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D.B.Ion

ISOSPIN CONSTRAINTS ON EXPERIMENTAL OBSERVABLES OF (0 1/2 – 0'0'1'/2) REACTIONS

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* On leave of absence from Institute of Atomic Physics, Bucharest, Romania.

1. Introduction

In a recent paper [1] we have presented a general method for derivation of all the constraints on the experimental data and amplitude analysis of three $(0\ 1/2 \to 0'1'/2)$ reactions related by internal symmetries. Then, we have suggested that this method can be extended to the three-body final state $(0\ 1/2 \to 0'0'1'/2)$ reactions and also to the cases when zero-spin particles are replaced by unpolarized J -spin particles. On the other hand, recently Cashmore and Hey [2] have developed a formalism to describe the three-body final state reactions. This formalism is particularly suitable for the reconstruction of all the helicity amplitudes from the experimental data using all possible types of polarization experiments in the three-body final state reactions.

In this paper (1) we investigate the isospin constraints on the experimental observables of three $(0\ 1/2 \to 0'0'1'/2)$ reactions using the formalism developed in ref. [1]. So in sect. 2, we define the polarized differential cross section σ_a and the spin rotation vectors $\vec{\xi}_a$, [see eqs. (3a,b,c,d), (4a,b,c,d), (5a,b,c,d), (6a,b,c,d), (7a,b,c,d), (8a,b,c,d), (9a) and (9b)], in terms of the unpolarized differential cross-sections I_0 and all possible types of polarization parameters discussed in ref. [2]. Then, in sect. 3, using the generalized helicity amplitudes (17a) and the bilinear forms defined by eqs. (19a,b,c,d), we prove that the isospin sum rules (10) alone imply the equalities: (12), (13a,b,c,d), (14), (15) and the isospin bounds (16a,b,c), valid for any unit vector $\vec{\kappa}$ for any values of the kinematical variables in the physical domain. The constraints on the experimental data and amplitude analysis as well as the Pomeranchuk-like theorems, when the isospin bounds are exactly saturated or degenerated, are discussed in sect. 4.

In the paper [3] we shall investigate other improvements of the isospin constraints on the experimental observables for the $(0\ 1/2 \to 0'0'1'/2)$ reactions, and their experimental consequences.

2. *Measurable Quantities for* $MB \to MMB$ *Reactions*

Recently Cashmore et al. [2] have developed a formalism to describe the reactions of the type

$$a + b \to c + d + e . \tag{1}$$

This formalism is particularly suitable for partial wave analyses at low energies. With this formalism they discuss all possible types of polarization experiments in the three-body final state reactions which are necessary for the reconstruction of the scattering amplitudes. They obtain expressions for all measurable quantities in terms of the helicity amplitudes $f^{\tau\mu}$, $\tau,\mu = \pm\frac{1}{2}$.

For the reactions

$$M_1 + B_1 \to M_2 + M_3 + B_2 \tag{2}$$

$(M_1, M_2, M_3 = 0^-$ - mesons, $B_1, B_2 = \frac{1^+}{2}$ - barions), there are four possible types of experiments:

(i) unpolarized differential cross section I_0 ;
(ii) polarization "asymmetry": $\vec{A} \equiv (A_x, A_y, A_z)$;
(iii) final polarization: $\vec{P} \equiv (P_X, P_Y, P_Z)$;
(iv) "depolarization tensor": D_{xX}, D_{xY}, D_{xZ}, D_{yX}, D_{yY}, D_{yZ}, D_{zX}, D_{zY}, D_{zZ}.

The polarization of the initial barion (B_1) and polarization "asymmetry" $\vec{A}$ are referred to the fixed 0_{xyz} reference frame and the final polarization $\vec{P}$ and "depolarization tensor" are referred to the moving 0_{XYZ} frame.

In terms of the above measurable quantities, using the results of ref. [2], we can write:

$$|f^{++}|^2 + |f^{+-}|^2 = (1 + P_Z)\, I_0 \equiv \Sigma^{(+)}, \tag{3a}$$

$$2\,\mathrm{Re}[f^{++}(f^{+-})^*] = (A_x + D_{xZ})\, I_0 \equiv A_{\Sigma^{(+)}}\Sigma^{(+)}, \tag{3b}$$

$$2\,\mathrm{Im}[f^{++}(f^{+-})^*] = (A_y + D_{yZ})\, I_0 \equiv P_{\Sigma^{(+)}}\Sigma^{(+)}, \tag{3c}$$

$$|f^{++}|^2 - |f^{+-}|^2 = (A_z + D_{zZ})\, I_0 \equiv R_{\Sigma^{(+)}}\Sigma^{(+)}, \tag{3d}$$

$$|f^{--}|^2 + |f^{-+}|^2 = (1 - P_Z)\, I_0 \equiv \Sigma^{(-)}, \tag{4a}$$

$$2\,\mathrm{Re}[f^{-+}(f^{--})^*] = (A_x - D_{xZ})\, I_0 \equiv A_{\Sigma^{(-)}}\Sigma^{(-)}, \tag{4b}$$

$$2\,\mathrm{Im}[f^{-+}(f^{--})^*] = (A_y - D_{yZ})\, I_0 \equiv P_{\Sigma^{(-)}}\Sigma^{(-)}, \tag{4c}$$

$$|f^{-+}|^2 - |f^{--}|^2 = (A_z - D_{zZ})\, I_0 \equiv R_{\Sigma^{(-)}}\Sigma^{(-)}, \tag{4d}$$

$$|f^{++}|^2 + |f^{-+}|^2 = (1 + A_z)\, I_0 \equiv \Omega^{(+)}, \tag{5a}$$

$$2\,\mathrm{Re}[f^{-+}(f^{++})^*] = (P_X + D_{zX})\, I_0 \equiv A_{\Omega^{(+)}}\Omega^{(+)}, \tag{5b}$$

$$2\,\mathrm{Im}[f^{-+}(f^{++})^*] = (P_Y + D_{zY})\, I_0 \equiv P_{\Omega^{(+)}}\Omega^{(+)}, \tag{5c}$$

$$|f^{++}|^2 - |f^{-+}|^2 = (P_Z + D_{zZ})\, I_0 \equiv R_{\Omega^{(+)}}\Omega^{(+)}, \tag{5d}$$

$$|f^{--}|^2 + |f^{+-}|^2 = (1 - A_z)\, I_0 \equiv \Omega^{(-)}, \tag{6a}$$

$$2\,\mathrm{Re}[f^{--}(f^{+-})^*] = (P_X - D_{zX})\, I_0 \equiv A_{\Omega^{(-)}}\Omega^{(-)}, \tag{6b}$$

$$2\mathrm{Im}\,[f^{--}(f^{+-})^*] = (P_Y - D_{zY})\, I_0 \equiv P_{\Omega^{(-)}}\Omega^{(-)}, \tag{6c}$$

$$|f^{+-}|^2 - |f^{--}| = (P_Z - D_{zZ}) I_0 \equiv R_{\Omega^{(-)}}\Omega^{(-)}, \tag{6d}$$

$$|f^{++}|^2 + |f^{--}|^2 = (1 + D_{zZ})\, I_0 \equiv \Delta^{(+)}, \tag{7a}$$

$$2\mathrm{Re}\,[f^{++}(f^{--})^*] = (D_{xX} + D_{yY})\, I_0 \equiv A_{\Delta^{(+)}}\Delta^{(+)}, \tag{7b}$$

$$2\mathrm{Im}\,[f^{++}(f^{--})^*] = (D_{yX} - D_{xY})\, I_0 \equiv P_{\Delta^{(+)}}\Delta^{(+)}, \tag{7c}$$

$$|f^{++}|^2 - |f^{--}|^2 = (A_z + P_Z)\, I_0 \equiv R_{\Delta^{(+)}}\Delta^{(+)}, \tag{7d}$$

$$|f^{+-}|^2 + |f^{-+}|^2 = (1 - D_{zZ})\, I_0 \equiv \Delta^{(-)}, \tag{8a}$$

$$2\mathrm{Re}\,[f^{+-}(f^{-+})^*] = (D_{xX} - D_{yY})\, I_0 \equiv A_{\Delta^{(-)}}\Delta^{(-)}, \tag{8b}$$

$$2\mathrm{Im}\,[f^{+-}(f^{-+})^*] = -(D_{yX} + D_{xY})\, I_0 \equiv P_{\Delta^{(-)}}\Delta^{(-)}, \tag{8c}$$

$$|f^{+-}|^2 - |f^{-+}|^2 = (P_z - A_Z)\, I_0 \equiv R_{\Delta^{(-)}}\Delta^{(-)}. \tag{8d}$$

Therefore, we have defined the following differential polarized cross sections

$$\sigma_\alpha \equiv [\Sigma^{(\pm)}, \Omega^{(\pm)}, \Delta^{(\pm)}], \tag{9a}$$

[see eqs. (3a), (4a), (5a), (6a), (7a) and (8a)] and the following "spin-rotation" vectors

$$\vec{\xi}_\alpha \equiv [A_\alpha, P_\alpha, R_\alpha], \quad \alpha \equiv \Sigma^{(+)}, \Sigma^{(-)}, \Omega^{(+)}, \Omega^{(-)}, \Delta^{(+)}, \Delta^{(-)}, \tag{9b}$$

[see eqs. (3b,c,d), (4b,c,d), (5b,c,d), (6b,c,d), (7b,c,d,) and (8b,c,d)] , respectively. We note of course that the spin rotation vectors $\vec{\xi}_\alpha$ have unit length ($|\vec{\xi}_\alpha| = 1$). Therefore, a large number of isospin constraints on the experimental observables of $(0\ 1/2 \to 0'0'1'/2)$ reactions can be derived just as in $(0\ 1/2 \to 0'1'/2)$ scattering case /1/.

3. Isospin Constraints on Differential Cross-Sections $\sigma_{\alpha k}$ and on Spin-Rotation Vectors $\vec{\xi}_{\alpha k}$

In order to obtain all the isospin constraints on the differential cross-section $\sigma_{\alpha k}$ and on spin-rotation vectors $\vec{\xi}_{\alpha k}$, defined in sect. 2, we start with the following definitions. Let $f_k^{\tau\mu}$ $(\tau, \mu = \pm\frac{1}{2})$ be the helicity amplitudes for three $(0\ 1/2 \to 0'0'1'/2)$ reactions satisfying the isospin sum rules:

$$\sum_{k=1}^{3} c_k f_k^{\tau\mu} = 0, \qquad \text{for any } \tau, \mu = \pm\frac{1}{2} \tag{10}$$

where c are real numbers. Let $\lambda(\sigma_\alpha)$, $\lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha)$, $\lambda^{(\pm)}_{\alpha\kappa}$, $H_{\alpha ij}$, $M^{(\pm\kappa)}_{\alpha\ell\ell}$, $Z^{(\kappa)}_{\alpha\ell\ell}$ and $a_{\alpha\kappa}(ij)$ be the functions:

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz, \tag{11a}$$

$$\lambda(\sigma_\alpha) \equiv \lambda(c_1^2\sigma_{\alpha 1}, c_2^2\sigma_{\alpha 2}, c_3^2\sigma_{\alpha 3}), \tag{11b}$$

$$\lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha) \equiv \lambda(c_1^2\vec{\kappa}\cdot\vec{\xi}_{\alpha 1}\sigma_{\alpha 1}, c_2^2\vec{\kappa}\cdot\vec{\xi}_{\alpha 2}\sigma_{\alpha 2}, c_3^2\vec{\kappa}\cdot\vec{\xi}_{\alpha 3}\sigma_{\alpha 3}), \tag{11c}$$

$$\lambda^{(\pm)}_{\alpha\kappa} \equiv \lambda[c_1^2(1 \pm \vec{\kappa}\cdot\vec{\xi}_{\alpha 1})\sigma_{\alpha 1}, c_2^2(1 \pm \vec{\kappa}\cdot\vec{\xi}_{\alpha 2})\sigma_{\alpha 2}, c_3^2(1 \pm \vec{\kappa}\cdot\vec{\xi}_{\alpha 3})\sigma_{\alpha 3}], \tag{11d}$$

$$H_{\alpha ij} \equiv \frac{1}{2}[1 - \vec{\xi}_{\alpha i}\cdot\vec{\xi}_{\alpha j}]\sigma_{\alpha i}\sigma_{\alpha j}, \tag{11e}$$

$$M^{(\pm\kappa)}_{\alpha\ell\ell} \equiv (1 \pm \vec{\kappa}\cdot\vec{\xi}_{\alpha\ell})\sigma_{\alpha\ell}, \quad Z^{(\kappa)}_{\alpha\ell\ell} \equiv (\vec{\kappa}\cdot\vec{\xi}_{\alpha\ell})\sigma_{\alpha\ell}, \quad \ell = 1, 2, 3, \tag{11f}$$

$$a_{\alpha\kappa} \equiv \frac{2[\sigma_{\alpha i}\sigma_{\alpha j} - H_{\alpha ij}]^{1/2}}{\sigma_{\alpha i}\sigma_{\alpha j}} [c_k^2 Z_{\alpha kk}^{(\kappa)} - c_i^2 Z_{\alpha ii}^{(\kappa)} - c_j^2 Z_{\alpha jj}^{(\kappa)}] -$$

$$- \frac{\vec{\kappa}\cdot(\vec{\xi}_{\alpha i} + \vec{\xi}_{\alpha j})}{[\sigma_{\alpha i}\ \sigma_{\alpha j} - H_{\alpha ij}]^{1/2}} [c_k^2\sigma_{\alpha k} - c_i^2\sigma_{\alpha i} - c_j^2\sigma_{\alpha j}],$$

$$i \neq j \neq k = 1, 2, 3. \tag{11g}$$

Then, *the isospin sum rules (10) alone imply the following sets of equalities:*

$$H_\alpha \equiv c_1^2 c_2^2 H_{\alpha 12} = c_2^2 c_3^2 H_{\alpha 13} = c_3^2 c_1^2 H_{\alpha 13}, \tag{12}$$

$$\frac{1}{2}|\lambda_{\alpha\kappa}^{(+)} - \lambda_{\alpha\kappa}^{(-)}| = 2[-4H_\alpha - \lambda(\sigma_\alpha)]^{1/2}\ [4H_\alpha - \lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha)]^{1/2}, \tag{13a}$$

$$|2H_\alpha + \frac{1}{4}\lambda(\sigma_\alpha) - \frac{1}{4}\lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha)| = [-\frac{1}{4}\lambda_{\alpha\kappa}^{(+)}]^{1/2}\ [-\frac{1}{4}\lambda_{\alpha\kappa}^{(-)}]^{1/2}, \tag{13b}$$

$$|2H_\alpha + \frac{1}{4}\lambda(\sigma_\alpha) - \frac{1}{4}\lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha) + \frac{1}{4}\lambda_{\alpha\kappa}^{(\pm)}| =$$

$$= [-\frac{1}{4}\lambda_{\alpha\kappa}^{(\pm)}]^{1/2}\ [-4H_\alpha - \lambda(\sigma_\alpha)]^{1/2}, \tag{13c}$$

$$|2H_\alpha + \frac{1}{4}\lambda(\sigma_\alpha) - \frac{1}{4}\lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha) - \frac{1}{4}\lambda_{\alpha\kappa}^{(\pm)}| =$$

$$= [-\frac{1}{4}\lambda_{\alpha\kappa}^{(\pm)}]^{1/2}\ [4H_\alpha - \lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha)]^{1/2}, \tag{13d}$$

$$4H_\alpha[c_k^2 M_{\alpha kk}^{(+\kappa)} - c_i^2 M_{\alpha ii}^{(+\kappa)} - c_j^2 M_{\alpha jj}^{(+\kappa)}][c_k^2 M_{\alpha kk}^{(-\kappa)} - c_i^2 M_{\alpha ii}^{(-\kappa)} - c_j^2 M_{\alpha jj}^{(-\kappa)}] -$$

$$- 4c_i^4 c_j^4 \sigma_{\alpha i}^2 \sigma_{\alpha j}^2 [\vec{\kappa}\cdot(\vec{\xi}_{\alpha i}\times\vec{\xi}_{\alpha j})]^2 =$$

$$= \lambda(\sigma_\alpha)[c_k^2 Z_{\alpha kk}^{(\kappa)} - c_i^2 Z_{\alpha ii}^{(\kappa)} - c_j^2 Z_{\alpha jj}^{(\kappa)}]^2 + \lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha)[c_k^2\sigma_{\alpha k} - c_i^2\sigma_{\alpha i} - c_j^2\sigma_{\alpha j}] +$$

$$+ \frac{1}{2}[-\lambda_{\alpha\kappa}^{(+)} + \lambda_{\alpha\kappa}^{(-)}][c_k^2 Z_{\alpha kk}^{(\kappa)} - c_i^2 Z_{\alpha ii}^{(\kappa)} - c_j^2 Z_{\alpha jj}^{(\kappa)}] \times$$

$$\times [c_k^2\sigma_{\alpha k} - c_i^2\sigma_{\alpha i} - c_j^2\sigma_{\alpha j}], \qquad i \neq j \neq k = 1, 2, 3, \tag{14}$$

$$\frac{a_{\alpha\kappa'}(ij)}{a_{\alpha\kappa''}(ij)} = \frac{\vec{\kappa}'\cdot(\vec{\xi}_{\alpha i}\times\vec{\xi}_{\alpha j})}{\vec{\kappa}''\cdot(\vec{\xi}_{\alpha i}\times\vec{\xi}_{\alpha j})} \quad \text{for any } \vec{\kappa}' \text{ and } \vec{\kappa}'' \tag{15}$$

and $i \neq j = 1, 2, 3,$ *and the following inequalities:*

$$0 \leq -\frac{1}{4}\lambda_{\alpha\kappa}^{(\pm)} \leq \min_{(ij)} \{c_i^2 c_j^2 M_{\alpha ii}^{(\pm\kappa)} M_{\alpha jj}^{(\pm\kappa)}\}, \tag{16a}$$

$$\max_{\{ij\}}\{-c_i^2 c_j^2 Z_{\alpha ii}^{(\kappa)} Z_{\alpha jj}^{(\kappa)}\} \leq \frac{1}{4}\lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha) \leq H_\alpha, \tag{16b}$$

$$H_\alpha \leq -\frac{1}{4}\lambda(\sigma_\alpha) \leq \min_{\{ij\}}\{c_i^2 c_j^2 \sigma_{\alpha i}\sigma_{\alpha j}\}. \tag{16c}$$

These results can be proved as follows. We define the functions

$$F_{\alpha k}^{(+\kappa)} \equiv \frac{\sqrt{2}}{(1 + |w|^2)^{1/2}}[f_k^{\tau\mu} + w f_k^{\tau'\mu'}],$$

$$F_{\alpha k}^{(-\kappa)} \equiv \frac{\sqrt{2}}{(1 + |w|^2)^{1/2}}[-w^* f_k^{\tau\mu} + f_k^{\tau'\mu'}], \tag{17a}$$

where w is an arbitrary complex number, The helicity amplitudes $f_k^{\tau\mu}$ and $f_k^{\tau'\mu'}$ are chosen such that

$$|f_k^{\tau\mu}|^2 + |f_k^{\tau'\mu'}|^2 = \sigma_{\alpha k}, \tag{17b}$$

$$\{2\,\mathrm{Re}[f_k^{\tau\mu}(f_k^{\tau'\mu'})^*],\ 2\,\mathrm{Im}[f_k^{\tau\mu}(f_k^{\tau'\mu'})^*],\ |f_k^{\tau\mu}|^2 - |f_k^{\tau'\mu'}|^2\} = \vec{\xi}_{\alpha k}\sigma_{\alpha k}. \tag{17c}$$

Therefore, denoting by

$$\vec{\kappa} \equiv \left\{ \frac{2\,\mathrm{Re}\,w}{1+|w|^2},\ \frac{2\,\mathrm{Im}\,w}{1+|w|^2},\ \frac{1-|w|^2}{1+|w|^2} \right\}, \tag{17d}$$

from (17a) we obtain

$$|F_{\alpha k}^{(\pm\kappa)}|^2 = (1 \pm \kappa\cdot\vec{\xi}_{\alpha k})\,\sigma_{\alpha k}. \tag{18}$$

Next, let us define the following bilinear forms

$$M_{\alpha ij}^{(\pm\kappa)} \equiv [F_{\alpha i}^{(\pm\kappa)}]^* F_{\alpha j}^{(\pm\kappa)},\quad |M_{\alpha ij}^{(\pm\kappa)}|^2 = M_{\alpha ii}^{(\pm\kappa)} M_{\alpha jj}^{(\pm\kappa)}, \tag{19a}$$

$$Z_{\alpha ij}^{(0)} \equiv \frac{1}{2}[M_{\alpha ij}^{(+\kappa)} + M_{\alpha ij}^{(-\kappa)}],\quad |Z_{\alpha ij}^{(0)}|^2 = \frac{1}{2}[1+\vec{\xi}_{\alpha i}\cdot\vec{\xi}_{\alpha j}]\sigma_{\alpha i}\sigma_{\alpha j}, \tag{19b}$$

$$Z_{\alpha ij}^{(\kappa)} \equiv \frac{1}{2}[M_{\alpha ij}^{(+\kappa)} - M_{\alpha ij}^{(-\kappa)}],\quad |Z_{\alpha ij}^{(\kappa)}|^2 = H_{\alpha ij} + (\vec{\kappa}\cdot\vec{\xi}_{\alpha i})(\vec{\kappa}\cdot\vec{\xi}_{\alpha j})\sigma_{\alpha i}\sigma_{\alpha j}, \tag{19c}$$

$$Y_{\alpha ij}^{(0)} \equiv \frac{1}{2}[F_{\alpha i}^{(+\kappa)} F_{\alpha j}^{(-\kappa)} - F_{\alpha i}^{(-\kappa)} F_{\alpha j}^{(+\kappa)}],\quad |Y_{\alpha ij}^{(0)}|^2 = H_{\alpha ij}. \tag{19d}$$

Now, since the sum rules (10) are equivalent to

$$c_1 c_2 Y_{\alpha 12}^{(0)} = c_2 c_3 Y_{\alpha 23}^{(0)} = c_3 c_1 Y_{\alpha 31}^{(0)} \tag{20}$$

then we obtain directly the equalities (12) [see eq. (19d)]. Next, the equalities (13a,b,c,d) can be proved, observing that the sum rules (19) imply the relations:

$$\mathrm{Re}\,N_{\alpha ij} = (2c_i c_j)^{-1}[c_k^2 N_{\alpha kk} - c_i^2 N_{\alpha ii} - c_j^2 N_{\alpha jj}], \tag{21a}$$

for any $N_{\alpha ij} \equiv Z_{\alpha ij}^{(0)}$, $Z_{\alpha ij}^{(\kappa)}$ and $M_{\alpha ij}^{(\pm\kappa)}$ ($Z_{\alpha\ell\ell}^{(0)} = \sigma_{\alpha\ell}$, $Z_{\alpha\ell\ell}^{(\kappa)} = (\vec{\kappa}\cdot\vec{\xi}_{\alpha\ell})\,\sigma_{\alpha\ell}$),

and

$$c_i^2 c_j^2[\mathrm{Im}\,M_{\alpha ij}^{(\pm\kappa)}]^2 = -\frac{1}{4}\lambda_{\alpha\kappa}^{(\pm)} = \{[-H_\alpha - \frac{1}{4}\lambda(\sigma_\alpha)]^{1/2} \pm \eta_{\alpha\kappa}[H_\alpha - \frac{1}{4}\lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha)]^{1/2}\}^2, \tag{21b}$$

$$4c_i^2 c_j^2[\mathrm{Im}\,Z_{\alpha ij}^{(0)}]^2 = -4H_\alpha - \lambda(\sigma_\alpha) = \frac{1}{4}\{[-\lambda_{\alpha\kappa}^{(+)}]^{1/2} + \epsilon_{\alpha\kappa}[-\lambda_{\alpha\kappa}^{(-)}]^{1/2}\}^2, \tag{21c}$$

$$4c_i^2 c_j^2[\mathrm{Im}\,Z_{\alpha ij}^{(\kappa)}]^2 = 4H_\alpha - \lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha) = \frac{1}{4}\{[-\lambda_{\alpha\kappa}^{(+)}]^{1/2} - \epsilon_{\alpha\kappa}[-\lambda_{\alpha\kappa}^{(-)}]^{1/2}\}^2, \tag{21d}$$

where $\eta_{\alpha\kappa}$ and $\epsilon_{\alpha\kappa}$ are defined by

$$\eta_{\alpha\kappa} \equiv \mathrm{sign}\{\mathrm{Im}\,Z_{\alpha ij}^{(0)}\,\mathrm{Im}\,Z_{\alpha ij}^{(\kappa)}\} = \mathrm{sign}\{-\lambda_{\alpha\kappa}^{(+)} + \lambda_{\alpha\kappa}^{(-)}\}, \tag{21e}$$

$$\epsilon_{\alpha\kappa} \equiv \mathrm{sign}\{\mathrm{Im}\,M_{\alpha ij}^{(+\kappa)}\,\mathrm{Im}\,M_{\alpha ij}^{(-\kappa)}\} = \mathrm{sign}\{-8H_\alpha - \lambda(\sigma_\alpha) + \lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha)\}. \tag{21f}$$

Now, since the equalities (21a,b,c,d) are equivalent to

$$c_i^2 c_j^2\{[\mathrm{Re}\,N_{\alpha ij}]^2 - N_{\alpha ii}N_{\alpha jj}\} = \frac{1}{4}\lambda[c_1^2 N_{\alpha 11}, c_2^2 N_{\alpha 22}, c_3^2 N_{\alpha 33}], \tag{22a}$$

$$-\lambda^{(\pm)}_{\alpha\kappa} = -\lambda(\sigma_\alpha) - \lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha) \pm 2\eta_{\alpha\kappa}[-4H_\alpha - \lambda(\sigma_\alpha)]^{1/2}\,[4H_\alpha - \lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha)]^{1/2}, \tag{22b}$$

$$8H_\alpha = -\lambda(\sigma_\alpha) + \lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha) - \epsilon_{\alpha\kappa}\,[-\lambda^{(+)}_{\alpha\kappa}]^{1/2}\,[-\lambda^{(-)}_{\alpha\kappa}]^{1/2}\ . \tag{22c}$$

respectively, then we obtain the equalities (13a,b,c,d) while the bounds (16a,b,c) are derived directly from (21b,c,d) and (22a).

Next, from the definitions (19a,b,c), it is easy to obtain

$$Z^{(\kappa)}_{\alpha ij}[Z^{(0)}_{\alpha ij}]^* = \frac{\sigma_{\alpha i}\sigma_{\alpha j}}{2}\,[\vec{\kappa}\cdot(\vec{\xi}_{\alpha i}+\vec{\xi}_{\alpha j}) - i\vec{\kappa}\cdot(\vec{\xi}_{\alpha i}\times\vec{\xi}_{\alpha j})]\ . \tag{23a}$$

On the other hand, if we introduce the phases $\phi^{(0)}_{\alpha ij}$ of the bilinear form $Z^{(0)}_{\alpha ij} = |Z^{(0)}_{\alpha ij}|\exp\{i\phi^{(0)}_{\alpha ij}\}$, then we can express $Z^{(\kappa)}_{\alpha ij}$ in the form:

$$\mathrm{Re}\,Z^{(\kappa)}_{\alpha ij} = \frac{\sigma_{\alpha i}\sigma_{\alpha j}}{2|Z^{(0)}_{\alpha ij}|}\{\vec{\kappa}\cdot(\vec{\xi}_{\alpha i}+\vec{\xi}_{\alpha j})\cos\phi^{(0)}_{\alpha ij} + \vec{\kappa}\cdot(\vec{\xi}_{\alpha i}\times\vec{\xi}_{\alpha j})\sin\phi^{(0)}_{\alpha ij}\}, \tag{23b}$$

$$\mathrm{Im}\,Z^{(\kappa)}_{\alpha ij} = \frac{\sigma_{\alpha i}\sigma_{\alpha j}}{2|Z^{(0)}_{\alpha ij}|}\{\vec{\kappa}\cdot(\vec{\xi}_{\alpha i}+\vec{\xi}_{\alpha j})\sin\phi^{(0)}_{\alpha ij} - \vec{\kappa}\cdot(\vec{\xi}_{\alpha i}\times\vec{\xi}_{\alpha j})\cos\phi^{(0)}_{\alpha ij}\}, \tag{23c}$$

$$\cos\phi^{(0)}_{\alpha ij} = \frac{c_k^2\sigma_{\alpha k} - c_i^2\sigma_{\alpha i} - c_j^2\sigma_{\alpha j}}{2c_i c_j[\sigma_{\alpha i}\sigma_{\alpha j} - H_{\alpha ij}]^{1/2}}, \tag{23d}$$

Then, the equalities (14) can be derived using the following relation:

$$\{\mathrm{Im}[Z^{(\kappa)}_{\alpha ij}(Z^{(0)}_{\alpha ij})^*]\}^2 = \frac{1}{4}[\vec{\kappa}\cdot(\vec{\xi}_{\alpha i}\times\vec{\xi}_{\alpha j})]^2\sigma^2_{\alpha i}\sigma^2_{\alpha j} \tag{24}$$

and eqs. (2a,b,c,d,e), while the equalities (15) are obtained from (23b,d) and (21a).

It is interesting to note that the result (23b) can be written in the form:

$$\frac{c_k^2\vec{\xi}_{\alpha k}\sigma_{\alpha k} - c_i^2\vec{\xi}_{\alpha i}\sigma_{\alpha i} - c_j^2\vec{\xi}_{\alpha j}\sigma_{\alpha j}}{2c_i c_j[\sigma_{\alpha i}\sigma_{\alpha j}]^{1/2}} = \frac{\vec{\xi}_{\alpha i}+\vec{\xi}_{\alpha j}}{|\vec{\xi}_{\alpha i}+\vec{\xi}_{\alpha j}|}\cos\phi^{(0)}_{\alpha ij} + \left[\frac{H_\alpha}{c_i^2c_j^2\sigma_{\alpha i}\sigma_{\alpha j}}\right]^{1/2}\frac{\vec{\xi}_{\alpha i}\times\vec{\xi}_{\alpha j}}{|\vec{\xi}_{\alpha i}\times\vec{\xi}_{\alpha j}|}\sin\phi^{(0)}_{\alpha ij}, \tag{25}$$

where H, is defined by eq. (12) and $\cos\phi^{(0)}_{\alpha ij}$ by eq. (23d).

4. *Experimental Consequences of Isospin Constraints*

The results obtained in this section may be conveniently summarized by the following interesting consequences:

(i) *Let* $\phi^{(0)}_{\alpha ij}$, $\phi^{(\kappa)}_{\alpha ij}$ *and* $\delta^{(\pm\kappa)}_{\alpha ij}$ *be the phases of the bilinear forms* $Z^{(0)}_{\alpha ij}$, $Z^{(\kappa)}_{\alpha ij}$ *and* $M^{(\pm\kappa)}_{\alpha ij}$ *respectively. Then, the sum rule (10) alone implies that all the phases* $\phi^{(0)}_{\alpha ij}$ *and hence* $\phi^{(\kappa)}_{\alpha ij}$, $\delta^{(\pm\kappa)}_{\alpha ij}$ *can unambiguously be determined from the experimental data.* Indeed, if $\sigma_{\alpha i}$, $\sigma_{\alpha j}$, $\sigma_{\alpha k}$ and $\vec{\xi}_{\alpha i}, \vec{\xi}_{\alpha j}$ are known, eq. (23d) alows to determine $\cos\phi^{(0)}_{\alpha ij}$ while the sign of $\sin\phi^{(0)}_{\alpha ij}$ can be obtained from [see eq. (25)] :

$$\mathrm{sign}\{\sin\phi^{(0)}_{\alpha ij}\} = \mathrm{sign}\{\vec{\xi}_{\alpha k}\cdot(\vec{\xi}_{\alpha i}\times\vec{\xi}_{\alpha j})\}. \tag{26}$$

Then, eqs. (23b,c) and $M^{(\pm\kappa)}_{\alpha ij} = Z^{(0)}_{\alpha ij} \pm Z^{(\kappa)}_{\alpha ij}$ yield the phases $\phi^{(\kappa)}_{\alpha ij}$ and $\delta^{(\pm\kappa)}_{\alpha ij}$.

(ii) *The lower bounds (16a,c) and the upper bounds (16b) are exactly saturated on the* $[\delta^{(\pm\kappa)}_{\alpha ij}, \phi^{(0)}_{\alpha ij}, \phi^{(\kappa)}_{\alpha ij}] = n\pi$ *phase contours* $(n = 0, 1, \dots)$ *respectively* [see eqs. (21b,c,d)] , *while the upper bounds (16a,c) and the lower bounds (16b) are exactly saturated on* $(n+\frac{1}{2})\pi$ *phase contours, respectively* [see eq. (22a)] .

The above phase contours lie on the zeros trajectories of $\mathrm{Im}N_{\alpha ij}$ and $\mathrm{Re}N_{\alpha ij}$, $N_{\alpha ij}\equiv M^{(\pm\kappa)}_{\alpha ij}, Z^{(0)}_{\alpha ij}, Z^{(\kappa)}_{\alpha ij}$ respectively. The zeros trajectories of $\mathrm{Im}N_{\alpha ij}$ are independent of channel indices (i,j) [see eqs. (21b,c,d)]. The bounds (16a,b,c) are degenerated if and only if $|N_{\alpha ij}|=0$ for $N_{\alpha ij}=M^{(\pm\kappa)}_{\alpha ij}$, $Z^{(\kappa)}_{\alpha ij}$ and $Z^{(0)}_{\alpha ij}$ respectively. We note that if $|N_{\alpha ij}|=0$, then the phases of the bilinear forms $N_{\alpha ij}$ are not determined.

(iii) *The exact saturation of the upper bounds (16b) or of the lower bounds (16c) imply the linear relations:*

$$c_k^2\vec{\kappa}\cdot\vec{\xi}_{\alpha k}\sigma_{\alpha k}[c_k^2\sigma_{\alpha k}-c_i^2\sigma_{\alpha i}-c_j^2\sigma_{\alpha j}]+$$

$$+c_i^2\vec{\kappa}\cdot\vec{\xi}_{\alpha i}\sigma_{\alpha i}[c_i^2\sigma_{\alpha i}-c_j^2\sigma_{\alpha j}-c_k^2\sigma_{\alpha k}]+$$

$$+c_j^2\vec{\kappa}\cdot\vec{\xi}_{\alpha j}\sigma_{\alpha j}[c_j^2\sigma_{\alpha j}-c_k^2\sigma_{\alpha k}-c_i^2\sigma_{\alpha i}]=0\,, \qquad (27)$$

between the projection of the spin-rotation vectors $\vec{\xi}_{\alpha\ell}$, $\ell=i\neq j\neq k$.

We remark that eq. (27) is valid for all $\vec{\kappa}$ if the lower bound (16c) is exactly saturated.

(iv) *If one of the conditions:*

1°) *the bounds (16b) are degenerated;*

2°) *the lower bounds (16b,c) are simultaneously saturated;*

3°) *the upper bounds (16b,c) are simultaneously saturated;*

4°) *the bounds (16c) are degenerated,*

holds for a given $\alpha,(i,j)$ *and* $\vec{\kappa}$, *then we obtain the mirror symmetry:*

$$(\vec{\kappa}\cdot\vec{\xi}_{\alpha i})=-(\vec{\kappa}\cdot\vec{\xi}_{\alpha j})\,. \qquad (28)$$

These consequences are obtained using eq. (27) and

$$c_k^2N_{\alpha kk}=c_i^2N_{\alpha ii}+c_j^2N_{\alpha jj} \qquad \text{for } N_{\alpha\ell\ell}\equiv Z^{(0)}_{\alpha\ell\ell}, Z^{(\kappa)}_{\alpha\ell\ell}\,,$$

$(\ell=i,j,k)$ respectively.

(v) *The exact saturation of the bound* $-\lambda(\sigma_\alpha)\geq 0$ *implies the equalities* $\vec{\xi}_{\alpha 1}=\vec{\xi}_{\alpha 2}=\vec{\xi}_{\alpha 3}$. This result is a direct consequence of the lower bound (16c).

(vi) *The lower bound (16c) is equivalent to*

$$[c_i^2\sigma_{\alpha i}-c_j^2\sigma_{\alpha j}]^2+4H_\alpha\leq 2c_k\sigma_{\alpha k}[c_i^2\sigma_{\alpha i}+c_j^2\sigma_{\alpha j}-\frac{1}{2}c_k^2\sigma_{\alpha k}] \qquad (29)$$

This bound requires that if

$$\sigma_{\alpha k}[c_i^2\sigma_{\alpha i}+c_j^2\sigma_{\alpha j}]\underset{s\to+\infty}{\to}0\,, \qquad (30a)$$

when the other kinematical variables are all fixed, then [see eqs. (11e) and (12)] ;

$$c_i^2\sigma_{\alpha i}-c_j^2\sigma_{\alpha j}\underset{s\to+\infty}{\to}0,\quad \vec{\xi}_{\alpha i}-\vec{\xi}_{\alpha j}\underset{s\to+\infty}{\to}0\,, \qquad (30b)$$

and conversely, the cross section $\sigma_{\alpha k}$ *cannot vanish for* $s\to+\infty$ *if one of the above relations (30b) does not hold for* $s\to+\infty$ [$\sqrt{s}$ is the c.m. energy] .

The Pomeranchuk-type theorems (30b) can be generalized in the following way. We observe that according to eqs. (22b,c) the bounds $-\lambda^{(\pm)}_{\alpha\kappa}\geq 0$ and $\lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha)\leq$ $\leq 4H_\alpha\leq-\lambda(\sigma_\alpha)$ are equivalent to

$$2[-4H_\alpha-\lambda(\sigma_\alpha)]^{1/2}[4H_\alpha-\lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha)]^{1/2}\leq-\lambda(\sigma_\alpha)-\lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha), \qquad (31a)$$

and

$$[-\lambda^{(+)}_{\alpha\kappa}]^{1/2}[-\lambda^{(-)}_{\alpha\kappa}]^{1/2}\leq-\lambda(\sigma_\alpha)-\lambda(\vec{\kappa}\cdot\vec{\xi}_\alpha\sigma_\alpha)\,, \qquad (31b)$$

respectively.

Therefore, let $X_{\alpha\kappa}$ and $B_{\alpha\kappa}$ be defined as

$$X_{\alpha\kappa} = \frac{2c_i^2\vec{\kappa}\cdot\vec{\xi}_{\alpha i}\sigma_{\alpha i} + 2c_j^2\vec{\kappa}\cdot\vec{\xi}_{\alpha j}\sigma_{\alpha j} - c_k^2\vec{\kappa}\cdot\vec{\xi}_{\alpha k}\sigma_{\alpha k}}{2c_i^2\sigma_{\alpha i} + 2c_j^2\sigma_{\alpha j} - c_k^2\sigma_{\alpha k}}, \tag{32a}$$

$$B_{\alpha\kappa} = \max\{[-\lambda_{\alpha\kappa}^{(+)}]^{1/2}[-\lambda_{\alpha\kappa}^{(-)}]^{1/2}, 2[-4H_\alpha - \lambda(\sigma_\alpha)]^{1/2}[4H_\alpha - \lambda(\kappa\cdot\xi_\alpha\sigma_\alpha)]^{1/2}\} \tag{32b}$$

then, the bounds (31a,b) can be written in the form

$$[c_i^2\sigma_{\alpha i} - c_j^2\sigma_{\alpha j}]^2 + [c_i^2\vec{\kappa}\cdot\vec{\xi}_{\alpha i}\sigma_{\alpha i} - c_j^2\vec{\kappa}\cdot\vec{\xi}_{\alpha j}\sigma_{\alpha j}]^2 \leq$$

$$\leq 2c_k^2\sigma_{\alpha k}[c_i^2\sigma_{\alpha i} + c_j^2\sigma_{\alpha j} - \frac{1}{2}c_k^2\sigma_{\alpha k}][1 + \vec{\kappa}\cdot\vec{\xi}_{\alpha k}X_{\alpha\kappa}] - B_{\alpha\kappa}. \tag{32c}$$

(vii) *The bound (32c) implies that, if the condition (30a) holds, then*

$$c_i^2\sigma_{\alpha i} - c_j^2\sigma_{\alpha j} \underset{s\to+\infty}{\to} 0, \ \vec{\xi}_{\alpha i} - \vec{\xi}_{\alpha j} \underset{s\to+\infty}{\to} 0, \ [\delta_{\alpha ij}^{(\pm\kappa)}, \phi_{\alpha ij}^{(\kappa)}, \phi_{\alpha ij}^{(0)} \underset{s\to+\infty}{\to} n\pi, \tag{33}$$

$n = 0, 1, \ldots,$ *when the other kinematical variables are taken to be constant, or conversely, the* $\sigma_{\alpha k}$ *cannot vanish for* $s \to +\infty$ *if one of the Pomeranchuk-like theorems (33) is violated.* The proof follows from eqs. (21a,b,c), (30a) and (32c).

(viii) *The bound (32c) also requires that, if*

$$2c_k^2\sigma_{\alpha k}[c_i^2\sigma_{\alpha i} + c_j^2\sigma_{\alpha j} - \frac{1}{2}c_k^2\sigma_{\alpha k}][1 + (\vec{\kappa}\cdot\vec{\xi}_{\alpha k})X_{\alpha k}] - B_{\alpha\kappa} \underset{s\to+\infty}{\to} 0 \tag{34a}$$

for a given $\vec{\kappa}$ *then*

$$c_i^2\sigma_{\alpha i} - c_j^2\sigma_{\alpha j} \underset{s\to+\infty}{\to} 0, \quad (\vec{\kappa}\cdot\vec{\xi}_{\alpha i}) - (\vec{\kappa}\cdot\vec{\xi}_{\alpha j}) \underset{s\to+\infty}{\to} 0. \tag{34b}$$

Next, starting with the bounds $4H_\alpha \leq -\lambda(\sigma_\alpha)$ and $4H_\beta \leq -\lambda(\sigma_\beta)$, , $\alpha \neq \beta$, and using the inequality

$$2a^{1/2}b^{1/2} \leq a + b, \quad a \geq 0, \quad b \geq 0 \tag{35}$$

for $a \equiv -4H_\alpha - \lambda(\sigma_\alpha)$, , $-\lambda(\sigma_\alpha)$ and $b \equiv -4H_\beta - \lambda(\sigma_\beta)$, $-\lambda(\sigma_\beta)$ then we obtain the bounds:

$$2[-\lambda(\sigma_\alpha)]^{1/2}[-\lambda(\sigma_\beta)]^{1/2} \leq -\lambda(\sigma_\alpha) - \lambda(\sigma_\beta), \tag{36a}$$

$$2[-4H_\alpha - \lambda(\sigma_\alpha)]^{1/2}[-4H_\beta - \lambda(\sigma_\beta)]^{1/2} + 4[H_\alpha + H_\beta] \leq$$

$$\leq -\lambda(\sigma_\alpha) - \lambda(\sigma_\beta). \tag{36b}$$

Let $(\alpha, \bar{\alpha})$ be chosen as $[\Sigma^{(+)}, \Sigma^{(-)}]$, $[\Omega^{(+)}, \Omega^{(-)}]$, $[\Delta^{(+)}, \Delta^{(-)}]$. Then, since $\sigma_{\alpha\ell} = (1 + X_\ell)I_{0\ell}$, $\sigma_{\bar{\alpha}} = (1 - X_\ell)I_{0\ell}$, $X \equiv P_Z, A_Z, D_{zZ}$, , respectively, the bounds (36a,b) can be expressed in the following equivalent forms:

$$[c_i^2 I_{0i} - c_j^2 I_{0j}]^2 + [c_i^2 X_i I_{0i} - c_j^2 X_j I_{0j}]^2 \leq$$

$$\leq 2c_k^2 I_{0k}(1 + X_k \cdot X_{ijk})[c_i^2 I_{0i} + c_j^2 I_{0j} - \frac{1}{2}c_k^2 I_{0k}] -$$

$$-[-\lambda(\sigma_\alpha)]^{1/2}[-\lambda(\sigma_{\bar{\alpha}})]^{1/2}, \tag{37a}$$

$$[c_i^2 I_{0i} - c_j^2 I_{0j}]^2 + [c_i^2 X_i I_{0i} - c_j^2 X_j I_{0j}]^2 + 2[H_\alpha + H_{\bar{\alpha}}] \leq$$

$$\leq 2c_k^2 I_{0k}(1 + X_k \cdot X_{ijk})[c_i^2 I_{0i} + c_j^2 I_{0j} - \frac{1}{2}c_k^2 I_{0k}] -$$

$$-[-4H_\alpha - \lambda(\sigma_\alpha)]^{1/2}[-4H_{\bar{\alpha}} - \lambda(\sigma_{\bar{\alpha}})]^{1/2}, \tag{37b}$$

where

$$[2c_i^2 I_{0i} + 2c_j^2 I_{0j} - c_k^2 I_{0k}]X_{ijk} = 2c_i^2 X_i I_{0i} + 2c_j^2 I_{0j} X_j - c_k^2 X_k I_{0k}. \tag{37c}$$

Now, it is easy to see that the bounds (37a,b) imply the following interesting results.

(ix) *If*

$$I_{0k}[c_i^2 I_{0i} + c_j^2 I_{0j}] \underset{s\to+\infty}{\to} 0, \tag{38a}$$

(all the other kinematical variables are fixed) *then the bounds (37a,b) imply:*

$$c_i^2 I_{0i} - c_j^2 I_{0j} \underset{s\to+\infty}{\to} 0, \tag{38b}$$

$$X_i - X_j \underset{s\to+\infty}{\to} 0, \tag{38c}$$

$$\vec{\xi}_{\alpha i} - \vec{\xi}_{\alpha i} \underset{s\to+\infty}{\to} 0, \quad \vec{\xi}_{\bar{\alpha} i} - \vec{\xi}_{\bar{\alpha} j} \underset{s\to+\infty}{\to} 0, \tag{38d}$$

$$[\delta^{(\pm\kappa)}_{\alpha ij}, \phi^{(\kappa)}_{\alpha ij}, \phi^{(0)}_{\alpha ij}] \underset{s\to+\infty}{\to} n\pi, \; [\delta^{(\pm\kappa)}_{\bar{\alpha} ij}, \phi^{(\kappa)}_{\bar{\alpha} ij}, \phi^{(0)}_{\bar{\alpha} ij}] \underset{s\to+\infty}{\to} n\pi,$$

$$n = 0, 1, \dots, \tag{38e}$$

for all $(\alpha, \bar{\alpha})$, *and conversely, the unpolarized cross section* I_{0k} *cannot vanish for* $s \to +\infty$ *if one of the Pomeranchuk-like theorems (38b,c,d,e) is violated for* $s \to +\infty$.

We note that, the results (38d) are equivalent to

$$\vec{A}_i - \vec{A}_j \underset{s\to+\infty}{\to} 0, \; \vec{P}_i - \vec{P}_j \underset{s\to+\infty}{\to} 0, \; \vec{D}_i - \vec{D}_j \underset{s\to+\infty}{\to} 0, \tag{38f}$$

where $\vec{A}$, $\vec{P}$ are the polarization "asymmetry" and final polarization vectors respectively and $\vec{D}$ is the "depolarization tensor" [see sect. 2] .

(x) *If*

$$2c_k^2 I_{0k}[c_i^2 I_{0i} + c_j^2 I_{0j} - \frac{1}{2} c_k^2 I_{0k}](1 + X_k X_{ijk}) -$$

$$-[-4H_\alpha - \lambda(\sigma_\alpha)]^{1/2} [-4H_{\bar{\alpha}} - \lambda(\sigma_{\bar{\alpha}})]^{1/2} \underset{s\to+\infty}{\to} 0, \tag{39}$$

then the bound (37c) requires the Pomeranchuk-like theorems (38b,c,d) only for a given $(\alpha, \bar{\alpha})$.

We note that the results (38d) can also be obtained if

$$-\lambda(I_0) - \lambda(XI_0) \underset{s\to+\infty}{\to} 0,$$

where

$$\lambda(I_0) \equiv \lambda(c_1^2 I_{01}, c_2^2 I_{02}, c_3^2 I_{03})$$

and

$$\lambda(XI_0) \equiv \lambda(c_1^2 X_1 I_{01}, c_2^2 X_2 I_{02}, c_3^2 X_3 I_{03})$$

[see the definition (11a)] .

(xi) *Let* $\vec{A}_\ell$ *and* $\vec{P}_\ell$ *be the "asymmetry" and final polarization, respectively* $(\ell = 1, 2, 3)$, *and let* $\vec{D}_{Z\ell}$, $\vec{D}_{Z\ell}$, $\vec{\Delta}_\ell$ *and* $\vec{\Lambda}_\ell$ *be defined as*

$$\vec{D}_{Z\ell} \equiv (D_{xZ\ell}, D_{yZ\ell}, D_{zZ\ell}), \; \vec{D}_{Z\ell} \equiv (D_{zX\ell}, D_{zY\ell}, D_{zZ\ell}), \tag{40a}$$

$$\vec{\Delta}_\ell \equiv (D_{xX\ell}, -D_{xY\ell}, P_{Z\ell}), \; \vec{\Lambda}_\ell \equiv (D_{yY\ell}, D_{yX}, A_z). \tag{40b}$$

Then, the equalities (12) for

$$[H_{\Sigma(+)}, H_{\Sigma(-)}], \; [H_{\Omega(+)}, H_{\Omega(-)}]$$

and $[H_{\Delta(+)}, H_{\Delta(-)}]$ *imply the following equalities:*

$$F_\Sigma \equiv c_1^2 c_2^2 [1 + P_{Z1} P_{Z2} - \vec{A}_1 \cdot \vec{A}_2 - \vec{D}_{Z1} \cdot \vec{D}_{Z2}] I_{01} I_{02} =$$

$$= c_2^2 c_3^2 [1 + P_{Z2} P_{Z3} - \vec{A}_2 \cdot \vec{A}_3 - \vec{D}_{Z2} \cdot \vec{D}_{Z3}] I_{02} I_{03} =$$

$$= c_3^2 c_1^2 [1 + P_{Z3} P_{Z1} - \vec{A}_3 \cdot \vec{A}_1 - \vec{D}_{Z3} \cdot \vec{D}_{Z1}] I_{03} I_{01}, \tag{41a}$$

$$c_1^2 c_2^2 [P_{Z1} + P_{Z2} - \vec{A}_1 \cdot \vec{D}_{Z2} - \vec{A}_2 \cdot \vec{D}_{Z1}] I_{01} I_{02} =$$

$$= c_2^2 c_3^2 [\, P_{Z2} + P_{Z3} - \vec{A}_2 \cdot \vec{D}_{Z3} - \vec{A}_3 \cdot \vec{D}_{Z2} \,] I_{02} I_{03} =$$

$$= c_3^2 c_1^2 [\, P_{Z3} + P_{Z1} - \vec{A}_3 \cdot \vec{D}_{Z1} - \vec{A}_1 \cdot \vec{D}_{Z3} \,] I_{03} I_{01} \; ; \qquad (41b)$$

$$E_\Omega = c_1^2 c_2^2 [\, 1 + A_{z1} A_{z2} - \vec{P}_1 \cdot \vec{P}_2 - \vec{D}_{z1} \cdot \vec{D}_{z2} \,] I_{01} I_{02} =$$

$$= c_2^2 c_3^2 [\, 1 + A_{z2} A_{z3} - \vec{P}_2 \cdot \vec{P}_3 - \vec{D}_{z2} \cdot \vec{D}_{z3} \,] I_{02} I_{03} =$$

$$= c_3^2 c_1^2 [\, 1 + A_{z3} A_{z1} - \vec{P}_3 \cdot \vec{P}_1 - \vec{D}_{z3} \cdot \vec{D}_{z1} \,] I_{03} I_{01} \; , \qquad (41c)$$

$$c_1^2 c_2^2 [\, A_{z1} + A_{z2} - \vec{P}_1 \cdot \vec{D}_{z2} - \vec{P}_2 \cdot \vec{D}_{z1} \,] I_{01} I_{02} =$$

$$= c_2^2 c_3^2 [\, A_{z2} + A_{z3} - \vec{P}_2 \cdot \vec{D}_{z3} - \vec{P}_3 \cdot \vec{D}_{z2} \,] I_{02} I_{03} =$$

$$= c_3^2 c_1^2 [\, A_{z3} + A_{z1} - \vec{P}_3 \cdot \vec{D}_{z1} - \vec{P}_1 \cdot \vec{D}_{z3} \,] I_{03} I_{01} \; , \qquad (41d)$$

$$E_\Delta = c_1^2 c_2^2 [\, 1 + D_{zZ1} D_{zZ2} - \vec{\Delta}_1 \cdot \vec{\Delta}_2 - \vec{\Lambda}_1 \cdot \vec{\Lambda}_2 \,] I_{01} I_{02} =$$

$$= c_2^2 c_3^2 [\, 1 + D_{zZ2} D_{zZ3} - \vec{\Delta}_2 \cdot \vec{\Delta}_3 - \vec{\Lambda}_2 \cdot \vec{\Lambda}_3 \,] I_{02} I_{03} =$$

$$= c_3^2 c_1^2 [\, 1 + D_{zZ3} D_{zZ1} - \vec{\Delta}_3 \cdot \vec{\Delta}_1 - \vec{\Lambda}_3 \cdot \vec{\Lambda}_1 \,] I_{03} I_{01} \; , \qquad (41e)$$

$$c_1^2 c_2^2 [\, D_{zZ1} + D_{zZ2} - \vec{\Delta}_1 \cdot \vec{\Lambda}_2 - \vec{\Delta}_2 \cdot \vec{\Lambda}_1 \,] I_{01} I_{02} =$$

$$= c_2^2 c_3^2 [\, D_{zZ2} + D_{zZ3} - \vec{\Delta}_2 \cdot \vec{\Lambda}_3 - \vec{\Delta}_3 \cdot \vec{\Lambda}_2 \,] I_{02} I_{03} =$$

$$= c_3^2 c_1^2 [\, D_{zZ3} + D_{zZ1} - \vec{\Delta}_3 \cdot \vec{\Lambda}_1 - \vec{\Delta}_1 \cdot \vec{\Lambda}_3 \,] I_{03} I_{01} \, . \qquad (41f)$$

Next, let us consider the bound (36b) for $(\alpha, \beta) = [\Sigma^{(+)}, \Sigma^{(-)}]$, $[\Omega^{(+)}, \Omega^{(-)}]$, $[\Delta^{(+)}, \Delta^{(-)}]$. **Then, we have**

$$[-4H_{\Sigma^{(+)}} - \lambda(\Sigma^{(+)})]^{1/2} \, [-4H_{\Sigma^{(-)}} - \lambda(\Sigma^{(-)})]^{1/2} + 2E_\Sigma \le$$

$$\le -\lambda(I_0) - \lambda(P_Z I_0) \, , \qquad (42a)$$

$$[-4H_{\Omega^{(+)}} - \lambda(\Omega^{(+)})]^{1/2} \, [-4H_{\Omega^{(-)}} - \lambda(\Omega^{(-)})]^{1/2} + 2E_\Omega \le$$

$$\le -\lambda(I_0) - \lambda(A_z I_0) \, , \qquad (42b)$$

$$[-4H_{\Delta^{(+)}} - \lambda(\Delta^{(+)})]^{1/2} \, [-4H_{\Delta^{(-)}} - \lambda(\Delta^{(-)})]^{1/2} + 2E_\Delta \le$$

$$\le -\lambda(I_0) - \lambda(D_{zZ} I_0) \, . \qquad (42c)$$

Now, it is easy to see that these bounds imply the following interesting consequences.

(xii) ***If the bound***

$$\lambda(X I_0) \le -\lambda(I_0) \, , \quad X \equiv P_Z, A_z, D_{zZ} \, , \qquad (43)$$

is exactly saturated, then

$$\vec{A}_1 = \vec{A}_2 = \vec{A}_3 \, , \quad \vec{D}_{Z1} = \vec{D}_{Z2} = \vec{D}_{Z3} \, , \qquad (44a)$$

$$\phi^{(0)}_{\Sigma^{(\pm)}12} = \phi^{(0)}_{\Sigma^{(\pm)}23} = \phi^{(0)}_{\Sigma^{(\pm)}31} = n\pi, \quad n = 0, 1, \ldots, \qquad (44b)$$

when $X \equiv A_z$,

$$\vec{P}_1 = \vec{P}_2 = \vec{P}_3 \, , \quad \vec{D}_{z1} = \vec{D}_{z2} = \vec{D}_{z3} \, , \qquad (44c)$$

$$\phi^{(0)}_{\Omega^{(\pm)}12} = \phi^{(0)}_{\Omega^{(\pm)}23} = \phi^{(0)}_{\Omega^{(\pm)}31} = n\pi, \quad n = 0, 1, \ldots, \qquad (44d)$$

when $X \equiv P_Z$,

$$\vec{\Delta}_1 = \vec{\Delta}_2 = \vec{\Delta}_3 \, , \quad \vec{\Lambda}_1 = \vec{\Lambda}_2 = \vec{\Lambda}_3 \, , \qquad (44e)$$

$$\phi^{(0)}_{\Delta^{(\pm)}12} = \phi^{(0)}_{\Delta^{(\pm)}23} = \phi^{(0)}_{\Delta^{(\pm)}31} = n\pi, \quad n = 0, 1, \dots, \tag{44f}$$

when $X = D_{ZZ}$, *where the vectors* $\vec{D}_Z, \vec{D}_z, \vec{\Delta}$ *and* $\vec{\Lambda}$ *are defined by eq. (40a,b).*

(xiii) *The exact saturation of the bound*

$$[-4H_\alpha - \lambda(\sigma_\alpha)]^{1/2} [-4H_{\bar{\alpha}} - \lambda(\sigma_{\bar{\alpha}})]^{1/2} \leq -\lambda(I_0) - \lambda(XI_0), \tag{45}$$

$X = P_Z, A_Z, D_{ZZ}$, *respectively, implies the results of form (44a), (44c) and (44e) respectively.*

We note that the experimental situations are more varied than the typical cases considered in this section. The results presented in sect. 3 are sufficient to obtain all the constraints on the experimental observables, implied by the sum rules (10), by specializing the unit vectors $\vec{\kappa}$.

Finally, we note that *the isospin sum rules (10) imply that each set of equalities (12) as well as the bounds (16a,b,c) have an integrated analogous.* Proof of this statement can be obtaines just as in $(1/2 \to 0' 1'/2)$ scattering case discussed in ref. /1/. Hence, the results (29), (30a,b), (31a,b), (32c), (33), (34a,b), (36a,b), (37a,b), (38a,b,c), (39), (41a,b,c,d,e,f), (42a,b,c), (43), (44a,b,c,e) are also valid for the integrated (partial or total) cross-sections and average values of the "asymmetry", final polarization and "depolarization tensor" components.

5. Conclusions

In this paper we have investigated the constraints on the experimental observables of three $(0\ 1/2 \to 0' 0' 1'/2)$ reactions. So, using the results of ref. /2/, in sect. 2, we have defined the polarized differential cross-sections σ_α and the spin-rotation vectors $\vec{\xi}_\alpha$. These definitions allow to discuss a large number of the isospin constraints by analogy with the isospin constrains for $(0\ 1/2 \to 0' 1'/2)$ reactions [see ref. /1/]. Then, all the constraints on σ_α and $\vec{\xi}_\alpha$ have been derived, in sect. 3, using the generalized functions $F^{(\pm\kappa)}_\alpha$ [see definitions (17a)] and the bilinear forms $M^{(\pm\kappa)}_{\alpha ij}$, $Z^{(0)}_{\alpha ij}$ and $Z^{(\kappa)}_{\alpha ij}$ defined by eqs. (19a,b,c,d). In this way, we have proved that the sum rules (10) alone imply the equalities (12), (13a,b,c,d), (14), (15) and the bounds (16a,b,c).

These results are presented in the most general form and are sufficient to obtain any particular constraints on $\sigma_{\alpha\ell}$ and $\vec{\xi}_{\alpha\ell}$, $\ell = 1, 2, 3$, by specializing the unit vectors $\vec{\kappa}$. Thus starting with these results, in sect. 4 we have derived a number of interesting experimental consequences. For example, we have shown that the exact saturation of the upper bounds (16b) and of the lower bound (16c) implies a linear relation [see eq. (27), sect. 4] on the projections of the spin rotation vectors $\vec{\xi}_{\alpha\ell}$, $\ell = 1, 2, 3$. We have obtained that the phases of all the bilinear forms: $N_{\alpha ij} \equiv M^{(\pm\kappa)}_{\alpha ij}$, $Z^{(\kappa)}_{\alpha ij}$, $Z^{(0)}_{\alpha ij}$ can be unumbiguously determined from the experimental data. Then, the exact saturation of the bounds (16a,b,c) is expressed in terms of the $[n\pi, (n+\frac{1}{2})\pi, n=0,1,\dots,]$ phase contours or equivalently in terms of the zeros trajectories of $\mathrm{Im}\,N_{\alpha ij}$ and $\mathrm{Re}\,N_{\alpha ij}$ respectively. Next, using eqs. (22b,c), we have obtained the bounds (32c) which are more stringent than the bounds (31a,b) and (16a,b,c) respectively. These bounds require the Pomeranchuk-like theorems (33) if the conditions (30a) hold for $s \to +\infty$ and the other kinematical variables are fixed. Also, using the inequality (35) and the lower bound (16c) we have derived the stringent bounds (36d) which imply the Pomeranchuk-type theorems (38b,c d,e) for all $(\alpha, \bar{\alpha}) \equiv [\Sigma^{(+)}, \Sigma^{(-)}], [\Omega^{(+)}, \Omega^{(-)}], [\Delta^{(+)}, \Delta^{(-)}]$ when the condition (38) holds. Moreover, the bounds (36b) imply the consequences (xii) and (xiii) when the bounds (43) and (45), respectively, are exactly saturated.

Therefore, the results obtained in sects.3,4, of this paper are of great interest for phenomenological description of the three-body final state reactions as well as in testing of the different theoretical models. These results are sufficient to obtain certain tests [see the equalities (12), (13a,b,c,d), (14), (15) and (41a,b,c,d,e,f)] of the isospin invariance, SU(3) -symmetry, quark models, etc., in the single meson production processes and to determine unumbiguously the symmetry-breaking parameters when the complete and accurate experimental data are available.

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