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# ASYMPTOTIC BEHAVIOUR OF FORM FACTOR AND INVARIANT DESCRIPTION OF PARTICLE SPATIAL DISTRIBUTION

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S U M M A R Y

It is shown that transition from the standard parametrization of electormagnetic current to the parametrization by the relativistic spin vector allows one to make a physical interpretation of a form factor in the rest frame of a particle (and not necessarily in the Breit frame!). A spatial distribution of particles is described in the configurational representation to which one goes over using expansions over unitary representations of the Lorentz group. For the proton form factor we have found a formula which gives the correct "almost dipole" asymptotical behaviour for its form factor.

## Introduction

The authors of paper /1/ have proposed a new relativistic generalization of relative coordinate which allows one to go over to the three-dimensional description of the relativistic two-body problem /2/. An analogous mathematical technique has been used in /3/ to describe the particle form factor,however,the meaning of a parameter N as a coordinate has not been found out. This has not allowed the author of /3/ to obtain the physical consequences from this approach. In paper /4/ the relativistic coordinate characterizing the proton distribution has been related to a rather important proton characteristic: its mean-square radius. It has been shown also that a new coordinate introduced in /1/ describes the proton distribution only at distances larger than its Compton wave length.

Besides, in /4/ it has been established that since the relativistic coordinate modulus is a relativistic invariant there is no need to go over to the Breit frame for three-dimensional spatial description of a particle distribution.

The present paper is a sequel to paper /4/ . In the first part we show that the Breit frame is not necessary to find out the physical meaning of the Sachs form factors. A transition to the parametrization of electromagnetic current by the relativistic spin vector /5,6/(the Pauli-Lubanski, or Bargmann-Shirokov vector) allows their direct interpretation in the rest frame of a particle itself.

In the second part we introduce the description of particle distribution in the relativistic coordinate space and present a new invariant definition of the particle mean-square radius. In the third sect. a new coordinate $\rho$ is introduced which describes distances smaller than the Compton wave length of a particle. A simple model is proposed based on the vector dominance model with allowing for a contribution from the particle central part. This model gives the correct "almost dipole" behaviour for the nucleon form factor.

## 2. Transition from the Standard Electromagnetic-Current Parametrization to that by the Relativistic Spin Vector.

The nucleon electromagnetic current, in quantum field theory, is given by the expression

$$j^{\mu}_{\sigma\sigma'}(\vec{p},\vec{k}) = e\,\bar{u}^{\sigma}(\vec{p})\left\{\gamma^{\mu}F_1(q^2) + \frac{\sigma^{\mu\nu}q_{\nu}}{2M}F_2(q^2)\right\}u^{\sigma'}(\vec{k})$$

$$\sigma^{\mu\nu} = \frac{\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}}{2}, \tag{1}$$

where $u^{\sigma}(\vec{p})$ are nucleon bispinors normalized by the condition $\bar{u}^{\sigma}(\vec{p})u^{\sigma'}(\vec{p}) = 2M\delta_{\sigma\sigma'}$ and $F_1(q^2)$ and $F_2(q^2)$ are the Dirac and Pauli form factors, resp., which depend on the 4-momentum transfer squared $t = q^2 = (p-k)^2$. The matrix $S_p$ of bispinor transformation from the rest frame $u^{\sigma}(\vec{p}) = S_p\,u^{\sigma}(0)$ has the form

$$S_p = \sqrt{\frac{p_0+M}{2M}}\left(1 + \frac{\vec{\alpha}\vec{p}}{p_0+M}\right) = \operatorname{ch}\chi_{p/2} + \vec{\alpha}\,\vec{n}_p\operatorname{sh}\chi_{p/2}, \tag{2}$$

where

$$p_0 = M\operatorname{ch}\chi_p \qquad \vec{n}_p = \frac{\vec{p}}{|\vec{p}|}$$
$$\vec{p} = \vec{n}_p M\operatorname{sh}\chi_p \qquad \vec{\alpha} = \gamma_0\vec{\gamma}. \tag{3}$$



By analogy with procedure in /4,8/ in eq. (1) we go over to the bispinors defined in the rest frame*):

$$j^{\mu}_{\sigma\sigma'}(\vec{p},\vec{k}) = e\,\bar{u}^{\sigma}(0)\,S_p^{-1}\left\{\gamma^{\mu}F_1(q^2) + \frac{\sigma^{\mu\nu}}{2M}q_{\nu}F_2(q^2)\right\}\cdot$$
$$\cdot S_p\,S_{\Lambda_p^{-1}k}\cdot D^{1/2}\{V^{-1}(\Lambda_p,k)\}\,u^{\sigma'}(0). \tag{4}$$

To obtain $S_{\Lambda_p^{-1}k}$ in (4) we have used the definition of the Wigner rotation

$$S_p^{-1}S_k = S_{\Lambda_p^{-1}k}\cdot D^{1/2}\{V^{-1}(\Lambda_p,k)\}. \tag{5}$$

Now we employ the formula from /8/

$$S_p^{-1}\gamma^{\mu}S_p = \frac{1}{M}\left\{p^{\mu} + 2\gamma^5 W^{\mu}(\vec{p})\right\}, \tag{6}$$

where $W^{\mu}(\vec{p})$ is the relativistic spin vector [5] (the Pauli--Lubanski or Bargman-Shirokov vector)**). As has been shown in /8/ eq. (6), with (2) considered, admits the part of current of (4) with $F_1(q^2)$ to be written as follows:

$$\bar{u}^{\sigma}(\vec{p})\gamma^{\mu}u^{\sigma'}(\vec{k}) =$$
$$= 2\sum_{\sigma_p=-1/2}^{1/2}\xi^{*\sigma}\left\{p^{\mu}\sqrt{1-t/4M^2} + \frac{2}{M^2}\frac{W^{\mu}(\vec{p})\left(W^{\nu}(\vec{p})\,q_{\nu}\right)}{\sqrt{1-t/4M^2}}\right\}\xi^{\sigma_p}D^{1/2}_{\sigma_p\sigma'}\{V^{-1}(\Lambda_p,k)\}.$$

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*) In the standard representation, where $\gamma_0 = \begin{pmatrix}1&0\\0&1\end{pmatrix}$; $\vec{\gamma} = \begin{pmatrix}0&\vec{\sigma}\\-\vec{\sigma}&0\end{pmatrix}$; $u^{\sigma}(0) = \sqrt{2M}\begin{pmatrix}\xi^{\sigma}\\0\end{pmatrix}$, in the spinor one, where $\gamma_0 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$; $\vec{\gamma} = \begin{pmatrix}0&-\vec{\sigma}\\\vec{\sigma}&0\end{pmatrix}$; $u^{\sigma}(0) = \sqrt{M}\begin{pmatrix}\xi^{\sigma}\\\xi^{\sigma}\end{pmatrix}$. The two-component spinors obey the normalization condition $\xi^{*\sigma}\xi_{\sigma'} = \delta_{\sigma\sigma'}$.

**) In the rest frame $(\vec{p} = 0)$ we have

$$W^0(0) = 0 \quad ; \quad \vec{W}(0) = M\frac{\vec{\sigma}}{2}. \tag{7}$$

Therefore the relativistic spin vector defined as $W^{\mu}(\vec{p}) = (\Lambda_p)^{\mu}_{\nu}W^{\nu}(0)$ has the following components

$$W^0(\vec{p}) = \frac{\vec{\sigma}\vec{p}}{2} \quad ; \quad \vec{W}(\vec{p}) = M\frac{\vec{\sigma}}{2} + \frac{\vec{p}\left(\frac{\vec{\sigma}}{2}\vec{p}\right)}{p_0+M}. \tag{8}$$

Making use of eq. (6) we also obtain

$$S_p^{-1}\sigma^{\mu\nu}S_p = \frac{2}{M^2}\left\{\gamma^5\left[p^\mu W^\nu(\vec{p}) - W^\mu(\vec{p})p^\nu\right] - 2\Sigma^{\mu\nu}(\vec{p})\right\}, \qquad (9)$$

where the quantity

$$\Sigma^{\mu\nu}(\vec{p}) = \frac{W^\mu(\vec{p})W^\nu(\vec{p}) - W^\nu(\vec{p})W^\mu(\vec{p})}{2} \qquad (10)$$

is constructed by analogy with $\sigma^{\mu\nu}$ , but $W^\mu(\vec{p})$ are taken instead of $\gamma$-matrices. Note that in contrast to $\gamma$ -matrices the vector $W^\mu(\vec{p})$ enters into an algebra of the Poincaré group and is related to observables directly: its square gives the particle spin by the formula $W^2 = -M^2 s(s+1)$. On substituting (6) and (9) into (1) and using the obtained in /8/ expression

$$W^\mu(\vec{p})W^\nu(\vec{p}) = \frac{1}{4}\left(p^\mu p^\nu - M^2 g^{\mu\nu}\right) + \Sigma^{\mu\nu}(\vec{p}) \qquad (11)$$

we arrive at the current (1) of the following form

$$j^\mu_{\sigma\sigma'}(\vec{p},\vec{\kappa}) = \frac{e}{\sqrt{1-t/4M^2}}\cdot\sum_{\sigma_p=-1/2}^{1/2}\overset{*}{\xi}{}^{\sigma}\Big\{(p+\kappa)^\mu G_E(t) + \frac{4}{M^2}\Sigma^{\mu\nu}(\vec{p})q_\nu G_M(t)\Big\}\xi_{\sigma_p} D^{1/2}_{\sigma_p\sigma'}\{V^{-1}(\Lambda_p,\kappa)\}. \qquad (12)$$

Then, the arising combinations $G_E(t) = F_1(t) + \frac{t}{4M^2}F_2(t)$ and $G_M(t) = F_1(t) + F_2(t)$ are, resp., the Sachs electric and magnetic form factors. A general parametrization of the currents by the relativistic spin vector which is valid for particles of arbitrary spin has been derived in paper /6/. The derivation of (12) can be regarded as a method for obtaining such a parametrization from the current of form (1) standard in quantum field theory.

Now let us find out what is a form of the interaction of nucleon with an external field if the expression (12) is accepted for the current. If one takes into account the invariant gauge



condition $q^\mu A^{ext}_\mu(q) = 0$, the energy of interaction of a nucleon with an external field takes the form

$$E = -j^\mu_{\sigma\sigma'}(\vec{p},\vec{\kappa})A^{ext}_\mu(q) = \frac{2}{\sqrt{1-t/4M^2}}\sum_{\sigma_p=-1/2}^{1/2}\overset{*}{\xi}{}^{\sigma}\Big\{e\left(p^\mu A_\mu(q)\right)G_E(t) + \frac{2e}{M^2}\Sigma^{\mu\nu}(\vec{p})A_\nu(q)q_\mu G_M(t)\Big\}\xi_{\sigma_p} D^{1/2}_{\sigma_p\sigma'}\{V^{-1}(\Lambda_p,\kappa)\}. \qquad (13)$$

Here the following should be recalled: In the Dirac equation, the term describing the interaction with electromagnetic field $\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}$ is transformed to the form $\sigma^{\mu\nu}F_{\mu\nu} = \vec{\Sigma}\vec{H} - i\vec{\alpha}\vec{E}$ by passing to the three-dimensional vectors $\vec{\Sigma} = -\vec{\alpha}\gamma_5$ and $\vec{\alpha} = \gamma^0\vec{\gamma}$ constructed of components of tensor $\sigma^{\mu\nu}$ . We shall make an analogous procedure in (13). To this end, we construct two-component analogs of $\vec{\Sigma}$ and $\vec{\alpha}$ but now from components of the tensor $\Sigma^{\mu\nu}(\vec{p})$ :

$$\vec{m}(\vec{p}) = \left(\Sigma^{32}(\vec{p}), \Sigma^{13}(\vec{p}), \Sigma^{21}(\vec{p})\right) \qquad (14)$$

$$\vec{a}(\vec{p}) = \left(\Sigma^{01}(\vec{p}), \Sigma^{02}(\vec{p}), \Sigma^{03}(\vec{p})\right). \qquad (15)$$

Then, by using the definition of the field strength vectors

$$\vec{H} = (F_{32}, F_{13}, F_{21}) \quad ; \quad \vec{E} = (F_{01}, F_{02}, F_{03}) \ ,$$

where following /9/ we put

$$F_{\mu\nu}(q) = q_\mu A_\nu(q) - q_\nu A_\mu(q) \ ,$$

the expression $\Sigma^{\mu\nu}(\vec{p})A_\mu q_\nu = \frac{1}{2}\Sigma^{\mu\nu}(\vec{p})F_{\mu\nu}$ entering into (13) can be reduced to the conventional form with separated electric and magnetic field interactions

$$\frac{e}{2}\Sigma^{\mu\nu}(\vec{p})F_{\mu\nu} = e\left(\vec{m}(\vec{p})\vec{H}\right) + e\left(\vec{a}(\vec{p})\vec{E}\right). \qquad (16)$$

As a result, (13) can be represented in the form

$$E = -j^\mu_{\sigma\sigma'}(\vec{p},\vec{\kappa})A_\mu(q) = -\frac{2}{\sqrt{1-t/4M^2}}\sum_{\sigma_p=-1/2}^{1/2}\overset{*}{\xi}_{\sigma}\Big\{e\left(p^\mu A_\mu(q)\right)G_E(t) + \frac{e}{M^2}\left(\vec{m}(\vec{p})\vec{H}\right)G_M(t) + \frac{e}{M^2}\left(\vec{a}(\vec{p})\vec{E}\right)\cdot G_M(t)\Big\}\xi_{\sigma_p} D^{1/2}_{\sigma_p\sigma'}\{V^{-1}(\Lambda_p,\kappa)\}. \qquad (17)$$

This general expression now will be considered for various cases.

Let the external field be of the pure magnetic nature, i.e., $A^{ext}_0 = \varphi = 0$ . Then from (17) we have

$$E = j^{\mu}_{\sigma\sigma'}(\vec{p},\vec{\kappa})A_{\mu}(q) = \frac{2}{\sqrt{1-t/4M^2}}\sum_{\sigma_p=-\frac{1}{2}}^{1/2}\xi^{*\sigma}\{-e(\vec{p}\vec{A}^{ext}(q))G_E(t) + \frac{e}{M^2}(\vec{\mathfrak{m}}(\vec{p})\vec{H})G_M(t)\}\xi_{\sigma_p}D^{1/2}_{\sigma_p\sigma'}\{V^{-1}(\Lambda_p,\kappa)\}. \quad (18)$$

As was noted in /9/, the first term of (18) with the external potential does not relate to the magnetic moment of a particle. Therefore the vector $\frac{e}{M^2}\vec{\mathfrak{m}}(\vec{p})$ constructed by formula (14) of the relativistic spin vectors $W^{\mu}(\vec{p})$ can be treated as the magnetic moment of a spinor particle moving with momentum $\vec{p}$. The function $D^{1/2}\{V^{-1}(\Lambda_p,\kappa)\}$ including the Wigner rotation $V(\Lambda_p,\kappa)$ describes the Tomas precession of spin resulting from the particle momentum change when interacting with the external field.

Due to the following from (14), (15) equalities

$$\vec{\mathfrak{m}}(0) = M^2\frac{\vec{\sigma}}{2}\ ; \quad \vec{a}(0) = 0$$

expression (17) takes the most simple form

$$E = -j^{\mu}(\vec{p},\vec{\kappa})A_{\mu}(q)/_{\vec{p}=0} = = \frac{2M}{\sqrt{1-t/4M^2}}\{e\Phi\cdot G_E(t) + \frac{e}{2M}(\vec{\sigma}\vec{H})G_M(t)\} \quad (19)$$

in the system where the nucleon was at rest before interaction, $\vec{p}=0,\ D^{1/2}\{V(\Lambda_p,\kappa)\}_{\vec{p}=0}=1$. Equation (19) demonstrates the fact that in the system, where the particle was at rest before interaction, the Sachs form factors $G_E(t)$ and $G_M(t)$ really describe "the charge density" and "magnetic moment distribution" of the nucleon. Note that such an interpretation is achieved without using the usual Breit frame.

## 3. Description of the Particle Distribution in the Relativistic Configurational Space.

The nucleon current parametrization (12) is used usually to interprete the form factors $G_E(t)$ and $G_M(t)$ as charge and magnetic moment distributions, resp., of a particle by means of the transition to the coordinate space in the Breit frame.



In this frame $\vec{p} = -\vec{\kappa}$ and because of that the time component of 4-vector of the momentum transfer $q = p - \kappa$ turns into zero $q_0 = p_0 - \kappa_0 = 0$ and, consequently, $F(t) = F(-\vec{q}^{\,2})$. Thus the 4-dimensional Fourier transformation reduces to the 3-dimensional one /10/

$$f(r) = \frac{1}{(2\pi)^3}\int d\vec{q}\ e^{-i\vec{q}\vec{r}}\ F(-\vec{q}^{\,2}). \quad (20)$$

However, as is well known, such a form of the spatial description of the nucleon distribution is not satisfactory since in the Breit frame the nucleon itself is moving, i.e., its internal motion and translational motion as a whole are not separated.

In paper /4/ it has been shown that this difficulty disappears if the nucleon spatial distribution is described in a new relativistic coordinate representation introduced in /1, 2/. A preliminary remark should be made that in the momentum space the three-dimensional description can be introduced in any coordinate frame if the language of the Lobachevsky space is used. Indeed, in (1) and (12) the momenta $\vec{p}$ and $\vec{k}$ are on the mass shell

$$p_0^2 - \vec{p}^{\,2} = M^2. \quad (21)$$

Equation (21) is the equation of hyperboloid on the upper sheet of which the Lobachevsky space is just realized.

The vector $\vec{\Delta} = \vec{p}(-)\vec{\kappa}$ **-the difference in the Lobachevsky space**

$$\vec{\Delta} \equiv \vec{p}(-)\vec{\kappa} = \overrightarrow{(\Lambda_{\kappa}^{-1}p)} = \vec{p} - \frac{\vec{\kappa}}{M}\left(p_0 - \frac{\vec{p}\vec{\kappa}}{\kappa_0+M}\right)$$

$$\Delta_0 \equiv (p(-)\kappa)_0 = (\Lambda_{\kappa}^{-1}p)^0 = \sqrt{M^2+\vec{\Delta}^2} = \frac{p_0\kappa_0 - \vec{p}\vec{\kappa}}{M} \quad (22)$$

can be considered as a relativistic generalization of the 3-dimensional vector of the nonrelativistic momentum transfer $\vec{q} = \vec{p} - \vec{\kappa}$. In the nonrelativistic limit, when the curvature of the Lobachevsky space tends to zero and it turns into the flat 3-dimensional

Euclidean space, the vector $\vec{\Delta}=\vec{p}(-)\vec{k}\rightarrow\vec{q}=\vec{p}-\vec{k}$. The four dimensional momentum transfer vector squared, as one can easily verify with the aid of (22), in any coordinate system can be expressed through $\vec{\Delta}^2$ by the formula /1/

$$t=(p-k)^2=2M^2-2M\sqrt{M^2+\vec{\Delta}^2}. \tag{23}$$

Consequently, in any reference frame a form factor $F(t)$ can be parametrized by the square of the three-dimensional momentum transfer $\vec{\Delta}^2=(\vec{p}(-)\vec{k})^2$ of the Lobachevsky space; $F(t)=F(\vec{\Delta}^2)$ *).

The relativistic coordinate space is introduced as canonically conjugate to the momentum space which geometry is the Lobachevsky geometry. The group of motion of the Lobachevsky space is the Lorentz group. Therefore, for transition to the relativistic coordinate representation the expansion over the principal series of unitary irreducible representations of the Lorentz group /1/, /2/ is used instead of the usual Fourier transformation. The mathematical aspect of the expansion procedure over the Lorentz group is well known /11/, /12/, /15/ . In papers /1/, /2/ this apparatus has been employed in the form developed in /13/. Due to the spherical symmetry of a form factor $F(t)$ such a transformation has the form /1,2/

$$F(r)=\frac{1}{2\pi^2}\int_0 \frac{\sin rM\chi}{rM\,sh\chi}\,F(\vec{\Delta}^2)\,sh^2\chi\,d\chi. \tag{24}$$

The hyperbolic angle $\chi$ parametrizing the vector $\vec{\Delta}=\vec{p}(-)\vec{k}$ in the spherical coordinates

$$\Delta_0=M\,ch\chi\;;\quad \vec{\Delta}=\vec{n}_\Delta M\,sh\chi\;;\quad \vec{n}_\Delta=\frac{\vec{\Delta}}{|\vec{\Delta}|}$$

is called "rapidity". The inverse to (24) transformation

$$F(t)=4\pi\int_0^\infty \frac{\sin rM\chi}{rM\,sh\chi}\,F(r)\,r^2dr \tag{25}$$

has the property

$$F(0)=4\pi\int_0^\infty F(r)\,r^2dr$$

due to the equality $t\big|_{\vec{\Delta}^2=0}=0$ resulting from (23).

In the Lorentz group there is the invariant Casimir operator $\hat{C}$

$$\hat{C}=-\frac{1}{4}M_{\mu\nu}M^{\mu\nu}=\vec{N}^2-\vec{M}^2=-\frac{1}{M^2}\frac{\partial^2}{\partial\chi^2}-\frac{2\,cth\chi}{M^2}\frac{\partial}{\partial\chi}-\frac{\Delta_{\theta,\varphi}}{M^2sh^2\chi}. \tag{26}$$

Its eigenvalues on the functions $\left(\frac{\sin rM\chi}{rM\,sh\chi}\right)$, which are elementary spherical functions of the principal series of unitary irreducible representations of the Lorentz group, are determined through the squared relativistic relative coordinate $r$ in the following way

$$\hat{C}\left(\frac{\sin rM\chi}{rM\,sh\chi}\right)=\left(\frac{1}{M^2}+r^2\right)\left(\frac{\sin rM\chi}{rM\,sh\chi}\right). \tag{27}$$

As the operator $\hat{C}$ is relativistic invariant, the modulus of the relativistic coordinate $r$ is a relativistic invariant too. In this way the function $F(r)$ in (24) gives the invariant description of the nucleon spatial distribution in any reference frame (and not only in the Breit one). Hence, the new description with the use of $F(r)$ is applicable also in the rest frame of a nucleon. To ascertain the physical meaning of the new coordinate we relate it to the important characteristic of a particle: its mean-square radius. It should be remarked beforehand that its definition by (20) in the Breit frame which has the nonrelativistic form

$$\langle r_0^2\rangle_{B.f.}=\frac{-6\,\frac{\partial F(-\vec{q}^2)}{\partial\vec{q}^2}\big|_{\vec{q}^2=0}}{F(0)}=\frac{\left\{\left(i\frac{\partial}{\partial\vec{q}}\right)^2F(-\vec{q}^2)\right\}\big|_{\vec{q}^2=0}}{F(0)}= \tag{28}$$

$$=\frac{\int r^2 f(r)\,d\vec{r}}{\int f(r)\,d\vec{r}}\;,$$

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*) For the first time form factors were parametrized by using the Lobachevsky space in paper /17/. The auther of /17/ like those of /3/ and /13/ have also used the expansion of the form (24) but they have not given the parameter in /24/ the sense of relative coordinate.

from the group-theoretical point of view has the meaning of expectation value of the eigenvalue of the Casimir operator of the Euclide group $\hat{C}_E = (i\frac{\partial}{\partial \vec{q}})^2$, the group of motion of the flat Euclidean space. The eigenvalues of the operator $(i\frac{\partial}{\partial \vec{q}})^2$ on the functions $e^{i\vec{q}\vec{r}}$ (or on the zero-order Bessel functions $\frac{\sin rq}{rq}$ ) which realize the unitary representations of the Euclide group, are the square of the nonrelativistic coordinate $r^2$

$$(i\frac{\partial}{\partial \vec{q}})^2 e^{i\vec{q}\vec{r}} = r^2 e^{i\vec{q}\vec{r}} .$$

By using (26) it can be checked easily that the usual formal invariant definition of the mean-square radius $\langle r_0^2 \rangle = \frac{6\frac{\partial F(t)}{\partial t}/_{t=0}}{F(0)}$ also has a group meaning: the meaning of expectation value of an eigenvalue of the Casimir operator of the Lorentz group /4/

$$\langle r_0^2 \rangle = \frac{6\frac{\partial F(t)}{\partial t}/_{t=0}}{F(0)} = \frac{\{\hat{C}\, F(t)\}/_{t=0}}{F(0)} . \qquad (29)$$

By virtue of (29) and (28) this definition can be written in the form

$$\langle r_0^2 \rangle = \frac{6\frac{\partial F(t)}{\partial t}/_{t=0}}{F(0)} = \frac{-\frac{6}{M^2}\frac{\partial F(t)}{\partial \chi^2}/_{\chi^2=0}}{F(0)} \qquad (30)$$

of direct geometrical generalization of the nonrelativistic definition of the mean-square radius (28), obtained by changing the modulus of vector of the momentum transfer $q$ by the corresponding rapidity $\chi = Ar\,ch(\frac{2M^2-t}{2M^2})$.

In terms of the invariant function $F(r)$ the mean-square radius, according to (27), has the following form *)

$$\langle r_0^2 \rangle = \frac{6\frac{\partial F(t)}{\partial t}/_{t=0}}{F(0)} = \frac{\hbar^2}{M^2c^2} + \frac{\int r^2 F(r) d\vec{r}}{\int F(r) d\vec{r}} = \frac{\hbar^2}{M^2c^2} + \langle r^2 \rangle . \qquad (31)$$

In the nonrelativistic limit the Casimir operator of the Lorentz

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*) Here we again write $\hbar$ and $c$ for more clear presentation.



group $\hat{C} = \vec{N}^2 - \vec{M}^2$ reduces to the Casimir operator of the group of motion of the Euclidean space $\hat{C} \xrightarrow[c\to\infty]{} \hat{C}_E = (i\frac{\partial}{\partial \vec{q}})^2$ and exp. (31) turns into (28). Thus, it can be said that in the relativistic generalization the group-theiretial **meaning** of the mean-square radius of a particle is conserved.

From expression (31) it follows that for particles for which the Compton wave length squared is small as compared with the experimentally measured value of the mean-square radius $\langle r_0^2 \rangle$ the quantity $\langle r^2 \rangle$ should be positive. An example of such particles is a proton. For a proton, as follows from (31), the new coordinate and the function $F(r)$ describe not the whole size but only the region outside a sphere of the radius equal to the Compton wave length of a nucleon. This result is consistent with the Newton-Wigner conclusion that the relativistic particle cannot be localized in the space with accuracy better than its Compton wave length /14/. Besides, in our approach there is a rather definite prediction on a magnitude of the contribution to the form factor from the central part which is not described by the coordinate $r$ .

Indeed, to the sphere with $R = \frac{\hbar}{Mc}$ there corresponds $r = 0$, or $F(r) = \frac{\delta(r)}{4\pi r^2}$ . Substituting this function into (25) gives the following form factor:

$$F_{R=\frac{1}{Mc}}(t) = \frac{\sin rM\chi}{rM\, sh\chi}/_{r=0} = \left(\frac{\chi}{sh\chi}\right) = \frac{2M^2 \ln\left(1-\frac{t}{2M^2}+\frac{1}{2M^2}\sqrt{t(t-4M^2)}\right)}{\sqrt{t(t-4M^2)}} \qquad (32)$$

corresponding to the contribution of the central sphere $R = \frac{\hbar}{Mc}$ .

Accordingly, the standard form factor can be represented in the form

$$F(t) = \left(\frac{\chi}{sh\chi}\right) \Phi(\chi), \qquad (33)$$

where the "external" form factor $\Phi(\chi)$ obeying the same normalization $\Phi(0) = F(0) = 1$ corresponds to the nucleon distribution outside the sphere with $R = \frac{\hbar}{Mc}$ . Such a factoring of the standard nucleon form factor into the factors which correspond to contributions from the central and "external" regions is obtained according to their contributions to the mean-square radius of a particle. Indeed, using (29), (30) one can easily see that

$$\frac{\partial \left(\frac{\chi}{sh\chi}\right)}{\partial t}\Big/_{t=0} = \frac{\hbar^2}{M^2c^2} \quad ; \quad \frac{\frac{\partial \Phi(\chi)}{\partial t}\Big/_{t=0}}{F(0)} = \frac{\int r^2 F(r) d\vec{r}}{\int F(r) d\vec{r}} = \langle r^2 \rangle . \quad (34)$$

It is important to note that the central region with $R = \frac{\hbar}{Mc}$ and the corresponding contribution $\left(\frac{\chi}{sh\chi}\right)$ have no nonrelativistic analogs since as $c \to \infty$ $R = \frac{\hbar}{Mc} \xrightarrow[c\to\infty]{} 0$ and $\left(\frac{\chi}{sh\chi}\right) \to 1$. Hence, only the "external" form factor $\Phi(\chi)$ can be considered as a direct relativistic generalization of nonrelativistic form factors. It is interesting that in terms of the "external" form factor $\Phi(\chi)$ the transformations (24), (25) look like the usual transformations with the Bessel functions of zeroth order

$$F(r) = \frac{1}{2\pi^2} \int_0 \frac{\sin rM\chi}{rM\chi} \Phi(\chi) \chi^2 d\chi . \quad (24a)$$

The relativistic coordinate here, however, in distinction with the nonrelativistic case, is conjugated not to the modulus of the momentum transfer $q$ but to the rapidity $\chi$ .

A question naturally arises whether it is always possible for a form factor to be represented as a product of contribution from different regions of a particle (33). We here notice that such an interpretation of factors in (33) is based on the positivity of $\langle r^2 \rangle_p$ resulting from experimental data on the proton radius. This positivity is possible for example when the function $F(r)$ is of constant sign.

It is also known from experiment that a number of particles (e.q., $\pi$ -meson) have m.s.radius smaller than their Compton wave length. For them, obviously, the quantity $\langle r^2 \rangle$ should be



negative. Let us establish in which way it can be achieved in the vector dominance model (VDM) that well describes the pion form factor. In the VDM the pion form factor is described by the $\rho$ -meson pole $F(t) = \frac{1}{1 - t/\mu^2}$ . It is known /2/ that the transform of such a relativistic propagator in the relativistic configurational representation essentially depends on the relation between the mass of a particle itself M and that of an exchanged particle $\mu$ :

$$F(r) = \begin{cases} \frac{1}{4\pi r} \frac{ch(rMa_1)}{sh(rM\pi)} & \mu^2 < 4M^2 \\ & a_1 = \arccos\left(\frac{\mu^2 - 2M^2}{2M^2}\right) \\ \frac{1}{4\pi r} \cdot \frac{\cos(rMa_2)}{sh(rM\pi)} & \mu^2 > 4M^2 \\ & a_2 = Ar\,ch\left(\frac{\mu^2 - 2M^2}{2M^2}\right) \end{cases} , \quad (35)$$

From (35) it is seen that for pion the second inequality $\mu_\rho^2 > 4M_\pi^2$ holds and $F_\pi(r)$ is an alternating function. For a nucleon in the VDM the relation $\mu^2_{\rho,\omega,\varphi} < 4M_N^2$ holds for $\rho, \omega, \rho'$ and $\varphi$ -meson, and the function $F_N(r)$ is of constant sign. By using $F(r)$ from (35), we obtain from (34) the expression for $\langle r^2 \rangle$ :

$$\langle r^2 \rangle = \frac{6M^2 - \mu^2}{\mu^2 M^2}$$

within the VDM. One can easily see that for pion $(M = M_\pi ; \mu = \mu_\rho)$ $\langle r^2 \rangle_\pi$ is negative and for nucleon $(M = M_N ; \mu = \mu_\rho, \mu_\omega, \mu_\varphi)$ $\langle r^2 \rangle_N$ is positive. Thus, for pion $\langle r_0^2 \rangle_\pi = \frac{1}{M_\pi^2} - |\langle r^2 \rangle|$ and it is impossible to interprete, as before, the coordinate $r$ as describing the particle distribution at distances larger than its Compton wave length. Hence, for pion no analog of the central part exists and the factorizing (33) makes no sense.

However, the difference between the pion and nucleon mean-square radius has more fundamental grounds from the viewpoint of applicability of expansions over unitary representations of the Lorentz group. It will be shown that, in accordance with the general theorems proved in /11,15, 12/, these expansions differ essentially in form for pion and nucleon that is due to the experimentally observed asymptotic behaviour different for their form factors at large $-t$.

Indeed, the above considered case of nucleon is distinct since the nucleon form factor (as is seen from the fitting experimental data dipole formula $G_D(t)=\frac{1}{(1-t/0{,}71)^2}$) is a square-integrable function, i.e., $\int |F_N(t)|^2 sh^2\chi\, d\chi < \infty$. By a theorem proven in /12/ such functions are expanded over representations of the principal series only (see also ref. /15/ ), i.e., formulae for transition to the coordinate space are of the form (24), (25).

The pion form factor is not a square-integrable function since it is known from experiment to decrease as $\frac{1}{|t|}$. By a theorem proved in /11/ such functions are decomposed into a direct sum of representations of the principal and complementary series.

Eigenvalues of the Casimir operator of the Lorentz group $\hat{C}=X^2$ playing the role of square of a distance from the particle centre are not bounded from below by the quantity $\frac{\hbar^2}{M^2c^2}$ when taking account of the complementary series, since by results of /11, 12/ one has

$$\hat{C}=X^2=\begin{cases}\frac{1}{M^2}+r^2 & 0\le r<\infty \\ & \text{for the pincipal series} \\ \frac{1}{M^2}-\rho^2 & 0\le\rho\le 1/M \\ & \text{for the complementary series}\end{cases} \qquad (36)$$

The parameter $\rho$ can be treated as a coordinate describing the interior of a region with $R=\frac{\hbar}{MC}$ and measured from the sphere boundary to its centre.

To a particle localized at the center, i.e., at $X=0$ the value of the parameter $\rho=\frac{\hbar}{MC}$ corresponds. In this case, for an elementary spherical function of the complementary series with $\ell=0$ $\frac{sh\,\rho M\chi}{\rho M\, sh\chi}$ (which remains due to the form factor $F(t)$ spherical symmetry) the equality $\frac{sh\,\rho M\chi}{\rho M\, sh\chi}\Big|_{\rho=\frac{1}{M}}=1$ holds. The latter gives $F(t)=1$ (unlike eq. (32) for nucleon). Such a form factor corresponds to a point-like particle.



Thus, the pion form factor in the coordinate space can be described in terms of representations of the complementary series only, i.e., in term of the coordinates $\rho$ and for it one cannot separate the central part contribution.

For proton the transition to the coordinate space is realized with the use of the principal series of unitary representations the proton spatial distribution is described in terms of the coordinate $r$. In this case the squared distance from the particle centre $X^2=\frac{1}{M^2}+r^2$ is limited from below by the radius of the central sphere $X^2_{min}=R^2=\frac{\hbar^2}{M^2c^2}$ to which, according to (32), there corresponds the contribution $\chi/sh\chi$.

This distinction of description of proton and pion in the new coordinate space suggested an idea that an additional contribution from the proton central part should be taken into account when using the VDM for proton. Since in the nonrelativistic limit just the Fourier transforms of usual Yukawa potentials correspond to the vector meson propagators $\frac{1}{\mu_V^2-t}\to\frac{1}{\mu_V^2+(\vec{p}-\vec{k})^2}$, then it should be considered that they contribute only to the form factor part having a nonrelativistic analog, viz. to the "external" form factor*). Thus, if we want to map, in the momentum space, the whole proton spatial distribution conceivable in the new coordinate representation then it is necessary to add also a contribution

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*) Spatial distributions corresponding to the "external" form factor $\Phi(\chi)\sim\frac{a_V}{\mu_V^2-t}$ are of the form $F(r)=\frac{\pi\, ch\,\pi r\, sh\, ra - a\, ch\, ar\, sh\,\pi r}{4\pi r\, sh^2\pi r\, \sin a}$, in accordance with (24a), and in the nonrelativistic limit reduce to the Yukawa potentials $F(r)\to\frac{e^{-\mu r}}{4\pi r}$.

of the central region with $R = \frac{\hbar}{Mc}$ , that just results in the formula

$$F_p(t) = \left(\frac{\chi}{sh\chi}\right) \sum_{v=\rho,\omega,\varphi \ldots} \frac{a_v}{\mu_v^2 - t} . \tag{37}$$

This expression for the proton form factor, in accordance with (32), has the correct "almost dipole" asymptotic behaviour at large $-t$

$$F_p(t) \xrightarrow[|t| \gg M^2]{} \frac{2M^2 \ln \frac{|t|}{M^2}}{t^2} . \tag{38}$$

It is interesting to notice that the VDM provides good results of description of the reactions with pions but for the nucleon form factors it describes satisfactorily only tha data at small $-t < 1 \left(\frac{Gev}{c}\right)^2$. It is just the region $-t < 1 \left(\frac{Gev}{c}\right)^2$ where the central part contribution does not differ, in practice, from unity, i.e., $\left(\frac{\chi}{sh\chi}\right) \approx 1$. When it varies in the whole experimentally available region $0 \leq -t \leq 25 \left(\frac{Gev}{c}\right)^2$ the factor $\left(\frac{\chi}{sh\chi}\right)$ runs through the interval $1 \geqslant \left(\frac{\chi}{sh\chi}\right) \geqslant 0.2$. This result can be interpreted as follows: At small momentum transfers a region of the "external" form factor was considered and with growing momentum transfer the regions are reached where the central part (with $R = \frac{\hbar}{Mc}$ ) contribution becomes significant.

## 4. Conclusion.

Let us summarize our consideration. As has been shown, the use of the Cheskov-Shirokov invariant parametrization of currents allows one to make physical interpretation of the nucleon electromagnetic form factor in the system where the nucleon is at rest before its interaction with a photon, whereas for this purpose, as a rule, the Breit reference frame is used. This has become possible since in the Cheskov-Shirokov parametrization a "removing"



both of all spin indices and of spin variables is done on to one and the same momentum /6/ . It is clear, as well that the above consideration of interaction of a particle with an external electromagnetic field and the form factor interpretation remain valid also for particles with an arbitrary spin. Really, interpreting form factors $G_E$ and $G_M$ we preceed from the current parametrization of a type of (12) which does not change its form for particles with an arbitrary spin. In this case, according to /6/ , it is only necessary to consider $W^\mu(\vec{p})$ as a relativistic spin vector of an arbitrary value $s$ and to replace $G_E$ and $G_M$ by sets of appropriate form factors by the formula: $G_{E,M} \to \sum_{n=0}^{\{s\}} f_n(t) \left(W^\mu(\vec{p}) q_\mu\right)^n$ /6/.

Transition to the relativistic configurational representation allows one to introduce the invariant description of particle spatial distribution. An important feature of the relativistic configurational representation is that it introduces the new scale: particle Compton wave length. The harmonic analysis on the Lorentz group has more possibilities then the expansion on the Euclide group, i.e.,the Fourier-Bessel transformation. As has been shown above, including into consideration, **besides** the principle series, also the complementary one makes it possible to describe the whole interval from the origin up to infinity. In this approach, the particle distribution at distances larger than its Compton wave length is described in terms of representations of the principle series and that at distances smaller than the corresponding Compton wave length - on the basis of the complementary series. The use of this language leads to concept of a contribution of the proton central part with radius $R = \frac{\hbar}{Mc}$ . The consideration of this contribution and the use of the VDM describing the proton distribution outside the sphere with $R$ equal to its Compton wave length give rise to the new formula for the proton form factor (37). This formula provides the correct "almost dipole" asymptotic behaviour of the nucleon form factor (38). A detailed comparison of theoretical dependence of the proton form factor at space-like momentum transfers given by (37) with experimental data will be made in a subsequent paper.

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## References:

1. V.G.Kadyshevsky, R.M.Mir-Kasimov, N.B.Skachkov. Nuovo Cim., 55A, 233 (1968); Particles and Nucleus, vol.2., N3, 635, Atomizdat, Moscow, 1972.
2. V.G.Kadyshevsky, R.M.Mir-Kasimov, N.B.Skachkov. Yadern.Fiz., 9. 219 (1969); Yadern.Fiz.,9, 462 (1969).
3. M.I.Pavkovic.Phys.Rev.,D4,1724 (1971).
4. N.B.Skachkov. JINR Communications E2-7890, Dubna, 1974; E2-8007, Dubna, E2-8424, Dubna, 1974.
5. Yu.M.Shirokov.JETP,33, 861 (1957).
6. A.A.Cheshkov, Yu.M.Shirokov, JETP,44, 1982 (1963).
7. V.R.Kozhevnikov, V.E.Troitsky, S.V.Trubnikov, Yu.M. Shirokov. Teor. i Mat. Fiz.,10, 47 (1972).
8. N.B.Skachkov.JINR Commutations E2-7333, Dubna, 1974; E2-8285, Dubna 1974; Teor. i Mat.Fiz.,22, 213 (1975).
9. N.N.Bogolubov, D.V.Shirkov. Introduction to the Theory of Quantized Fields , Nauka, Moscow, 1973.
10. D.I.Blochintsev. Space and Time in Microworld, Nauka, Moscow, 1970.
11. I.M.Gelfand, M.A.Naimark. Izvestia Acad. Nauk SSSR, Ser. math., vol.11, 411 (1947).
12. M.A.Naimark. Linear Representations of the Lorentz Group. Pergamon, New-York, 1964.
13. J.S.Shapiro.Dokl.Acad.Nauk SSSR, 106, 647 (1956); JETP 43, 1727 (1962); Phys.Lett., 5, 253 (1962) ; A.Z.Dolginov. JETP, 30, 746 (1956);