$ with n = 0,2,4,... .In the Taylor series for T_1 ,however, instead of $\mu_{1,0}(Q^2)$,the subtraction term $T_0(Q^2)$ appears. We conclude that the zeroth moment of $W_1(Q^2,\xi)$ is indeed not defined as long as we suppose the on-shell behavior $W_1(\nu,Q^2) \sim \nu$ (Q^2 fixed).Connections with the so-called fixed pole problem[8] will be discussed below.

Having expressed the Compton amplitude $T_{\mu\nu}(p,q)$ in terms of the moments $\mu_{i,n}(Q^2)$ and $T_0(Q^2)$ we evaluate the Cottingham integral

$$\delta m^{p,n} = \frac{\alpha \pi}{i} \int_{-\infty}^{+\infty} d^4 q \; \frac{\Delta T^{p,n}(p,q)}{q^2 + i\varepsilon} \qquad (1.12)$$

with

$$\Delta T(q^2,\nu) \equiv g^{\mu\nu} T_{\mu\nu}(p,q) = -3\, T_1(q^2,\nu) + \left(1 - \frac{\nu^2}{q^2}\right) T_2(q^2,\nu) \, . \qquad (1.13)$$

After performing a Wick rotation[9] and changing the integration variables we obtain

$$\delta m^{p,n} = 2\pi^2 \alpha \int_0^\infty \frac{dQ^2}{Q^2} \int_0^{2\sqrt{Q^2}} d\nu' \sqrt{Q^2 - \frac{\nu'^2}{4}} \; \Delta T^{p,n}(i\nu', Q^2) \qquad (1.14)$$

where we have used $T_i(q^2,\nu) = T_i(q^2,-\nu)$ and the analyticity properties of $T_i(q^2,\nu)$ contained in the DJL representation. To determine the resulting mass corrections

$$\begin{aligned} \delta m^{p,n} = & - 6\pi^2 \alpha \int_0^\infty \frac{dQ^2}{Q^2} \int_0^{2\sqrt{Q^2}} d\nu' \sqrt{Q^2 - \frac{\nu'^2}{4}} \cdot T_1^{p,n}(Q^2, i\nu') \\ & + 2\pi^2 \alpha \int_0^\infty \frac{dQ^2}{Q^2} \int_0^{2\sqrt{Q^2}} d\nu' \sqrt{Q^2 - \frac{\nu'^2}{4}} \cdot T_2^{p,n}(Q^2, i\nu') \qquad (1.15) \\ & - \frac{\pi^2 \alpha}{2} \int_0^\infty \frac{dQ^2}{Q^4} \int_0^{2\sqrt{Q^2}} d\nu' \sqrt{Q^2 - \frac{\nu'^2}{4}} \cdot \nu'^2 \cdot T_2^{p,n}(Q^2, i\nu') \end{aligned}$$

we have to know the invariant amplitudes $T_i(Q^2, i\nu')$ in the region $0 \leq \nu' \leq 2\sqrt{Q^2}$ .For studying the convergence of $\delta m^p - \delta m^n$ it is sufficient to consider only the contribution to the integral from $Q^2 > 4$ .This restriction,however, guarantees convergence of the Taylor series (1.9),(1.10) for the $T_i(Q^2, i\nu')$: $Q^2 > 4$ means $\frac{\sqrt{Q^2}}{2} > 1$ and therefore

$$|\nu'| < |\nu'| \cdot \frac{\sqrt{Q^2}}{2} \leq Q^2 \, .$$

Now we are able to insert the expansions for the $T_i(Q^2, i\nu')$ into

$$\delta m^{p,n} = \delta m^{p,n}_{\infty} + \delta m^{p,n}_{\text{finite}}$$

$$\delta m_{\infty}^{p,n} = -6\pi^2\alpha \int_4^{\infty} \frac{dQ^2}{Q^2} T_0(Q^2) \int_0^{2\sqrt{Q^2}} d\nu' \cdot \sqrt{Q^2 - \frac{\nu'^2}{4}} \tag{1.16}$$

$$- 12\pi\alpha \int_4^{\infty} dQ^2 \sum_{n=1,2,\dots}^{\infty} (Q^2)^{-2n-1} \mu_{1,2n}(Q^2) \int_0^{2\sqrt{Q^2}} d\nu' \cdot \sqrt{Q^2 - \frac{\nu'^2}{4}} \cdot \nu'^{2n} (-1)^n$$

$$+ 4\pi\alpha \int_4^{\infty} dQ^2 \sum_{n=0,1,\dots}^{\infty} (Q^2)^{-2n-1} \mu_{2,2n}(Q^2) \int_0^{2\sqrt{Q^2}} d\nu' \cdot \sqrt{Q^2 - \frac{\nu'^2}{4}} \cdot \nu'^{2n} (-1)^n$$

$$- \pi\alpha \int_4^{\infty} dQ^2 \sum_{n=0,1,\dots}^{\infty} (Q^2)^{-2n-2} \mu_{2,2n}(Q^2) \int_0^{2\sqrt{Q^2}} d\nu' \cdot \sqrt{Q^2 - \frac{\nu'^2}{4}} \cdot \nu'^{2n+2} (-1)^n .$$

After performing the ν'-integration we obtain

$$\delta m_{\infty} = -3\pi^3\alpha \int_4^{\infty} dQ^2 T_0(Q^2) \tag{1.17}$$

$$- 6\pi^2\alpha \int_4^{\infty} dQ^2 \sum_{n=1}^{\infty} (Q^2)^{-n} \mu_{1,2n}(Q^2) \cdot 4^n (-1)^n \prod_{i=0}^{n-1} \frac{2i+1}{2i+4}$$

$$+ 2\pi^2\alpha \int_4^{\infty} dQ^2 \sum_{n=0}^{\infty} (Q^2)^{-n} \mu_{2,2n}(Q^2) \cdot 4^n (-1)^n \prod_{i=0}^{n-1} \frac{2i+1}{2i+4} \cdot \frac{3}{2n+4} .$$

From our assumptions (1.2) and (1.3) we conclude that the convergence of $\delta m_{\infty}^{p,n}$ depends only on the asymptotic properties of the moments $\mu_{1,2}(Q^2)$, $\mu_{2,0}(Q^2)$ and the subtraction term $T_0(Q^2)$ for large Q^2 :

$$\delta m = \delta m^p - \delta m^n \tag{1.18}$$

$$\sim \int^{\infty} dQ^2 \left[\mu_{2,0}(Q^2) + \frac{4}{Q^2} \mu_{1,2}(Q^2) - 2\pi T_0(Q^2) \right]^{p-n} < \infty .$$

Discussion

1.Comparison with the case of scaling.

There are one or two sum rules involving scaling functions in dependence of the underlying parton structure.Application of the DJL representation [2,10] is the most useful method in that case.It allows one to circumvent the problem of subtractions and yields the scaling functions $F_i(\xi)$ as distributions if defined in the limit $\nu \to \infty$ at ξ fixed. Accordingly,the integrations are mathematically well-defined, in particular,at $\xi = 0$ [2] .Working with the variables Q^2 and ξ is quite different.One cannot expect the limit $W(\xi, Q^2)$ for $Q^2 \to \infty$ to exist.For applications condition (1.18) and the scaling sum rules both are similar involving expressions $T_0(Q^2)$ or $\delta(\xi)$ resp. not experimentally attainable by deep-inelastic scattering.

2.Subtracted zeroth moment $\tilde{\mu}_{1,0}(Q^2)$ and fixed pole.

There is a connection between the subtraction term $T_0(Q^2)$

and the fixed pole contribution to W_1 [11].

Let us assume for $W_1(\nu, Q^2)$ the following Regge behavior [11]

$$W_1(\nu, Q^2) \sim \sum_{0 \leq \alpha \leq 1} C_\alpha(Q^2)\, \nu^\alpha \qquad \nu \longrightarrow \infty,\ Q^2 \text{ fixed.} \tag{1.19}$$

After adding and subtracting (1.19) from the structure function $W_1(Q^2, \nu)$ in (1.6) we perform the first integration and obtain

$$T_1(Q^2, \nu) = T_0(Q^2) + \frac{2\nu^2}{\pi} \int_{Q^2}^{\infty} \frac{d\nu'}{\nu'(\nu'^2 - \nu^2)} \Big\{ W_1(Q^2, \nu') - \sum_{0 \leq \alpha \leq 1} C_\alpha(Q^2)\, \nu'^\alpha \Big\}$$

$$- \frac{2\nu^2}{\pi} \int_0^{Q^2} \frac{d\nu' \sum_{0 < \alpha \leq 1} C_\alpha(Q^2)\, \nu'^\alpha}{\nu'(\nu'^2 - \nu^2)} - \sum_{0 < \alpha \leq 1} \nu^\alpha C_\alpha(Q^2) \frac{\cos \frac{\pi\alpha}{2}}{\sin \frac{\pi\alpha}{2}}$$

$$+ i \sum_{0 < \alpha \leq 1} \nu^\alpha C_\alpha(Q^2) + i\, C_0(Q^2) + \frac{2}{\pi} C_0(Q^2) \left[\ln Q^2 - \ln \nu \right] \tag{1.20}$$

If we now take the real part of this equation for $\nu \longrightarrow \infty$

$$T_0(Q^2) = \lim_{\nu \to \infty} \Big\{ \mathrm{Re}\, T_1(Q^2, \nu) + \sum_{0 < \alpha \leq 1} C_\alpha(Q^2)\, \nu^\alpha \frac{\cos \frac{\pi\alpha}{2}}{\sin \frac{\pi\alpha}{2}} + \frac{2}{\pi} C_0(Q^2) \cdot \ln \nu \Big\}$$

$$+ \frac{2}{\pi} \int_0^{\infty} \frac{d\nu'}{\nu'} \Big\{ W_1(Q^2, \nu') - \sum_{0 < \alpha \leq 1} C_\alpha(Q^2)\, \nu'^\alpha - C_0(Q^2) \cdot \Theta(\nu' - Q^2) \Big\}$$

$$- \frac{2}{\pi} C_0(Q^2) \ln Q^2 \tag{1.21}$$

the subtraction term $T_0(Q^2)$ appears as the sum of the so-called fixed pole (defined as the asymptotically constant contribution to the real part of T_1) and the "subtracted" zeroth moment of $W_1(Q^2, \nu)$. Therefore theory could provide the missing information either in terms of $T_0(Q^2)$, i.e., from light-cone expansions (see Section 2), or by means of the fixed pole contribution, i.e., from Feynman diagrams of the vertex type.

3. Application to asymptotically free field theories.

At the end of this section we want to apply the condition (1.18) to so-called AFT, in particular the three-triplet model studied by Gross and Wilczek [3], which predicts canonical behavior up to logarithmic corrections

$$\mu_{1,n}(Q^2) \underset{Q^2 \to \infty}{\sim} C_{1,n} \frac{1}{[\ln Q^2]^{\gamma_{1,n}}}$$
$$\mu_{2,n}(Q^2) \underset{Q^2 \to \infty}{\sim} C_{2,n} \frac{1}{Q^2 [\ln Q^2]^{\gamma_{2,n}}} \tag{1.22}$$

The condition (1.18) for finite nucleon mass difference now reads

$$\int^{\infty} dQ^2 \Big\{ C_{2,0}\, Q^{-2} [\ln Q^2]^{-\gamma_{2,0}} + 4\, C_{1,2}\, Q^{-2} [\ln Q^2]^{-\gamma_{1,2}} - 2\pi\, T_0(Q^2) \Big\}^{p-n} < \infty \tag{1.23}$$

with $\gamma_{1,2} = \gamma_{2,0} = \frac{16}{81}$, where only the non-singlet parts contribute to the p-n mass difference. If there exist no gauge-invariant scalar operator of twist two (compare eq. (2.1)) in the Gross-Wilczek model and consequently $T_0(Q^2) \equiv 0$ condition (1.23) reads

$$\int^{\infty} dQ^2 \, \frac{\Delta C_{2,0}^{NS} + 4\Delta C_{1,2}^{NS}}{Q^2 [\ln Q^2]^{\frac{16}{81}}} < \infty$$

which can be valid for $\Delta C_{2,0}^{NS} + 4\Delta C_{1,2}^{NS} = 0$ only.

2. Relations between Moments and Light-Cone Singularities

In the last time non-perturbative field theoretic methods have been developed (renormalization group or Callan-Symanzik equations applied to asymptotically free field theories) in order to obtain predictions for deep inelastic scattering.

But asymptotically free field theory (AFT) does not immediately evaluate measurable quantities like structure functions. On the contrary AFT concerns the concrete singularity structure of the coefficients $C_{i,n}(x^2)$ in the operator product expansion of electromagnetic currents [3]

$$T\{j_\mu(x)\, j_\nu(0)\} \sim g_{\mu\nu} \Box \frac{1}{x^2 - i\varepsilon} \sum_n C_{1,n}(x^2)\, x^{\mu_1 \cdots \mu_n}\, O^{(n)}_{\mu_1 \cdots \mu_n}(0)$$

$$+ \frac{1}{x^2 - i\varepsilon} \sum_n C_{2,n}(x^2)\, x^{\mu_1 \cdots \mu_n}\, O^{(n)}_{\mu\nu\mu_1 \cdots \mu_n}(0) \qquad (2.1)$$

at the light cone $x^2 = 0$.

To obtain really predictions from the AFT for deep inelastic scattering one has to make use of well-defined correlations between the expansion of $T\{j_\mu(x)\, j_\nu(0)\}$ on the light cone (2.1) and characteristic attributes of the virtual Compton scattering amplitude. Usually one asserts that the Fourier transform of $C_n(x^2)$ is related to asymptotic behavior of the moments of the structure functions W_i [12]. Because of the methodical importance of this point it is worthwhile to recapitulate the underlying assumptions and to ask for what can be proved in the general framework of QFT.

2.1 Light-Cone Expansion and Moments

Let us start from the one-nucleon matrix element (spin averaged) of the LCE (2.1).

The operator $O^{(n)}_{\mu_1 \cdots \mu_n}$ is of spin n and its matrix element leads in a standard way to a polynomial of order n in the variable px. Thus each of the invariant amplitudes $\tilde{T}(x^2, px)$ (whose imaginary parts are just the structure functions $\tilde{W}_i(x)$) has the LCE

$$\tilde{T}(x^2, px) \sim \sum_n C_n(x^2)\, P_n(x^2, px) \qquad (2.2)$$

$$P_n(x^2, px) = x^{\mu_1 \cdots \mu_n} \langle p | O^{(n)}_{\mu_1 \cdots \mu_n}(0) | p \rangle = \mathrm{const}\, (px)^n + \ldots \; .$$

The usual procedure [13,14] consists in termwise Fourier transformation (for simplicity we assume power behavior $C_n(x^2) = (x^2)^{\kappa_n}$):

$$
\begin{aligned}
&\int dx\, e^{iqx} \Big\{ \sum_n c_n (x^2 - i0)^{k_n} (px)^n \Big\} \qquad (2.3) \\
&= \sum_n c_n \Big(ip\frac{\partial}{\partial q} \Big)^n (q^2)^{-k_n - 2} \\
&= \sum_n \Big\{ d_n \Big(\frac{i\nu}{2} \Big)^n (q^2)^{-k_n - n - 2} + d_n' \Big(\frac{i\nu}{2} \Big)^{n-2} (q^2)^{-k_n - n} + \dots \Big\} \\
&= \sum_n \Big\{ b_n \Big(\frac{\nu}{q^2} \Big)^n + b_n' \Big(\frac{\nu}{q^2} \Big)^{n-2} + \dots \Big\} (q^2)^{-k_n - 2}
\end{aligned}
$$

and comparison with the Taylor series for $T(q)$ (1.8)

$$
T(Q^2,\nu) = \frac{2}{\pi} \sum_n \Big(\frac{\nu}{Q^2} \Big)^n \mu_n(Q^2) \qquad \text{for} \quad Q^2 = -q^2 > 0 , \quad \Big| \frac{\nu}{Q^2} \Big| < 1 .
$$

(Here the zeroth moment eventually has to be replaced by a subtraction term $T_0(Q^2)$).

The basic idea is that the coefficients of $\left(\frac{\nu}{Q^2}\right)^n$ in the two series for $T(q)$ (the classical and the asymptotic one) can be identified.+) The result is then

$$
\mu_n(Q^2) \sim (Q^2)^{-k_n - 2} \qquad \text{for } Q^2 \to \infty . \quad (2.4)
$$

where the inequalities

$$
k_{n+2} \geq k_n \qquad (2.5)
$$

have to be fulfilled as a consequence of positivity of the $W_i(\nu, Q^2)$.

+) It makes no difference if instead of power series expansions with respect to orthogonal polynomials are applied.[14]

There are three assumptions, one has made to obtain (2.4) :

a) Term by term Fourier transformation of (2.2) is allowed and sums up to the Fourier transformed matrix element $\tilde{T}(x^2, px)$ at least in the region $q^2 = -Q^2 < 0$, $|\nu| < Q^2$.

b) Identification of the Fourier transformed LCE with the Taylor series for $T(q)$ in the region $|\nu| < Q^2$.

c) Validity of the inequalities (2.5) to have the possibility to determine the leading term among the infinite many coefficients of $\left(\frac{\nu}{Q^2}\right)^n$ in the Fourier transform (2.3).

It is obviously not enough quantitatively to determine the LC singularities $C_n(x^2)$ in (2.1) or (2.2), one should rather know the mathematical meaning of the LCE, i.e., how to work with it, too.

Of course, only a derivation of the LCE (2.1) or (2.2) from QFT can finally determine its mathematical properties and, correspondingly, the conditions of its validity (compare the proof for short distance expansion [15]). Without such a proof*) one should understand the LCE in a restricted sense which justifies the procedure to get the correspondence (2.4) :

The LCE (2.2) is an asymptotic expansion defined on the space of test functions $f(x)$ such that the Fourier transforms $\tilde{f}(q)$ are infinitely differentiable functions with support contained in K.
Here K is an arbitrary compact manifold in the region $\{ q^2 < 0 , -|q^2| < \nu < |q^2| \}$ of momentum space.

***)The investigations by Zimmermann [21] restricted to finite order of perturbation theory do not cover the more interesting case of removed dimension degeneration.**

Thus the support of the LCE in momentum space is contained in the region of convergence of the Taylor series (1.8). Then having two power series valid in the region of convergence uniqueness of coefficients emerges at once. The support puzzle[*] (the individual terms in (2.2) have timelike support only) disappears simply because by definition the LCE is meaningful outside the physical spectrum only. But the question about compatibility of QFT and so specified LCE remains open.

2.2 An Asymptotic Series on the Light Cone

In this part we represent another possible asymptotic series on the light cone developed in connection with the scaling case.

Let us outline the general procedure to determine the LC singularities to arbitrary order for the matrix element of the current commutator in case of the invariants $\tilde{C}_i(x)$ which are the Fourier transforms of $W_1(p,q)$ and $W_2(p,q)$.

It has been shown[16], that the $\tilde{C}_i(x)$ are causal and fulfill all necessary conditions for a DJL representation. For this purpose we consider now the distribution $\bar{C}(x^2,\vec{x}) = \varepsilon(x_0)\tilde{C}(x)$ defined by

$$\left\{ \tilde{C}(x), \varphi(x) \right\} = \left\{ \bar{C}(x^2,\vec{x}) , \frac{\varphi(\sqrt{x^2+\vec{x}^2},\vec{x}) - \varphi(-\sqrt{x^2+\vec{x}^2},\vec{x})}{2\sqrt{x^2+\vec{x}^2}} \right\} \tag{2.6}$$

Because of antisymmetry of $\tilde{C}(x)$ the test function $\varphi(x) \in S(R_4)$ has to be antisymmetric, too, so that on the right hand side of (2.6) there is a test function with respect to x^2. Notice, that the support of $\bar{C}(x^2,\vec{x})$ is restricted to $x^2 \geq 0$. For the so defined $\bar{C}(x^2,\vec{x})$ the DJL representation reads

$$\bar{C}(x^2,\vec{x}) = \frac{1}{4i\pi^2} \frac{\partial}{\partial x^2} \left\{ \Theta(x^2) \int_0^\infty d\lambda^2 \, J_0(\lambda\sqrt{x^2}) \, \tilde{\psi}(\vec{x},\lambda^2) \right\} \tag{2.7}$$

with
$$\tilde{\psi}(\vec{x},\lambda^2) = \int_0^1 d\vec{\mu} \, e^{i\vec{u}\vec{x}} \, \psi(|\vec{u}|,\lambda^2) \tag{2.8}$$

Further it has been shown[17]:

> If there exists a real number k_0, such that the sequence of the distributions $\frac{1}{\varrho^{2k_0}} \bar{C}(\frac{1}{\varrho^2}x^2,\vec{x})$ for $\varrho^2 \to \infty$ approaches some nonzero distribution $\bar{g}(x^2,\vec{x})$ (if integrated with a test function $f(x^2) \in S(R_1)$), then the limit has the structure $\bar{g}(x^2,\vec{x}) = G_0(\vec{x}) \frac{(x^2)_+^{k_0}}{\Gamma(k_0+1)}$ and is called the most singular part of $\bar{C}(x^2,\vec{x})$ at the light cone $x^2 = 0$.

After subtracting this leading part $\bar{g}(x^2,\vec{x})$ from $\bar{C}(x^2,\vec{x})$ one has to repeat this procedure described with an appropriate $k_1 > k_0$ and so on. Of course, there are generalizaof an asymptotic series by the well-defined Fourier transformation of an integral representation for the matrix element $\widetilde{C}(x)$ itself. For reasons of simplicity we restrict the consideration to a DGS representation

$$\widetilde{C}(x) = \frac{1}{2\pi i}\int_0^\infty d\lambda^2\, \widetilde{\Psi}(x_0,\lambda^2)\,\Delta(x,\lambda^2) \tag{2.12}$$

with

$$\widetilde{\Psi}(x_0,\lambda^2) = \int_{-1}^{+1} d\mu\, e^{i\mu x_0}\,\Psi(\mu,\lambda^2) \tag{2.13}$$

and $\Delta(x,\lambda^2) = \frac{i}{(2\pi)^3}\int dq\, e^{-iqx}\varepsilon(q_0)\,\delta(q^2-\lambda^2) = \frac{\varepsilon(x_0)}{2\pi}\frac{\partial}{\partial x^2}\left\{\Theta(x^2)\, J_0(\lambda\sqrt{x^2})\right\}$.

Because of the finite support in μ, $\widetilde{\Psi}(x_0,\lambda^2)$ is an entire function with respect to x_0 if integrated with a test function $\varphi(\lambda^2)\in S(R_+)$. Therefore $\widetilde{\Psi}(x_0,\lambda^2)$ can be expanded in a Taylor series

$$\widetilde{\Psi}(x_0,\lambda^2) = \sum_{n=0}^{\infty} x_0^n\, \frac{i^n}{n!}\, h_n(\lambda^2) \tag{2.14}$$

with

$$h_n(\lambda^2) = \int_{-1}^{+1} d\mu\cdot \mu^n\, \Psi(\mu,\lambda^2) \tag{2.15}$$

To derive a similar series for the commutator itself it is useful to consider the symmetrically extended distribution $\overline{C}(x) = \varepsilon(x_0)\widetilde{C}(x)$. Remembering that the functional $\{\varepsilon(x_0)\Delta(x,\lambda^2),\ \chi(x^2)\}$ is a test function with respect to λ^2 [17] (where $\chi(x^2)\in S(R_+)$) we obtain from (2.12) and (2.14) a Taylor series for $\overline{C}(x)$:

$$\overline{C}(x) = \frac{1}{2\pi i}\sum_{n=0}^{\infty} x_0^n\, \frac{i^n}{n!}\, f_n(x^2) \tag{2.16}$$

$$f_n(x^2) = \int_0^\infty d\lambda^2\, [\varepsilon(x_0)\Delta(x,\lambda^2)]\, h_n(\lambda^2) \tag{2.17}$$

$$= \frac{1}{2\pi}\frac{\partial}{\partial x^2}\left\{\Theta(x^2)\int_0^\infty d\lambda^2\, J_0(\lambda\sqrt{x^2})\, h_n(\lambda^2)\right\}$$

which has to be understood as a functional with respect to x^2.

The distributions $f_n(x^2)$ and $h_n(\lambda^2)$ are connected by the special Bessel transformation studied in [17].

It should be mentioned that (2.16) has not been derived as an expansion on the light cone. Nevertheless, this power series can be connected with the moments $\mu_n(Q^2)$ expressed in terms of the amplitude at $q^2 = -Q^2 < 0$

$$\mu_n(Q^2) = \frac{\pi}{2}\,\frac{Q^{2n}}{n!}\left(\frac{\partial}{\partial\nu}\right)^n T(q)\Big|_{\substack{q^2\ \text{fixed}\\ \nu=0}}, \quad n = 0,2,4,\ldots$$

From the DGS representation+) of the amplitude

$$T(q) = -\frac{1}{\pi}\int_0^\infty d\lambda^2\int_{-1}^{+1} d\mu\, \frac{\Psi(\mu,\lambda^2)}{(q_0-\mu)^2-\vec{q}^{\,2}-\lambda^2+i0} \tag{2.18}$$

+) For the amplitudes T_i corresponding to the structure functions W_i there seems to be no need for subtractions in the DGS-representation.

$$T(q) = \frac{1}{\pi}\int_0^\infty d\bar{\lambda}^2 \int_{-1}^{+1} d\mu \, \frac{\Psi(\mu,\bar{\lambda}^2+\mu^2)}{-q^2+\mu\nu+\bar{\lambda}^2-i\varepsilon}$$

we obtain

$$\mu_n(Q^2) = \frac{1}{2}\, Q^{2n}\int_0^\infty d\lambda^2 \, \frac{\bar{h}_n(\lambda^2)}{\{Q^2+\lambda^2\}^{n+1}} \qquad (2.19)$$

with
$$\bar{h}_n(\lambda^2) = \int_{-1}^{+1} d\mu \cdot \mu^n \, \Psi(\mu,\lambda^2+\mu^2) \qquad . \qquad (2.20)$$

In such a way we have established a relation between the moments and the uniquely defined coefficients in the Taylor series $f_n(x^2)$ mediated by the spectral function $\Psi(\mu,\lambda^2)$ in $h_n(\lambda^2)$ and $\bar{h}_n(\lambda^2)$, respectively,

$$f_n(x^2) = \int_0^\infty d\lambda^2 \, [\varepsilon(x_0)\,\Delta(x,\lambda^2)]\, h_n(\lambda^2) \qquad (2.21)$$

$$\mu_n(Q^2) = \frac{1}{2}\, Q^{2n}\int_0^\infty d\lambda^2 \, \frac{\bar{h}_n(\lambda^2)}{\{Q^2+\lambda^2\}^{n+1}} \qquad . \qquad (2.22)$$

It should be noted that the second of the above relations has been obtained by Cornwall and Norton [19] already. These authors, however, considering neither Taeylor series nor expansions on the light cone, tried to connect the moments with the B JL commutators (i.e., equal-time

commutators involving higher derivatives of the current components $[(\frac{\partial}{\partial x_0})^n j_i(x), j_k(0)]_{x_0=0}$). In any case existence of the infinite set of BJL commutators is a rather strong assumption whereas the connections contained in eqs. (2.21), (2.22) have general validity.

In particular, eqs. (2.21), (2.22) yield relations between the asymptotic behavior of $\mu_n(Q^2)$ at $Q^2 \to \infty$ and $f_n(x^2)$ at $x^2 \to 0$.

For this purpose it is useful to apply the method of the quasi-limit, i.e., to consider in the sense of functionals $f_n(\frac{1}{\rho^2}x^2)$, $h_n(\rho^2\lambda^2)$ and $\bar{h}_n(\rho^2\lambda^2)$ for $\rho^2 \to \infty$, respectively. In reference [17] it has been shown that the transformation (2.21) constitutes an unique correspondence between the singularities at $x^2 \to 0$ and the asymptotic behavior at $\lambda^2 \to \infty$ (e.g., $f_n(x^2) \sim x^{2k_n}$ equivalent to $h_n(\lambda^2) \sim (\lambda^2)^{-k_n-2}$). The large Q^2 and large λ^2 behavior in (2.22), on the other hand, are uniquely related, too. (e.g., $\bar{h}_n(\lambda^2) \sim \lambda^{2\alpha}$ equivalent to $\mu_n(Q^2) \sim Q^{2\alpha}$) [20].

On this basis a direct connection between the asymptotic behavior of $f_n(x^2)$ and $\mu_n(Q^2)$ will be established if $h_n(\lambda^2)$ and $\bar{h}_n(\lambda^2)$ show the same asymptotic behavior. This is the case if positivity is fulfilled (see Appendix II). Therefore we have derived relations between the asymptotic behavior of the moments $\mu_n(Q^2)$ and the LC singularities of $f_n(x^2)$ which are similar to (2.4) but refer to the Taylor series (2.16).

The question could arise about the existence of a physical meaningful amplitude with moments showing different asymptotic Q^2-behavior. In Appendix I we give an example for an amplitude fulfilling causality and spectrum conditions such that the moments differ asymptotically by powers of $\log Q^2$.

3. Conclusions

There are, in principle, three types of series which could be understood as expansions on the light cone.
The first is the canonical LCE (2.1), (2.2) abstracted from Lagrangian Field Theory. Its mathematical meaning is rather unclear and correspondingly additional assumptions must be imposed to get useful information from this LCE.
Second, an asymptotic series (2.10) in the strict sense of generalized functions can be defined. The physical significance of its individual terms, except for the leading one, remains obscure, however.
There is finally an approach via the Taylor series of the matrix element $\bar{C}(x)$.

On the basis of general QFT we have established a connection between the moments $\mu_n(Q^2)$ and the coefficient distributions $f_n(x^2)$ of the well-defined Taylor series for the matrix element $\bar{C}(x)$.
Need for subtractions may spoil the definition of some

of the lowest moments. In this case the same connection is mediated by the Taylor coefficients of the amplitude.

The LC behavior of $f_n(x^2)$ is uniquely related to asymptotic behavior of the moments, at least if positivity conditions for the structure functions are fulfilled. In this case the LC singularities in eq. (2.16) appear in decreasing order. Nevertheless, this series must not necessarily coincide with the asymptotic series on the light cone in the strict sense (eq. (2.10)). From present investigations it seems reasonable to identify the series (2.16) near $x^2=0$ with the LCE (eq. (2.2)) used in Lagrangian Field Theory (if (2.2) accordingly is rewritten for the matrix element $\bar{C}(x)$).

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Appendix I Example for an amplitude with different asymptotic behavior for its moments

Attempts to construct structure functions $W(\xi,Q^2)$ corresponding to given asymptotics of $\mu_n(Q^2)$ have been undertaken more than once [3],[14]. We consider it desirable, however, to give an example in the DGS representation in order to verify compatibility of causal and spectral conditions with particular asymptotic behavior of $\mu_n(Q^2)$ (differing by powers of $\log Q^2$).

The example is described by spectral function

$$\psi(\mu,\lambda^2) = \frac{(\lambda^2-\mu^2)_+^{k(\mu)}}{\Gamma(k(\mu)+1)} ,$$

$$k(\mu) = k_0 - c|\mu|^p , \quad k(\mu) < 0 , \quad k_0 \neq 0$$

to be inserted into the DGS integral (2.18)

$$T(q) = \frac{1}{\pi}\int_{-1}^{+1} d\mu \int_0^\infty d\bar\lambda^2 \frac{[\Gamma(k+1)]^{-1}(\bar\lambda^2)_+^k}{\bar\lambda^2+\mu\nu-q^2-i0}$$

$$= \frac{1}{\pi}\int_{-1}^{+1} d\mu\, \Gamma(-k)\left\{\mu\nu-q^2-i0\right\}^k .$$

Applying formula (1.9) we obtain for the moments

$$\mu_n(Q^2) = \frac{\pi}{2}\frac{(Q^2)^n}{n!}\left(\frac{\partial}{\partial\nu}\right)^n T(q) \Bigg|_{\substack{q^2=-Q^2 \text{ fixed} \\ \nu=0}}$$

$$= \frac{1}{2n!}\int_{-1}^{+1} d\mu \cdot \mu^n\, \Gamma(-k(\mu)+n+1)\left(Q^2\right)^{k(\mu)} .$$

Asymptotic behavior is determined by the contributions from $\mu \approx 0$:

$$\mu_n(Q^2) \approx \frac{\Gamma(n+1-k_0)}{n!}\, Q^{2k_0}\int_0^\infty d\mu \cdot \mu^n e^{-\mu^p c \log Q^2}$$

$$\sim \frac{Q^{2k_0}}{[\log Q^2]^{\frac{n+1}{p}}} .$$

Remark that the amplitude has time-like and space-like support which is guaranteed by the finite extension of the support of $\psi(\mu,\lambda^2)$ around $\mu=0$.

The stronger individual moments differ from one another ($p\to 0$), the narrower becomes the support of $\psi(\mu,\lambda^2)$. We expect that stronger differences in asymptotic behavior (by powers of Q^2) would spoil support properties at $q^2<0$.

Appendix II

We have to show that $\bar h_n(\lambda^2)$ and $h_n(\lambda^2)$ have the same asymptotic behavior if positivity of the structure functions $W(\xi,Q^2)$ is fulfilled.

Positivity of the $W(\xi,Q^2)$ is equivalent to the relations (1.2)

$$\mu_{i,n+2}(Q^2) \leq \mu_{i,n}(Q^2)$$

and therefore equivalent to

$$\bar h_{i,n+2}(\lambda^2) \leq \bar h_{i,n}(\lambda^2) \qquad \text{for } \lambda^2 \to \infty . \quad \text{(A2.1)}$$

Let us assume that $\bar{h}_n(\lambda^2)$ has an q-limit of order α_n

$$\frac{\{\bar{h}_n(\varrho\lambda^2),\ \varphi(\lambda^2)\}}{\varrho^{\alpha_n}} \xrightarrow[\varrho\to\infty]{} \text{const} \neq 0$$

with $\varphi(\lambda^2) \in S(R_1)$.

Then for the difference of $\bar{h}_n(\lambda^2)$ and $h_n(\lambda^2)$ it holds (comp.(2.20),(2.15))

$$\begin{aligned}
&\{[\bar{h}_n(\varrho\lambda^2) - h_n(\varrho\lambda^2)],\ \varphi(\lambda^2)\} = \\
&= \int_{-1}^{+1} d\mu \cdot \mu^n \int_0^\infty d\lambda^2 \left[\psi(\mu, \varrho\lambda^2 + \mu^2) - \psi(\mu, \varrho\lambda^2) \right] \varphi(\lambda^2) \\
&= \int_{-1}^{+1} d\mu \cdot \mu^n \int_0^\infty d\bar{\lambda}^2\, \psi(\mu, \varrho\bar{\lambda}^2 + \mu^2) \left[\varphi(\bar{\lambda}^2) - \varphi\left(\bar{\lambda}^2 + \frac{\mu^2}{\varrho}\right) \right] \\
&= -\frac{1}{\varrho} \int_{-1}^{+1} d\mu \cdot \mu^{n+2} \int_0^\infty d\bar{\lambda}^2\, \psi(\mu, \varrho\bar{\lambda}^2 + \mu^2)\, \varphi'\left(\bar{\lambda}^2 + \Theta \frac{\mu^2}{\varrho}\right)
\end{aligned} \tag{A2.2}$$

where we have performed the variable transformation $\bar{\lambda}^2 = \lambda^2 - \frac{\mu^2}{\varrho}$ and used the mean value theorem.

Now at the right-hand side of (A2.2) we derived the $\bar{h}_{n+2}(\lambda^2)$ because $\varphi'(\lambda^2 + \Theta \frac{\mu^2}{\varrho})$ is a test function again. Therefore

$$\frac{\{[\bar{h}_n(\varrho\lambda^2) - h_n(\varrho\lambda^2)],\ \varphi(\lambda^2)\}}{\varrho^{\alpha_n}} = \frac{1}{\varrho^{\alpha_n+1}}\, \bar{h}_{n+2}(\lambda^2) \longrightarrow 0$$

because $\bar{h}_{n+2}(\lambda^2)$ has a q-limit of order α_{n+2} with $\alpha_{n+2} \leq \alpha_n$ (compare (A2.1)).

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