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STRINGENT ISOSPIN CONSTRAINTS ON EXPERIMENTAL OBSERVABLES OF (0 1/2 → 0'0'1'/2) REACTIONS

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1. *Introduction*

This paper is a direct continuation of ref.[1] where we have initiated the investigation of the stringent constraints on experimental observables of three $(0\,1/2 \to 0'\,0'\,1'/2)$ reactions[2] related by the isospin invariance via two isospin channels.

Therefore, in sect. 1, using the generalized helicity amplitudes (1a,b) and the bilinear forms (2a,b,c,d) and (11a,b,c) we prove that the isospin invariance alone implies the equalities (7), (8a,b), (16), (17a,b) and the isospin bounds (9a,b), (10), (18a,b) and (20). In sect. 3, the saturation of isospin bounds and some experimental consequences for experimental data and amplitude analysis are discussed. The method of investigation of isospin bounds presented here and in refs.[1,5,6] have advantage that the exact saturation of isospin bounds can be expressed in terms of the zeros of real or imaginary parts of a specific bilinear form. Thus, all the constraints on experimental data and amplitude analyses, when the isospin bounds are exactly saturated or degenerated, can be unambiguously determined.

The results are presented in a general form, and are sufficient to obtain any constraints (equalities or bounds) by specializing the unit vectors $\vec{\kappa}$ and using tables I, II, III from sect. 3. These results improve in the most general form the usual triangle inequalities and are useful to obtain certain tests of the isospin invariance and to determine the breaking effects when the experimental data, on differential cross sections and on the spin density matrices of the single meson production processes, are available.

2. *Equalities and Bounds from Isospin Invariance*

In order to obtain new isospin constraints on the experimental data of $(0 1/2 \to 0'0'1'/2)$ reactions we start with the following definitions.

Let f_ℓ^{++}, f_ℓ^{+-}, f_ℓ^{-+}, f_ℓ^{--}, $\ell = 1,2,3$ be the helicity amplitudes of three $(0 1/2 \to 0'0'1'/2)$ reactions related by isospin invariance via two channels. Let $F_\alpha^{(\pm\kappa)}$ and $F_\beta^{(\pm\kappa)}$ be the generalized helicity amplitudes defined as

$$F_{\alpha\ell}^{(+\kappa)} = \frac{\sqrt{2}}{[1+|w|^2]^{1/2}} [f_\ell^{\tau\mu} + w f_\ell^{\tau'\mu'}],$$

$$F_{\alpha\ell}^{(-\kappa)} = \frac{\sqrt{2}}{[1+|w|^2]^{1/2}} [-w^* f_\ell^{\tau\mu} + f_\ell^{\tau'\mu'}], \tag{1a}$$

$$F_{\beta\ell}^{(+\kappa)} = \frac{\sqrt{2}}{[1+|w|^2]^{1/2}} [f_\ell^{\nu\lambda} + w f_\ell^{\nu'\lambda'}],$$

$$F_{\beta\ell}^{(-\kappa)} = \frac{\sqrt{2}}{[1+|w|^2]^{1/2}} [-w^* f_\ell^{\nu\lambda} + f_\ell^{\nu'\lambda'}]. \tag{1b}$$

Also, let us define the following bilinear forms:

$$M_{\gamma ij}^{(\pm\kappa)} = [F_{\gamma i}^{(\pm\kappa)}]^* F_{\gamma j}^{(\pm\kappa)}, \quad \gamma = \alpha, \beta, \tag{2a}$$

$$G_{\alpha\beta ij}^{(\pm\kappa)} = \frac{1}{2} [M_{\alpha ij}^{(\pm\kappa)} + M_{\beta ij}^{(\pm\kappa)}], \tag{2b}$$

$$W_{\alpha\beta ij}^{(\pm\kappa)} = \frac{1}{2} [M_{\alpha ij}^{(\pm\kappa)} - M_{\beta ij}^{(\pm\kappa)}], \tag{2c}$$

$$Y_{\alpha\beta ij}^{(\pm\kappa)} = \frac{1}{2} \{ F_{\alpha i}^{(\pm\kappa)} F_{\beta j}^{(\pm\kappa)} - F_{\beta i}^{(\pm\kappa)} F_{\alpha j}^{(\pm\kappa)} \}. \tag{2d}$$

Now, using the relations (see ref. [1])

$$| F_{\gamma\ell}^{(\pm\kappa)} |^2 = (1 \pm \vec{\kappa}\cdot\vec{\xi}_{\gamma\ell}) \, \sigma_{\gamma\ell}, \quad \gamma = \alpha, \beta, \tag{3a}$$

and the definitions

$$I_{0\ell} S_{\alpha\beta\ell} = \mathrm{Re}\{ f_\ell^{\tau\mu} [f_\ell^{\nu\lambda}]^* \} + \mathrm{Re}\{ f_\ell^{\tau'\mu'} [f_\ell^{\nu'\lambda'}]^* \}, \tag{3b}$$

$$I_{0\ell} T_{\alpha\beta\ell} = \mathrm{Im}\{ f_\ell^{\tau\mu} [f_\ell^{\nu\lambda}]^* \} + \mathrm{Im}\{ f_\ell^{\tau'\mu'} [f_\ell^{\nu'\lambda'}]^* \}, \tag{3c}$$

$$\begin{aligned} I_{0\ell} \vec{V}_{\alpha\beta\ell} = \{ & \mathrm{Re}\{ f_\ell^{\tau'\mu'} [f_\ell^{\nu\lambda}]^* \} + \mathrm{Re}\{ f_\ell^{\tau\mu} [f_\ell^{\nu'\lambda'}]^* \}, \\ & - \mathrm{Im}\{ f_\ell^{\tau'\mu'} [f_\ell^{\nu\lambda}]^* \} + \mathrm{Im}\{ f_\ell^{\tau\mu} [f_\ell^{\nu'\lambda'}]^* \}, \\ & \mathrm{Re}\{ f_\ell^{\tau\mu} [f_\ell^{\nu\lambda}]^* \} - \mathrm{Re}\{ f_\ell^{\tau'\mu'} [f_\ell^{\nu'\lambda'}]^* \} \}, \end{aligned} \tag{3d}$$

$$\begin{aligned} I_{0\ell} \vec{U}_{\alpha\beta\ell} = \{ & \mathrm{Im}\{ f_\ell^{\tau'\mu'} [f_\ell^{\nu\lambda}]^* \} + \mathrm{Im}\{ f_\ell^{\tau\mu} [f_\ell^{\nu'\lambda'}]^* \}, \\ & \mathrm{Re}\{ f_\ell^{\tau'\mu'} [f_\ell^{\nu\lambda}]^* \} - \mathrm{Re}\{ f_\ell^{\tau\mu} [f_\ell^{\nu'\lambda'}]^* \}, \\ & \mathrm{Im}\{ f_\ell^{\tau\mu} [f_\ell^{\nu\lambda}]^* \} + \mathrm{Im}\{ f_\ell^{\tau'\mu'} [f_\ell^{\nu'\lambda'}]^* \} \}, \end{aligned} \tag{3e}$$

when $F_{\gamma\ell}^{(\pm\kappa)}$, $\gamma = \alpha, \beta$ are given by (1a,b) and the unit $\vec{\kappa}$ vector is chosen as

$$\vec{\kappa} \equiv \left\{ \frac{2\,\mathrm{Re}\,w}{1+|w|^2},\ \frac{2\,\mathrm{Im}\,w}{1+|w|^2},\ \frac{1-|w|^2}{1+|w|^2} \right\}, \tag{3f}$$

then, by straightforward development, we obtain

$$|G^{(\pm\kappa)}_{\alpha\beta ij}|^2 = G^{(\pm\kappa)}_{\alpha\beta ii}\, G^{(\pm\kappa)}_{\alpha\beta jj} - H^{(\pm\kappa)}_{\alpha\beta ij}, \tag{3g}$$

$$|W^{(\pm\kappa)}_{\alpha\beta ij}|^2 = W^{(\pm\kappa)}_{\alpha\beta ii}\, W^{(\pm\kappa)}_{\alpha\beta jj} + H^{(\pm\kappa)}_{\alpha\beta ij}, \tag{3h}$$

$$\begin{aligned} |Y^{(\pm\kappa)}_{\alpha\beta ij}|^2 \equiv H^{(\pm\kappa)}_{\alpha\beta ij} = \frac{1}{4}\{ & [1\pm\vec{\kappa}\cdot\vec{\xi}_{\alpha i}][1\pm\vec{\kappa}\cdot\vec{\xi}_{\beta j}]\sigma_{\alpha i}\,\sigma_{\beta j} + \\ & + [1\pm\vec{\kappa}\cdot\vec{\xi}_{\beta i}][1\pm\vec{\kappa}\cdot\vec{\xi}_{\alpha j}]\sigma_{\beta i}\,\sigma_{\alpha j} - \\ & - 2I_{0i}I_{0j}[(S_{\alpha\beta i}\pm\vec{\kappa}\cdot\vec{V}_{\alpha\beta i})(S_{\alpha\beta j}\pm\vec{\kappa}\cdot\vec{V}_{\alpha\beta j}) + \\ & + (T_{\alpha\beta i}\pm\vec{\kappa}\cdot\vec{U}_{\alpha\beta i})(T_{\alpha\beta j}\pm\vec{\kappa}\cdot\vec{U}_{\alpha\beta j})]\}, \end{aligned} \tag{3i}$$

where, according to the definitions (2b,c) and eq. (3a) $G^{(\pm\kappa)}_{\alpha\beta\ell\ell}$ and $W^{(\pm\kappa)}_{\alpha\beta\ell\ell}$ are given by

$$G^{(\pm\kappa)}_{\alpha\beta\ell\ell} = \frac{1}{2}[(1\pm\vec{\kappa}\cdot\vec{\xi}_{\alpha\ell})\sigma_{\alpha\ell} + (1\pm\vec{\kappa}\cdot\vec{\xi}_{\beta\ell})\sigma_{\beta\ell}], \tag{3k}$$

$$W^{(\pm\kappa)}_{\alpha\beta\ell\ell} = \frac{1}{2}[(1\pm\vec{\kappa}\cdot\vec{\xi}_{\alpha\ell})\sigma_{\alpha\ell} - (1\pm\vec{\kappa}\cdot\vec{\xi}_{\beta\ell})\sigma_{\beta\ell}]. \tag{3l}$$

Now, since the isospin sum rules on the helicity amplitudes are equivalent to

$$\sum_{\ell=1}^{3} c_\ell\, F^{(\pm\kappa)}_{\gamma\ell} = 0, \quad \gamma=\alpha,\beta,\dots, \tag{4a}$$

and also to

$$c_1c_2Y^{(\pm\kappa)}_{\alpha\beta 12} = c_2c_3\,Y^{(\pm\kappa)}_{\alpha\beta 23} = c_3c_1\,Y^{(\pm\kappa)}_{\alpha\beta 31}, \tag{4b}$$

then, we obtain (for eqs. (5d,e) see ref. [1])

$$\mathrm{Re}\,N_{\alpha\beta ij} = (2c_ic_j)^{-1}[c_k^2N_{\alpha\beta kk} - c_i^2N_{\alpha\beta ii} - c_j^2N_{\alpha\beta jj}], \tag{5a}$$

for $N_{\alpha\beta ij} \equiv G^{(\pm\kappa)}_{\alpha\beta ij}$, $W^{(\pm\kappa)}_{\alpha\beta ij}$, $M^{(\pm\kappa)}_{\gamma ij}$, $\gamma=\alpha,\beta,\dots$, and

$$\begin{aligned} c_i^2c_j^2[\mathrm{Im}\,G^{(\pm\kappa)}_{\alpha\beta ij}]^2 &= -H^{(\pm\kappa)}_{\alpha\beta} - \frac{1}{4}\lambda[G^{(\pm\kappa)}_{\alpha\beta}] \\ &= \frac{1}{16}\{[-\lambda^{(\pm)}_{\alpha\kappa}]^{1/2} + \epsilon^{(\pm)}_{\alpha\beta\kappa}[-\lambda^{(\pm)}_{\beta\kappa}]^{1/2}\}^2, \end{aligned} \tag{5b}$$

$$\begin{aligned} c_i^2c_j^2[\mathrm{Im}\,W^{(\pm\kappa)}_{\alpha\beta ij}]^2 &= H^{(\pm\kappa)}_{\alpha\beta} - \frac{1}{4}\lambda[W^{(\pm\kappa)}_{\alpha\beta}] \\ &= \frac{1}{16}\{[-\lambda^{(\pm)}_{\alpha\kappa}]^{1/2} - \epsilon^{(\pm)}_{\alpha\beta\kappa}[-\lambda^{(\pm)}_{\beta\kappa}]^{1/2}\}^2, \end{aligned} \tag{5c}$$

$$\begin{aligned} c_i^2c_j^2[\mathrm{Im}\,M^{(\pm\kappa)}_{\alpha ij}]^2 = -\frac{1}{4}\lambda^{(\pm)}_{\alpha\kappa} & \{[-H^{(\pm\kappa)}_{\alpha\beta} - \frac{1}{4}\lambda[G^{(\pm\kappa)}_{\alpha\beta}]]^{1/2} + \eta^{(\pm)}_{\alpha\beta\kappa}[H^{(\pm\kappa)}_{\alpha\beta} - \\ & - \frac{1}{4}\lambda[W^{(\pm\kappa)}_{\alpha\beta}]]^{1/2}\}^2, \end{aligned} \tag{5d}$$

$$\begin{aligned} c_i^2c_j^2[\mathrm{Im}\,M^{(\pm\kappa)}_{\beta ij}]^2 = -\frac{1}{4}\lambda^{(\pm)}_{\beta\kappa} & \{[-H^{(\pm\kappa)}_{\alpha\beta} - \frac{1}{4}\lambda[G^{(\pm\kappa)}_{\alpha\beta}]]^{1/2} - \eta^{(\pm)}_{\alpha\beta\kappa}[H^{(\pm\kappa)}_{\alpha\beta} - \\ & - \frac{1}{4}\lambda[W^{(\pm\kappa)}_{\alpha\beta}]]^{1/2}\}^2, \end{aligned} \tag{5e}$$

$$\eta_{\alpha\beta\kappa} = \mathrm{sign}\,\{ c_i^2 c_j^2\, \mathrm{Im}\, G^{(\pm\kappa)}_{\alpha\beta ij}\, \mathrm{Im}\, W^{(\pm\kappa)}_{\alpha\beta ij} \} = \mathrm{sign}\,\{ -\lambda^{(\pm)}_{\alpha\kappa} + \lambda^{(\pm)}_{\beta\kappa} \}, \quad (5f)$$

$$\epsilon^{(\pm)}_{\alpha\beta\kappa} = \mathrm{sign}\,\{ c_i^2 c_j^2\, \mathrm{Im}\, M^{(\pm\kappa)}_{\alpha ij}\, \mathrm{Im}\, M^{(\pm\kappa)}_{\beta ij} \} = \quad (5g)$$

$$= \mathrm{sign}\,\{ -8H^{(\pm\kappa)}_{\alpha\beta} - \lambda[G^{(\pm\kappa)}_{\alpha\beta}] + \lambda[W^{(\pm\kappa)}_{\alpha\beta}] \},$$

where

$$\lambda(x,y,z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz, \quad (6a)$$

$$\lambda[G^{(\pm\kappa)}_{\alpha\beta}] \equiv \lambda\,(c_1^2 G^{(\pm\kappa)}_{\alpha\beta 11},\ c_2^2 G^{(\pm\kappa)}_{\alpha\beta 22},\ c_3^2 G^{(\pm\kappa)}_{\alpha\beta 33}), \quad (6b)$$

$$\lambda[W^{(\pm\kappa)}_{\alpha\beta}] \equiv \lambda\,(c_1^2 W^{(\pm\kappa)}_{\alpha\beta 11},\ c_2^2 W^{(\pm\kappa)}_{\alpha\beta 22},\ c_3^2 W^{(\pm\kappa)}_{\alpha\beta 33}), \quad (6c)$$

$$\lambda^{(\pm)}_{\gamma\kappa} \equiv \lambda\,[c_1^2(1 \pm \vec{\kappa}\cdot\vec{\xi}_{\gamma 1})\sigma_{\gamma 1},\ c_2^2(1 \pm \vec{\kappa}\cdot\vec{\xi}_{\gamma 2})\sigma_{\gamma 2},\ c_3^2(1 \pm \vec{\kappa}\cdot\vec{\xi}_{\gamma 3})\sigma_{\gamma 3}], \quad (6d)$$

$\gamma = \alpha, \beta$ and $H^{(\pm\kappa)}_{\alpha\beta}$ are defined by eq. (7).

Therefore, using eqs. (4b) and (5b,c,d,e) we can prove the following interesting results.

Constraints 1: The isospin invariance alone implies that the equalities:

$$H^{(\pm\kappa)}_{\alpha\beta} \equiv c_1^2 c_2^2 H^{(\pm\kappa)}_{\alpha\beta 12} = c_2^2 c_3^2 H^{(\pm\kappa)}_{\alpha\beta 23} = c_3^2 c_1^2 H^{(\pm\kappa)}_{\alpha\beta 31}, \quad (7)$$

$$| 8H^{(\pm\kappa)}_{\alpha\beta} + \lambda[G^{(\pm\kappa)}_{\alpha\beta}] - \lambda[W^{(\pm\kappa)}_{\alpha\beta}] | = [-\lambda^{(\pm)}_{\alpha\kappa}]^{1/2} [-\lambda^{(\pm)}_{\beta\kappa}]^{1/2}, \quad (8a)$$

$$\frac{1}{4} |\lambda^{(\pm)}_{\alpha\kappa} - \lambda^{(\pm)}_{\beta\kappa}| = [-4H^{(\pm\kappa)}_{\alpha\beta} - \lambda[G^{(\pm\kappa)}_{\alpha\beta}]]^{1/2} [4H^{(\pm\kappa)}_{\alpha\beta} - \lambda[W^{(\pm\kappa)}_{\alpha\beta}]]^{1/2}, \quad (8b)$$

etc. (see the eqs. (13a,b,c,d) from sect. 3 of ref. /1/ *and the bounds:*

$$\max_{\{ij\}} \{ -c_i^2 c_j^2\, W^{(\pm\kappa)}_{\alpha\beta ii} W^{(\pm\kappa)}_{\alpha\beta jj} \} \le \frac{1}{4}\lambda[W^{(\pm\kappa)}_{\alpha\beta}] \le H^{(\pm\kappa)}_{\alpha\beta}, \quad (9a)$$

$$H_{\alpha\beta} \le -\frac{1}{4}\lambda[G^{(\pm\kappa)}_{\alpha\beta}] \le \min_{\{ij\}} \{ c_i^2 c_j^2\, G^{(\pm\kappa)}_{\alpha\beta ii} G^{(\pm\kappa)}_{\alpha\beta jj} \}, \quad (9b)$$

are valid at any values of the kinematical variables in the physical domain, for any α, β *and* $\vec{\kappa}$.

Now, it is easy to obtain (see eqs. (5b,c,d,e) the following stringent bounds.

Constraints 1′

$$\max\{ [-\lambda^{(\pm)}_{\alpha\kappa}]^{1/2} [-\lambda^{(\pm)}_{\beta\kappa}]^{1/2},\ 2[-4H^{(\pm\kappa)}_{\alpha\beta} - \lambda[G^{(\pm\kappa)}_{\alpha\beta}]]^{1/2} [4H^{(\pm\kappa)}_{\alpha\beta} - \quad (10)$$

$$- \lambda[W^{(\pm\kappa)}_{\alpha\beta}]]^{1/2} \} \le -\lambda[G^{(\pm\kappa)}_{\alpha\beta}] - \lambda[W^{(\pm\kappa)}_{\alpha\beta}],$$

valid for any kinematical values in the physical domain and for any α, β *and unit vector* $\vec{\kappa}$.

Next, let us define the bilinear forms:

$$Z^{(0)}_{ij} = \frac{1}{2} \{ [f^{++}_i]^* f^{++}_j + [f^{+-}_i]^* f^{+-}_j + [f^{-+}_i]^* f^{-+}_j + [f^{--}_i]^* f^{--}_j \}, \quad (11a)$$

$$Z^{(0)}_{\alpha\bar{\alpha} ij} = \frac{1}{2} \{ Z^{(0)}_{\alpha ij} - Z^{(0)}_{\bar{\alpha} ij} \}, \quad (11b)$$

$$Z^{(0)}_{\gamma ij} = \frac{1}{2} \{ M^{(+\kappa)}_{\gamma ij} + M^{(-\kappa)}_{\gamma ij} \}, \quad \gamma = \alpha, \bar{\alpha}, \ldots, \quad (11c)$$

where $(\alpha,\bar{\alpha}) \equiv [\Sigma^{(+)}, \Sigma^{(-)}], [\Omega^{(+)}, \Omega^{(-)}], [\Delta^{(+)}, \Delta^{(-)}]$,

(see sect. 2 of ref. [1]). Then we observe that $Z^{(0)}_{\ell\ell} = I_{0\ell}$,

($I_{0\ell}$ - unpolarized differential cross section),

$$Z^{(0)}_{\alpha\bar{\alpha}\ell\ell} = \frac{1}{2}[\sigma_{\alpha\ell} - \sigma_{\bar{\alpha}\ell}] = X_\ell \, I_{0\ell} \qquad (12b)$$

where $X_\ell \equiv \hat{P}_{Z\ell}$, $A_{Z\ell}$, $D_{zZ\ell}$, for $\alpha \equiv \Sigma, \Omega, \Delta$, respectively.

Now, by the straightforward calculation we get

$$|Z^{(0)}_{ij}|^2 = \frac{1}{4}[1 + \vec{A}_i \cdot \vec{A}_j + \vec{P}_i \cdot \vec{P}_j + \vec{D}_i \cdot \vec{D}_j] I_{0i} I_{0j} = \qquad (13a)$$

$$= I_{0i} I_{0j} - H_{ij} ,$$

$$|Z^{(0)}_{\alpha\bar{\alpha}ij}|^2 = Z^{(0)}_{\alpha\bar{\alpha}ii} Z^{(0)}_{\alpha\bar{\alpha}jj} + H_{ij} - \frac{1}{2}[H_{\alpha ij} + H_{\bar{\alpha}ij}], \qquad (13b)$$

$$H_{ij} = \frac{1}{4}[3 - \vec{A}_i \cdot \vec{A}_j - \vec{P}_i \cdot \vec{P}_j - \vec{D}_i \cdot \vec{D}_j] I_{0i} I_{0j}, \qquad (13c)$$

where

$$H_{\gamma ij} = \frac{1}{2}[1 - \vec{\xi}_{\gamma i} \cdot \vec{\xi}_{\gamma j}]\sigma_{\gamma i}\sigma_{\gamma j}, \quad \gamma = \alpha, \bar{\alpha}, \ldots, \qquad (13d)$$

$$\vec{D}_i \cdot \vec{D}_j \equiv D_{xXi} D_{xXj} + D_{xYi} D_{xYj} + D_{xZi} D_{xZj} +$$

$$+ D_{yXi} D_{yXj} + D_{yYi} D_{yYj} + D_{yZi} D_{yZj} + \qquad (13e)$$

$$+ D_{zXi} D_{zXj} + D_{zYi} D_{zYj} + D_{zZi} D_{zZj} .$$

$$\vec{A}_i \cdot \vec{A}_j \equiv A_{xi} A_{xj} + A_{yi} A_{yj} + A_{zi} A_{zj} , \qquad (13f)$$

$$\vec{P}_i \cdot \vec{P}_j \equiv P_{Xi} P_{Xj} + P_{Yi} P_{Yj} + P_{Zi} P_{Zj} , \qquad (13g)$$

$\vec{A}_\ell , \vec{P}_\ell$ and $(D_{xX\ell}, D_{xY\ell}, D_{xZ\ell}, D_{yX\ell}, D_{yY\ell}, D_{yZ\ell}, D_{zX\ell}, D_{zY\ell} D_{zZ\ell})$ are the "polarization asymmetry" and "final polarization" vectors and "depolarization" tensor components, respectively. (see [1,2]).

Therefore, since the isospin sum rules (4a) imply a relation (5a) for each $N_{\alpha\beta ij} \equiv Z^{(0)}_{ij}, Z^{(0)}_{\alpha\bar{\alpha}ij}, Z^{(0)}_{\gamma ij}$, $\gamma = \alpha, \bar{\alpha}, \ldots$, then we obtain:

$$c_i^2 c_j^2 [\mathrm{Im} Z^{(0)}_{ij}]^2 = -H - \frac{1}{4}\lambda[I_0] \qquad (14a)$$

$$= \frac{1}{16}\{[-4H_\alpha - \lambda(\sigma_\alpha)]^{1/2} + \epsilon_{\alpha\bar{\alpha}}[-4H_{\bar{\alpha}} - \lambda(\sigma_{\bar{\alpha}})]^{1/2}\}^2,$$

$$c_i^2 c_j^2 [\mathrm{Im} Z^{(0)}_{\alpha\bar{\alpha}ij}]^2 = H - \frac{1}{2}[H_\alpha + H_{\bar{\alpha}}] - \frac{1}{4}\lambda[XI_0] \qquad (14b)$$

$$= \frac{1}{16}\{[-4H_\alpha - \lambda(\sigma_\alpha)]^{1/2} - \epsilon_{\alpha\bar{\alpha}}[-4H_{\bar{\alpha}} - \lambda(\sigma_{\bar{\alpha}})]^{1/2}\}^2,$$

$$c_i^2 c_j^2 [\mathrm{Im} Z^{(0)}_{\alpha ij}]^2 = -H_\alpha - \frac{1}{4}\lambda(\sigma_\alpha) \qquad (14c)$$

$$= \{[-H - \frac{1}{4}\lambda[I_0]]^{1/2} + \eta_{\alpha\bar{\alpha}}[H - \frac{1}{2}[H_\alpha + H_{\bar{\alpha}}] - \frac{1}{4}\lambda[XI_0]]^{1/2}\}^2,$$

$$c_i^2 c_j^2 [\mathrm{Im} Z^{(0)}_{\bar{\alpha}ij}]^2 = -H_{\bar{\alpha}} - \frac{1}{4}\lambda(\sigma_{\bar{\alpha}}) = \qquad (14d)$$

$$= \{[-H - \frac{1}{4}\lambda[I_0]]^{1/2} - \eta_{\alpha\bar{\alpha}}[H - \frac{1}{2}[H_\alpha + H_{\bar{\alpha}}] - \frac{1}{4}\lambda[XI_0]]^{1/2}\}^2,$$

$$\epsilon_{\alpha\bar\alpha} = \text{sign}\{ c_i^2 c_j^2 \,\text{Im}\, Z_{\alpha ij}^{(0)} \,\text{Im}\, Z_{\bar\alpha ij}^{(0)} \}$$

$$= \text{sign}\{ -2H + \frac{1}{2}[H_\alpha + H_{\bar\alpha}] - \frac{1}{4}\lambda[I_0] + \frac{1}{4}\lambda[XI_0] \}, \qquad (14e)$$

$$\eta_{\alpha\bar\alpha} = \text{sign}\{ - H_\alpha + H_{\bar\alpha} - \frac{1}{4}\lambda(\sigma_\alpha) + \frac{1}{4}\lambda(\sigma_{\bar\alpha}) \}, \qquad (14f)$$

where

$$\lambda[I_0] \equiv \lambda[c_1^2 I_{01}, c_2^2 I_{02}, c_3^2 I_{03}], \qquad (15a)$$

$$\lambda[XI_0] \equiv \lambda[c_1^2 X_1 I_{01}, c_2^2 X_2 I_{02}, c_3^2 X_3 I_{03}], \qquad (15b)$$

and H is defined by eq. (16).

Hence, from the above results, we obtain the following constraints.

Constraints 2: The isospin invariance alone implies that the differential cross sections I_0, σ_α, $\sigma_{\bar\alpha}$ *and the polarization parameters* X *and* $\vec\xi_\alpha$, $\vec\xi_{\bar\alpha}$, *must obey the equalities:*

$$H \equiv c_1^2 c_2^2 H_{12} = c_2^2 c_3^2 H_{23} = c_3^2 c_1^2 H_{31}, \qquad (16)$$

$$|8H - 2(H_\alpha + H_{\bar\alpha}) + \lambda[I_0] - \lambda[XI_0]| = [-4H_\alpha - \lambda(\sigma_\alpha)]^{1/2} \times [-4H_{\bar\alpha} - \lambda(\sigma_{\bar\alpha})]^{1/2} \qquad (17a)$$

$$\frac{1}{2}|4(H_\alpha - H_{\bar\alpha}) + \lambda(\sigma_\alpha) - \lambda(\sigma_{\bar\alpha})| = 2[-4H - \lambda[I_0]]^{1/2} \times [4H - 2(H_\alpha + H_{\bar\alpha}) - \lambda[XI_0]]^{1/2} \qquad (17b)$$

(and the other equalities similar to eqs. (13c,d), (14), (15) from ref. [1]), *and the bounds*

$$\max_{\{ij\}} \{ -c_i^2 c_j^2 X_i X_j I_{0i} I_{0j} \} \le \frac{1}{4}\lambda[XI_0] \le H - \frac{1}{2}[H_\alpha + H_{\bar\alpha}], \qquad (18a)$$

$$H \le -\frac{1}{4}\lambda[I_0] \le \min_{\{ij\}} \{ c_i^2 c_j^2 I_{0i} I_{0j} \} \qquad (18b)$$

valid for any $(\alpha, \bar\alpha) \equiv [\Sigma^{(+)}, \Sigma^{(-)}], [\Omega^{(+)}, \Omega^{(-)}], [\Delta^{(+)}, \Delta^{(-)}]$, *in any spin reference frame.*

We note, of course, that the equality (13a) is a consequence of the Lagrange identity (see ref. [3]). Hence, the sum rules (4a) imply the equalities (16) since H can be rewritten in the equivalent form

$$H = \frac{1}{4}[H_{\Sigma(+)} + H_{\Sigma(-)} + H_{\Omega(+)} + H_{\Omega(-)} + H_{\Delta(+)} + H_{\Delta(-)}] \qquad (19a)$$

where (see eq. (13d))

$$H_\gamma \equiv c_1^2 c_2^2 H_{\gamma 12} = c_2^2 c_3^2 H_{\gamma 23} = c_3^2 c_1^2 H_{\gamma 31}, \quad \gamma = \alpha, \bar\alpha, \ldots, \qquad (19b)$$

(see also eqs. (12) from ref. [1]).

Now, using eqs. (14a,b,c,d) we obtain the following stringent bounds.

Constraints 2.′

$$\max\{ [-4H_\alpha - \lambda(\sigma_\alpha)]^{1/2} [-4H_{\bar\alpha} - \lambda(\sigma_{\bar\alpha})]^{1/2}, 2[-4H - \lambda[I_0]]^{1/2} \times [4H - 2(H_\alpha + H_{\bar\alpha}) - \lambda[XI_0]]^{1/2} \} + 2(H_\alpha + H_{\bar\alpha}) \le -\lambda[I_0] - \lambda[XI_0]. \qquad (20)$$

These bounds improve the upper bounds (18a) and the lower bound (18b) respectively.

3. *Saturation of Isospin Bounds and Experimental Consequences*

Now, from the results obtained in sect. 2, we obtain the following consequences.

Consequence 1. Let $\phi^{(\pm\kappa)}_{\alpha\beta ij}$, $\Phi^{(\pm\kappa)}_{\alpha\beta ij}$ *and* $\delta^{(\pm\kappa)}_{\gamma ij}$ *be the phases of the bilinear forms* $G_{\alpha\beta ij}$, $W^{(\pm\kappa)}_{\alpha\beta ij}$ *and* $M^{(\pm\kappa)}_{\gamma ij}$ *(see eqs. (2a,b,c) respectively.*

Then, according to eqs. (5b,c,d,e), the bounds:

$$4H^{(\pm\kappa)}_{\alpha\beta} \leq -\lambda[G^{(\pm\kappa)}_{\alpha\beta}],\ \lambda[W^{(\pm\kappa)}_{\alpha\beta}] \leq 4H^{(\pm\kappa)}_{\alpha\beta} \text{ and } -\lambda^{(\pm)}_{\gamma\kappa} \geq 0 \qquad (21a)$$

are exactly saturated if and only if

$$[\phi^{(\pm\kappa)}_{\alpha\beta ij},\ \Phi^{(\pm\kappa)}_{\alpha\beta ij},\ \delta^{(\pm\kappa)}_{\gamma ij}] = n\pi,\ n=0,1,\ldots,\ \gamma=\alpha,\beta,\ldots \qquad (21b)$$

respectively. The $n\pi$ -phase contours lie on the zeros trajectories of the $\mathrm{Im}N_{ij}$ for $N_{ij} = G^{(\pm\kappa)}_{\alpha\beta ij}$, $W^{(\pm\kappa)}_{\alpha\beta ij}$ and $M^{(\pm\kappa)}_{\gamma ij}$, respectively. The zeros trajectories of $\mathrm{Im}N_{ij}$ are all independent of the channel indices (i,j).

Next, in order to obtain the explicit expressions of $H^{(\pm\kappa)}_{\alpha\beta}$ defined by eq. (3i) and the equalities (7), we have expressed the quantities $S_{\alpha\beta}$, $T_{\alpha\beta}$, $\vec{V}_{\alpha\beta}$ and $\vec{U}_{\alpha\beta}$ (see eqs. (3b,c,d,e)) in terms of the experimental observables. These results for $(\alpha,\beta)=(\alpha,\bar{\alpha})\equiv[\Sigma^{(+)},\Sigma^{(-)}],[\Omega^{(+)},\Omega^{(-)}]$ are given in table I. Then, using eqs. (3i), (7) and table I, we obtain

$$H^{(+\kappa)}_{\alpha\bar{\alpha}} + H^{(-\kappa)}_{\alpha\bar{\alpha}} = c_i^2 c_j^2 I_{0i} I_{0j}[1 - \vec{P}_i \cdot \vec{P}_j + (\vec{\kappa}\cdot\vec{A}_i)(\vec{\kappa}\cdot\vec{A}_j) - (\vec{\kappa}\cdot\vec{D}_{Xi})(\vec{\kappa}\cdot\vec{D}_{Xj}) - (\vec{\kappa}\cdot\vec{D}_{Yi})(\vec{\kappa}\cdot\vec{D}_{Yj}) - (\vec{\kappa}\cdot\vec{D}_{Zi})(\vec{\kappa}\cdot\vec{D}_{Zj})], \qquad (22a)$$

$$H^{(+\kappa)}_{\alpha\bar{\alpha}} - H^{(-\kappa)}_{\alpha\bar{\alpha}} = c_i^2 c_j^2 I_{0i} I_{0j}[\vec{\kappa}\cdot(\vec{A}_i + \vec{A}_j) - P_{Xi}(\vec{\kappa}\cdot\vec{D}_{Xj}) - P_{Xj}(\vec{\kappa}\cdot\vec{D}_{Xi}) - P_{Yi}(\vec{\kappa}\cdot\vec{D}_{Yj}) - P_{Yj}(\vec{\kappa}\cdot\vec{D}_{Yi}) - P_{Zi}(\vec{\kappa}\cdot\vec{D}_{Zj}) - P_{Zj}(\vec{\kappa}\cdot\vec{D}_{Zi})], \qquad (22b)$$

Table I

The explicit expressions of $S_{\alpha\bar{\alpha}}$, $T_{\alpha\bar{\alpha}}$, $\vec{V}_{\alpha\bar{\alpha}}$, $\vec{U}_{\alpha\bar{\alpha}}$, $G^{(\pm\kappa)}_{\alpha\bar{\alpha}}$, $W^{(\pm\kappa)}_{\alpha\bar{\alpha}}$ and $Z^{(0)}_{\alpha\bar{\alpha}}$ in terms of the experimental observables for $[\alpha,\bar{\alpha}]$ $[\Sigma^{(+)},\Sigma^{(-)}]$, $[\Omega^{(+)},\Omega^{(-)}]$ and $[\Lambda^{(+)},\Lambda^{(-)}]$.

$\alpha\bar{\alpha}$	$S_{\alpha\bar{\alpha}}$	$T_{\alpha\bar{\alpha}}$	$\vec{V}_{\alpha\bar{\alpha}}$	$\vec{U}_{\alpha\bar{\alpha}}$	$G^{(\pm\kappa)}_{\alpha\bar{\alpha}}$	$W^{(\pm\kappa)}_{\alpha\bar{\alpha}}$	$Z^{(0)}_{\alpha\bar{\alpha}}$
$\Sigma^{(+)}\Sigma^{(-)}$	P_x	P_y	D_{xx}, D_{yx}, D_{zx}	D_{xy}, D_{yy}, D_{yz}	$(1\pm\vec{x}\cdot\vec{A})I_0$	$(P_z\pm\vec{x}\cdot\vec{D}_z)I_0$	$P_z I_0$
$\Omega^{(+)}\Omega^{(-)}$	A_x	A_y	$D_{xx}, -D_{xy}, D_{xz}$	$D_{yx}, -D_{yy}, D_{yz}$	$(1\pm\vec{x}\cdot\vec{P})I_0$	$(A_z\pm\vec{x}\cdot\vec{D}_z)I_0$	$A_z I_0$
$\Delta^{(+)}\Delta^{(-)}$	A_x	D_{yz}	$P_x, -P_y, D_{zz}$	D_{zy}, D_{zx}, A_y	$(1\pm\vec{x}\cdot\vec{\Delta})I_0$ *)	$(D_{zz}\pm\vec{x}\cdot\vec{\Lambda})I_0$ *)	$D_{zz} I_0$

* The vectors $\vec{\Delta}$ and $\vec{\Lambda}$ are defined according to: $\frac{1}{2}(\xi_\alpha\sigma_\alpha \pm \xi_{\bar{\alpha}}\sigma_{\bar{\alpha}})$ resp., (see ref. [1]).

for $(\alpha,\bar\alpha)\equiv[\Sigma^{(+)},\Sigma^{(-)}]$, $(i,j)\equiv(1,2),(2,3),(3,1)$, and

$$H^{(+\kappa)}_{\alpha\bar\alpha}+H^{(-\kappa)}_{\alpha\bar\alpha}=c_i^2c_j^2 I_{0i}I_{0j}[1-\vec A_i\cdot\vec A_j+(\vec\kappa\cdot\vec P_i)(\vec\kappa\cdot\vec P_j)- \tag{22c}$$

$$-(\vec\kappa\cdot\vec D_{xi})(\vec\kappa\cdot\vec D_{xj})-(\vec\kappa\cdot\vec D_{yi})(\vec\kappa\cdot\vec D_{yj})-(\vec\kappa\cdot\vec D_{zi})(\vec\kappa\cdot\vec D_{zj})],$$

$$H^{(+\kappa)}_{\alpha\bar\alpha}-H^{(-\kappa)}_{\alpha\bar\alpha}=c_i^2c_j^2 I_{0i}I_{0j}[\vec\kappa\cdot(\vec P_i+\vec P_j)-A_{xi}(\vec\kappa\cdot\vec D_{xj})-A_{xj}(\vec\kappa\cdot\vec D_{xi})-$$

$$-A_{yi}(\vec\kappa\cdot\vec D_{yj})-A_{yj}(\vec\kappa\cdot\vec D_{yi})-A_{zi}(\vec\kappa\cdot\vec D_{zj})-A_{zj}(\vec\kappa\cdot\vec D_{zi})], \tag{22d}$$

for $(\alpha,\bar\alpha)\equiv[\Omega^{(+)},\Omega^{(-)}]$ and $(i,j)=(1,2),(2,3),(3,1)$, where using the components of the "depolarization tensor" we have defined the vectors

$$\vec D_u\equiv(D_{uX},D_{uY},D_{uZ}),\quad \vec D_U\equiv(D_{zU},D_{yU},D_{zU}) \tag{22e}$$

for $u\equiv x,y,z$ and $U\equiv X,Y,Z$.

We remark that eqs. (7) can be rewritten as equalities between quantities of form: $c_i^2c_j^2[H^{(+\kappa)}_{\alpha\beta ij}\pm H^{(-\kappa)}_{\alpha\beta ij}]$. The explicite expressions of these terms, for $\vec\kappa=(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ and $(\alpha,\bar\alpha)\equiv[\Sigma^{(+)},\Sigma^{(-)}],[\Omega^{(+)},\Omega^{(-)}]$ are presented in table II.

Therefore, a large number of isospin constraints can be obtained explicitly, using eqs. (7), (8a,b), the bounds (9a,b), (10) and eqs. (22a,b,c,d,e) by specializing the unit vectors $\vec\kappa$ and (α,β). For example, the bounds

$$4H^{(+\kappa)}_{\alpha\bar\alpha}\le-\lambda[G^{(+\kappa)}_{\alpha\bar\alpha}],\quad 4H^{(-\kappa)}_{\alpha\bar\alpha}\le-\lambda[G^{(-\kappa)}_{\alpha\bar\alpha}] \tag{23a}$$

can be combined to a stringent bound

$$[-4H^{(+\kappa)}_{\alpha\bar\alpha}-\lambda[G^{(+\kappa)}_{\alpha\bar\alpha}]]^{1/2}[-4H^{(-\kappa)}_{\alpha\bar\alpha}-\lambda[G^{(-\kappa)}_{\alpha\bar\alpha}]]^{1/2}+ \tag{23b}$$

$$+2(H^{(+\kappa)}_{\alpha\bar\alpha}+H^{(-\kappa)}_{\alpha\bar\alpha})\le-\lambda\left[\frac{G^{(+\kappa)}_{\alpha\bar\alpha}+G^{(-\kappa)}_{\alpha\bar\alpha}}{2}\right]-\lambda\left[\frac{G^{(+\kappa)}_{\alpha\bar\alpha}-G^{(-\kappa)}_{\alpha\bar\alpha}}{2}\right].$$

Table II

The quantities $c_i^2c_j^2[H^{(+\kappa)}_{\alpha\bar\alpha ij}\pm H^{(-\kappa)}_{\alpha\bar\alpha}]$ in terms of the experimental observables for $[\alpha,\bar\alpha]\equiv[\Sigma^{(+)},\Sigma^{(-)}][\Omega^{(+)},\Omega^{(-)}]$ and $\vec\kappa\equiv(1,0,0),(0,1,0),(0,0,1)$.

$\alpha\bar\alpha$	$\vec\kappa$*	$c_i^2c_j^2[H^{(+\kappa)}_{\alpha\bar\alpha ij}+H^{(-\kappa)}_{\alpha\bar\alpha ij}]$	$c_i^2c_j^2[H^{(+\kappa)}_{\alpha\bar\alpha ij}-H^{(-\kappa)}_{\alpha\bar\alpha ij}]$
	(1,0,0)	$c_i^2c_j^2[1+A_{xi}A_{xj}-\vec P_i\cdot\vec P_j-\vec D_{xi}\cdot\vec D_{xj}]I_{0i}I_{0j}$	$c_i^2c_j^2[A_{xi}+A_{xj}-\vec P_i\cdot\vec D_{xj}-\vec P_j\cdot\vec D_{xi}]I_{0i}I_{0j}$
$\Sigma^{(+)}\Sigma^{(-)}$	(0,1,0)	$c_i^2c_j^2[1+A_{yi}A_{yj}-\vec P_i\cdot\vec P_j-\vec D_{yi}\cdot\vec D_{yj}]I_{0i}I_{0j}$	$c_i^2c_j^2[A_{yi}+A_{yj}-\vec P_i\cdot\vec D_{yj}-\vec P_j\cdot\vec D_{yi}]I_{0i}I_{0j}$
	(0,0,1)	$c_i^2c_j^2[1+A_{zi}A_{zj}-\vec P_i\cdot\vec P_j-\vec D_{zi}\cdot\vec D_{zj}]I_{0i}I_{0j}$	$c_i^2c_j^2[A_{zi}+A_{zj}-\vec P_i\cdot\vec D_{zj}-\vec P_j\cdot\vec D_{zi}]I_{0i}I_{0j}$
	(1,0,0)	$c_i^2c_j^2[1+P_{Xi}P_{Xj}-\vec A_i\cdot\vec A_j-\vec D_{Xi}\cdot\vec D_{Xj}]I_{0i}I_{0j}$	$c_i^2c_j^2[P_{Xi}+P_{Xj}-\vec A_i\cdot\vec D_{Xj}-\vec A_j\cdot\vec D_{Xi}]I_{0i}I_{0j}$
$\Omega^{(+)}\Omega^{(-)}$	(0,1,0)	$c_i^2c_j^2[1+P_{Yi}P_{Yj}-\vec A_i\cdot\vec A_j-\vec D_{Yi}\cdot\vec D_{Yj}]I_{0i}I_{0j}$	$c_i^2c_j^2[P_{Yi}+P_{Yj}-\vec A_i\cdot\vec D_{Yj}-\vec A_j\cdot\vec D_{Yi}]I_{0i}I_{0j}$
	(0,0,1)	$c_i^2c_j^2[1+P_{Zi}P_{Zj}-\vec A_i\cdot\vec A_j-\vec D_{Zi}\cdot\vec D_{Zj}]I_{0i}I_{0j}$	$c_i^2c_j^2[P_{Zi}+P_{Zj}-\vec A_i\cdot\vec D_{Zj}-\vec A_j\cdot\vec D_{Zi}]I_{0i}I_{0j}$

* See eqs. (22e).

The sign of equality holds in the inequality (23b) if and only if

$$4[H_{a\bar{a}}^{(-\kappa)} - H_{a\bar{a}}^{(-\kappa)}] = -\lambda[G_{aa}^{(+\kappa)}] + \lambda[G_{a\bar{a}}^{(-\kappa)}]. \qquad (23c)$$

We note of course that in the derivation of eq. (23b) we have used the inequality: $a^{1/2} b^{1/2} \leq a+b, a \geq 0, b \geq 0$ for $a \equiv -4H_{a\bar{a}}^{(+\kappa)} - \lambda[G_{a\bar{a}}^{(+\kappa)}]$, $b = -4H_{a\bar{a}}^{(-\kappa)} - \lambda[G_{a\bar{a}}^{(-\kappa)}]$ and the identity

$$\lambda[G_{a\bar{a}}^{(+\kappa)}] + \lambda[G_{a\bar{a}}^{(-\kappa)}] = 2\lambda\left[\frac{G_{a\bar{a}}^{(+\kappa)} + G_{a\bar{a}}^{(-\kappa)}}{2}\right] + 2\lambda\left[\frac{G_{a\bar{a}}^{(+\kappa)} - G_{a\bar{a}}^{)(-\kappa)}}{2}\right]. \qquad (23d)$$

Now, using the explicit expressions of $G_{a\bar{a}}^{(\pm\kappa)}$ given in table I and the bound (23b), we obtain the following interesting constraints.

Consequence 2. The experimental data on unpolarized differential cross sections $I_{0\ell}$, *"polarization asymmetry* $\vec{A}_\ell$, *"final polarization"* $\vec{P}_\ell$ *and "depolarization tensor" components of three* $(0\ 1/2 \to 0'0'1'/2)$ *reactions related by isospin invariance, via two isospin channels, must obey the inequalities:*

$$\{-4H_{a\bar{a}}^{(+\kappa)} - \lambda[(1+\vec{\kappa}\cdot\vec{A})I_0]\}^{1/2}\{-4H_{a\bar{a}}^{(-\kappa)} - \lambda[(1-\vec{\kappa}\cdot\vec{A})I_0]\}^{1/2} + \qquad (24a)$$

$$+ 2(H_{a\bar{a}}^{(+\kappa)} + H_{a\bar{a}}^{(-\kappa)}) \leq -\lambda[I_0] - \lambda[\vec{\kappa}\cdot\vec{A}I_0], \quad (a,\bar{a}) \equiv [\Sigma^{(+)}, \Sigma^{(-)}],$$

$$\{-4H_{a\bar{a}}^{(+\kappa)} - \lambda[(1+\vec{\kappa}\cdot\vec{P})I_0]\}^{1/2}\{-4H_{a\bar{a}}^{(-\kappa)} - \lambda[(1-\vec{\kappa}\cdot\vec{P})I_0]\}^{1,2} + \qquad (24b)$$

$$2(H_{a\bar{a}}^{(+\kappa)} + H_{a\bar{a}}^{(-\kappa)}) \leq -\lambda[I_0] - \lambda[\vec{\kappa}\cdot\vec{P}I_0], \quad (a,\bar{a}) \equiv [\Omega^{(+)}, \Omega^{(-)}],$$

at any kinematical values in the physical domain and any unit vector $\vec{\kappa}$, where (see eq. (6a)

$$\lambda[(1 \pm \vec{\kappa}\cdot\vec{L})I_0] \equiv \lambda[c_1^2(1\pm\vec{\kappa}\cdot\vec{L}_1)I_{01}, c_2^2(1\pm\vec{\kappa}\cdot\vec{L}_2)I_{02}, c_3^2(1\pm\vec{\kappa}\cdot\vec{L}_3)I_{03}] \qquad (24c)$$

$$\lambda[\vec{\kappa}\cdot\vec{L}I_0] \equiv \lambda[c_1^2\vec{\kappa}\cdot\vec{L}_1 I_{01}, c_2^2\vec{\kappa}\cdot\vec{L}_2 I_{02}, c_3^2\vec{\kappa}\cdot\vec{L}_3 I_{03}], \qquad (24d)$$

for $\vec{L} \equiv \vec{A}, \vec{P}$, respectively. The quantities $H_{a\bar{a}}^{(\pm\kappa)}$ from eq. (24a,b) can be in general obtained from eqs. (22a,c) for any unit vector $\vec{\kappa}$. But, the inequalities (24a,b) for $\vec{\kappa} \equiv (1,0,0), (0,1,0)$ and $(0,0,1)$ are of great interest since giving only the unpolarized differential cross sections and a single component for each vector $\vec{A}_\ell$ or $\vec{P}_\ell$ $\ell = 1,2,3$, we can obtain strong constraints on the experimental data and amplitude analyses. Indeed, using the bound (24a,b) and the result from table II, we can prove the following interesting consequences.

Consequence 3. Let $\vec{D}_u$ *and* $\vec{D}_U$, $u \equiv x,y,z, U \equiv X,Y,Z$ *be the vectors defined by eq. (22e).*
Then,
1°) the exact saturation of the inequality

$$\lambda[A_u I_0] \leq -\lambda[I_0], \qquad (25a)$$

implies the equalities

$$\vec{P}_1 = \vec{P}_2 = \vec{P}_3, \qquad (25b)$$

$$\vec{D}_{u1} = \vec{D}_{u2} = \vec{D}_{u3}, \qquad (25c)$$

2°) The exact saturation of the bound

$$\lambda[P_U I_0] \leq -\lambda[I_0], \qquad (26a)$$

implies the constraints

$$\vec{A}_1 = \vec{A}_2 = \vec{A}_3 \qquad (26b)$$

$$\vec{D}_{U1} = \vec{D}_{U2} = \vec{D}_{U3}. \qquad (26c)$$

We see that eqs. (25b) , (26b) are independent of u and U respectively. Next, using eqs. (22a,c) and the bounds (24a,b) we can show that the equalities (25b) and (26b) can be obtained if the bounds $\lambda[\vec{\kappa}\cdot\vec{L}\, I_0] \le -\lambda[I_0]$ for $\vec{L} \equiv \vec{A}, \vec{P}$ respectively, are exactly saturated for an arbitrary unit vector $\vec{\kappa}$. We remark that the exact saturation of the bounds (25a) or (26a) implies also strong constraints on the phases of the corresponding bilinear forms $G^{(\pm\kappa)}_{\alpha\bar{\alpha}}$ since, then the bounds (23a) are also saturated (see eq. (5b) and (21a,b)).

Now, let $\sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}}$ and $\xi^{(\pm\kappa)}_{\alpha\bar{\alpha}}$ be the "polarized" differential cross sections and "spin-rotation" vectors defined in table III. Then, using table II, we observe that $H^{(+\kappa)}_{\alpha\bar{\alpha}}$ and $H^{(-\kappa)}_{\alpha\bar{\alpha}}$ can be rewritten in a form similar to H_α and $H_{\bar{\alpha}}$ (see eq. (13d))

$$H^{(\pm\kappa)}_{\alpha\bar{\alpha}} = \frac{1}{2} c_i^2 c_j^2 \left[1 - \xi^{(\pm\kappa)}_{\alpha\bar{\alpha}i}\cdot \xi^{(\pm\kappa)}_{\alpha\bar{\alpha}j}\right] \sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}i}\, \sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}j} \tag{27a}$$

where

$$\sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}\ell} = G^{(\pm\kappa)}_{\alpha\bar{\alpha}\ell} \tag{27b}$$

and $\xi^{(\pm\kappa)}_{\alpha\bar{\alpha}\ell}$ are defined in table III for $(\alpha,\bar{\alpha}) \equiv [\Sigma^{(+)},\Sigma^{(-)}]$, $[\Omega^{(+)},\Omega^{(-)}]$.

This observation is important for a proof of the consequences 3 and also to derive (just as in sect. 4 of ref. /1/) other interesting results. For example we obtain the following consequences.

Consequence 4. The exact saturation of the bound

$$4H^{(\pm\kappa)}_{\alpha\bar{\alpha}} \le -\lambda[G^{(\pm\kappa)}_{\alpha\bar{\alpha}}], \tag{28a}$$

implies the linear relation

$$c_1^2 \sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}1}\, \xi^{(\pm\kappa)}_{\alpha\bar{\alpha}1} [c_1^2 \sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}1} - c_2^2 \sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}2} - c_3^2 \sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}3}] +$$
$$c_2^2 \sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}2}\, \xi^{(\pm\kappa)}_{\alpha\bar{\alpha}2} [c_2^2 \sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}2} - c_3^2 \sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}3} - c_1^2 \sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}1}] + \tag{28b}$$

Table III

The "polarized" differential cross sections $\sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}}$ the "spin-rotation" vectors $\xi^{(\pm\kappa)}_{\alpha\bar{\alpha}}$ and $Z^{(\kappa)}_{\alpha\bar{\alpha}}$ for $[\alpha,\bar{\alpha}] \equiv [\Sigma^{(+)},\Sigma^{(-)}], [\Omega^{(+)},\Omega^{(-)}]$, and $\vec{\kappa} \equiv (1,0,0), (0,1,0), (0,0,1)$.

$\alpha\bar{\alpha}$	κ	$\sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}}$	$\xi^{(\pm\kappa)}_{\alpha\bar{\alpha}}\sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}}$	$Z^{(\kappa)}_{\alpha\bar{\alpha}}$
	(1,0,0)	$(1 \pm A_x) I_0$	$(\vec{P} \pm \vec{D}_x) I_0$ *)	$A_x I_0$
$\Sigma^{(+)}\Sigma^{(-)}$	(0,1,0)	$(1 \pm A_y) I_0$	$(\vec{P} \pm \vec{D}_y) I_0$	$A_y I_0$
	(0,0,1)	$(1 \pm A_z) I_0$	$(\vec{P} \pm \vec{D}_z) I_0$	$A_z I_0$
	(1,0,0)	$(1 \pm P_x) I_0$	$(\vec{A} \pm \vec{D}_x) I_0$	$P_x I_0$
$\Omega^{(+)}\Omega^{(-)}$	(0,1,0)	$(1 \pm P_y) I_0$	$(\vec{A} \pm \vec{D}_y) I_0$	$P_y I_0$
	(0,0,1)	$(1 \pm P_z) I_0$	$(\vec{A} \pm \vec{D}_z) I_0$	$P_z I_0$

*See eqs. (22e).

$$+ c_3^2 \sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}3} \xi^{(\pm\kappa)}_{\alpha\bar{\alpha}3} [c_3^2 \sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}} - c_1^2 \sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}1} - c_2^2 \sigma^{(\pm\kappa)}_{\alpha\bar{\alpha}2}] = 0,$$

between the "spin-rotation" vectors $\vec{\xi}^{(\pm\kappa)}_{\alpha\bar{\alpha}\ell}$, $\ell = 1,2,3$, *defined in table III.*

Next, let us define the bilinear forms (see ref. /1/)

$$Z^{(\kappa)}_{\alpha\bar{\alpha}ij} = \frac{1}{2}[Z^{(\kappa)}_{\alpha ij} + Z^{(\kappa)}_{\bar{\alpha}ij}], \quad Z^{(\kappa)}_{\alpha\bar{\alpha}\ell\ell} = \frac{1}{2}[G^{(+\kappa)}_{\alpha\bar{\alpha}} - G^{(-\kappa)}_{\alpha\bar{\alpha}}] \qquad (29a)$$

Then, it is easy to see that the isospin sum rules (4a) imply that

$$c_i^2 c_j^2 [\operatorname{Im} Z^{(\kappa)}_{\alpha\bar{\alpha}ij}]^2 = H - \frac{1}{2}[H^{(+\kappa)}_{\alpha\bar{\alpha}} + H^{(-\kappa)}_{\alpha\bar{\alpha}}] - \frac{1}{4}\lambda[Z^{(\kappa)}_{\alpha\bar{\alpha}}], \qquad (29b)$$

where H is defined by eq. (16). Hence, we obtain the following stringent constraints.

Cosntraints 3

$$\max_{\{ij\}} \{ -c_i^2 c_j^2 Z^{(\kappa)}_{\alpha ii} Z^{(\kappa)}_{\alpha jj} \} \le \frac{1}{4}\lambda [Z^{(\kappa)}_{\alpha\bar{\alpha}}] \le H - \frac{1}{2}(H^{(+\kappa)}_{\alpha\bar{\alpha}} + H^{(-\kappa)}_{\alpha\bar{\alpha}}) \qquad (30a)$$

$$\max\{ [-4H^{(+\kappa)}_{\alpha\bar{\alpha}} - \lambda[G^{(+\kappa)}_{\alpha\bar{\alpha}}]]^{1/2} [-4H^{(-\kappa)}_{\alpha\bar{\alpha}} - \lambda[G^{(-\kappa)}_{\alpha\bar{\alpha}}]]^{1/2},$$

$$2[-4H - \lambda[I_0]]^{1/2} [4H - 2(H^{(+\kappa)}_{\alpha\bar{\alpha}} + H^{(-\kappa)}_{\alpha\bar{\alpha}}) - \lambda[Z^{(\kappa)}_{\alpha\bar{\alpha}}]]^{1/2} \} +$$

$$+ 2[H^{(+\kappa)}_{\alpha\bar{\alpha}} + H^{(-\kappa)}_{\alpha\bar{\alpha}}] \le -\lambda[I_0] - \lambda[Z^{(\kappa)}_{\alpha\bar{\alpha}}] \qquad (30b)$$

where $Z^{(\kappa)}_{\alpha\bar{\alpha}}$ is given by eq. (29a) (see also table III). We note that in derivation of eq. (30b) we have used the results (18b), (23b) and (30 a). We observe that, the bounds (30a,b) for $\vec{\kappa} = (0,0,1)$ are just the inequalities (18a) and (20), respectively. Therefore, these results improve in the most general form all the bounds previously obtained in sect. 2 and ref. /1/.

The exact saturations of the above bounds are strictly connected with the zeros of the imaginary part of the corresponding bilinear forms (see eqs. (5b,c,d,e), (14 a, b,c,d) and (29b). We remark, that, *the Pomeranchuk-like theorems can also be improved (just as in sect. 4 of ref. /1/) using different stringent isospin bounds equivalent to the bounds (30b).*

Finally, let us discuss some implications of the lower bound (18b). We observe that this bound is similar to the bound derived by Doncel et al. /4/ for three $(0\,1/2 \to 0'1'/2)$ reactions connected by the isospin invariance via two isospin channels.

Therefore, the bound

$$4H \le -\lambda[I_0], \qquad (31a)$$

is equivalent to

$$[c_i^2 I_{0i} - c_j^2 I_{0j}]^2 + 4H \le 2c_k^2 I_{0k}[c_i^2 I_{0i} + c_j^2 I_{0j} - \frac{1}{2} c_k^2 I_{0k}]. \qquad (31b)$$

We see that, according to eq. (14a), *the sign of equality holds in the ineqialities (3la,b) if and only if* $\operatorname{Im} Z^{(0)}_{ij} = 0$ Then, the *constraints on experimental data are* (see eq. (17b))

$$4[H_\alpha - H_{\bar{\alpha}}] = \lambda(\sigma_\alpha) - \lambda(\sigma_{\bar{\alpha}}), \qquad (32)$$

valid for any $(\alpha, \bar{\alpha}) = [\Sigma^{(+)}, \Sigma^{(-)}], [\Omega^{(+)}, \Omega^{(-)}], [\Delta^{(+)}, \Delta^{(-)}]$.

The zeros trajectories of $\operatorname{Im} Z^{(0)}_{ij}$, as well as the zeros of all the other bilinear forms (see eqs. (5b,c,d,e), (14a,b,c,d) and (29b)), are independent of the channel indices (i,j). The bilinear forms $Z^{(0)}_{ij}$, defined by eq. (11a), are invariant under rotations of the spin reference frame.

Therefore, using the bounds (30b) or (31) we can obtain other proofs of the Pomeranchuk-like theorems (see ref. /1/).

4. Conclusions

In this paper, as a continuation of ref. [1], we have investigated the constraints on experimental data and amplitude analysis of three $(0\,1/2 \to 0'0'1'/2)$ reactions related by isospin invariance via two isospin channels. So, in sect. 2, using the generalized helicity amplitudes $F^{(+\kappa)}_{\gamma\ell}$ (see also ref. [1,5,6]) and the bilinear forms (2a,b,c,d) and (11a,b,c), *we have proved that the isospin sum rules (4a) alone imply the equalities (7), (8a,b), (16), (17a,b) and the inequalities (9a,b), (10), (18a,b) and (20) valid for any values of kinematical variables in the physical domain and for any unit vector* $\vec{\kappa}$.

These results are presented in a general and compact form and are sufficient to obtain any particular constraints (equalities or bounds) by specializing the unit vectors $\vec{\kappa}$ and the indices (α,β) or $(\alpha,\bar{\alpha})$. A large number of isospin constraints can be written in an explicit form by using the results presented in sect. 3 and tables I, II, III. For example, a number of 36 equalities can be written explicitly by using the relations of form

$$c_k^2 c_\ell^2 [H^{(+\kappa)}_{\alpha\bar{\alpha}k\ell} \pm H^{(-\kappa)}_{\alpha\bar{\alpha}k\ell}] = c_n^2 c_m^2 [H^{(+\kappa)}_{\alpha\bar{\alpha}nm} \pm H^{(-\kappa)}_{\alpha\bar{\alpha}nm}] \qquad (32)$$

$k \neq \ell$, $n \neq m$, $k,\ell,m,n = 1,2,3$, and the results of table II. Hence, the results presented (this paper and ref. [1]) are sufficient to obtain certain tests of the isospin invariance and to determine the breaking effects (e.g., the indirect effects due to mass and width differences, mixing between Δ'_s and N^*_{1s}, differences in coupling constants, etc.), when the experimental data on differential cross sections and spin density matrices are available. On the other hand, our results will be useful for determination of all constraints on experimental data and amplitude analyses when different weak isospin bounds, such as: $-\lambda[I_0] \geq 0$, or $\lambda[\kappa \cdot \vec{L}\, \vec{I}_0] \leq -\lambda[I_0]$ $\vec{L} \equiv \vec{A}, \vec{P}, \ldots,$ are exactly saturated or degenerated. Moreover, these results enable us to understand the small differences between diffractive cross sections at high energies in terms of small charge exchange cross sections and also to prove the Pomeranchuk-like theorem for: differential cross sections, "asymmetry" parameters, polarization and "depolarization tensor" components, in the region of the asymptotic energies.

Finally, we note that all the results obtained here can be extended to cases when zero-spin particles are replaced by unpolarized J -spin particles.

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