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1. Introduction

In the first part of this paper [1] we dealt with the canonical realizations in the Weyl algebra of the Lie algebra $gl(n,R)$ $n \geq 2$, ,i.e., with polynomials in a certain number of quantum mechanical canonical pairs p_i , q_i which commute as the generators of $gl(n,R)$. The defined realizations form a set uniquely classified by sequences of real numbers (signatures) $(d; 0, \dots, \alpha_{n-d}, \dots, \alpha_n)$ with $d = 1, 2, \dots, n-1$. Various properties of these realizations were derived, and particularly we proved that all Casimir operators are realized by multiples of the identity element (we call these realizations by Schur-realizations). Now we are interested in their eigenvalues.

We give simple formulas for calculation of eigenvalues of the generating Casimir operators in our realizations. We show that these eigenvalues are certain symmetric polynomials in variables $y_1^{(n-d)}, \dots, y_{n-d}^{(n-d)}$,

$i\alpha_{n-d+1}, \dots, i\alpha_n$, where $y_k^{(n-d)} = i \cdot \frac{\alpha_{n-d}}{n-d} + (\frac{n-d+1}{2} - k)$

(theorem 1). Due to symmetry property there is only a finite number of realizations in our set with the same eigenvalues of any Casimir operator (theorem 2).

In conclusion we discuss the question of independence of eigenvalues of generating Casimir operators as functions of signatures in subsets of realizations with signatures with fixed d.

2. Preliminaries

A. If $F_{\mu\nu}$, $\mu, \nu = 1, 2, \dots, n-1$ denote the canonical realization of generators of the Lie algebra $gl(n-1,R)$ in the Weyl algebra W_{2m} (m - the number of canonical pairs), then the following formulae

$$
\begin{aligned}
E_{\mu\nu} &= q_\mu p_\nu + F_{\mu\nu} + \frac{1}{2}\delta_{\mu\nu}\cdot 1, \\
E_{n\mu} &= -p_\mu, \\
E_{\mu n} &= q_\mu(q_\nu p_\nu + \frac{n}{2} - i\alpha) + q_\nu F_{\mu\nu}, \\
E_{nn} &= -q_\nu p_\nu - \frac{n-1}{2} + i\alpha\cdot 1, \qquad \alpha \in R
\end{aligned}
\tag{1}
$$

(summation over ν)*

define a canonical realization of generators of $gl(n,R)$ in $W_{2(n-1+m)}$.
The generators E_{ij} satisfy the commutation relations of $gl(n,R)$

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}, \tag{2}$$

and the subalgebra with generators

$$A_{ij} = E_{ij} - \frac{1}{n}\delta_{ij} E_{kk} \tag{3}$$

is $sl(n,R)$. In the first part of this paper we used formulae (1) in an iterative way to obtain a set of canonical realizations of $gl(n,R)$ for $n \geq 2$. Every realization from this set is characterized uniquely by a signature. The realization with signature $(d;0,\ldots,0,\alpha_{n-d},\ldots,\alpha_n)$ is defined recurrently by formulae (1) in such a way that $(1;0,\ldots,0,\alpha_{n-1},\alpha_n)$ denotes the realization (1) with $F_{\mu\nu} = i\alpha_{n-1}\cdot\frac{\delta_{\mu\nu}}{n-1}\cdot 1$, , $\alpha = \alpha_n$ and $(d;0,\ldots,0,\alpha_{n-d},\ldots,\alpha_n)$ denotes the realization (1) with $\alpha = \alpha_n$, where the realization of $gl(n-1,R)$ has the signature $(d-1;0,\ldots 0,\alpha_{n-d},\ldots,\alpha_{n-1})$.

B. The center of the enveloping algebra $U[gl(n,R)]$ contains n generating Casimir operators which we choose as follows. Let (D_{ij}) be the formal matrix with the matrix elements

$$(D)_{ij} = E_{ij}. \tag{4}$$

The Casimir operator C_n^p is the trace of the p-th power of D

$$C_n^p = Tr D^p \tag{5}$$

or explicitly written

$$C_n^p = E_{i_1 i_2} E_{i_2 i_3} \ldots E_{i_p i_1}, \qquad C_n^0 = n. \tag{6}$$

The operators $C_n^1, C_n^2, \ldots, C_n^n$ generate the center of $U[gl(n,R)]$ and the commutation relations for matrix elements of D^p with E_{ij} have the form

$$[E_{ij}, (D^p)_{kl}] = \delta_{jk}(D^p)_{il} - \delta_{li}(D^p)_{kj}, \quad p = 0, 1, \ldots \tag{7}$$

3. σ_n^p -Polynomials

Let us introduce now the set of polynomials by which the eigenvalues of C_n^p will be expressed* .

Let $S_n = S_n(x_1, x_2, \ldots, x_n)$, $n > 1$, be the $n \times n$ - matrix with elements

$$(S_n)_{kl} = (\frac{n-1}{2} + x_k)\delta_{kl} - \Lambda_{kl}, \tag{8}$$

---

* In what follows the greek indices will run always from 1 to $n-1$ latin from 1 to n and twice occuring indices mean summation.

* Such polynomials were defined in /2/ , see also /3/.

where

$$\Delta_{kl} = \begin{cases} 1 & \text{for } k < l, \\ 0 & \text{for } k \geq l. \end{cases}$$

We define polynomials σ_n^p, $p = 0, 1, \dots,$ as

$$\sigma_n^p \equiv \sigma_n^p(x_1, x_2, \dots, x_n) = e_n^+ S_n^p e_n , \qquad (9)$$

where $e_n^+ = (1, 1, \dots, 1)$ is the n-dimensional row and e_n denotes its transposed.

These polynomials have the following five main properties.

Property 1. $\sigma_n^p(x_1, x_2, \dots, x_n)$ is a symmetric polynomial in the variables $x_1, x_2, \dots, x_n$.

The proof of this assertion is given essentially in /3/ where it is shown that $e_n^+ S_n^p e_n$ is a symmetric polynomial in the matrix elements l_k on the main diagonal of S_n. Since $x_k = l_k - \frac{n-1}{2}$ and $\frac{n-1}{2}$ does not depend on k, $e_n^+ S_n^p e_n$ is also symmetric in the variables $x_1, x_2, \dots, x_n$.

Property 2. The set of polynomials $\sigma_n^p(c_1, \dots, c_{n-k}, x_1, \dots, x_k)$ $p \leq k$ with fixed constants $c_i \in C$ is a generating system in the algebra of symmetric polynomials in the variables $x_1, x_2, \dots, x_k$.

For the proof it is sufficient to show that the Newton sums

$$s_r = \sum_{j=1}^{k} x_j^r , \quad r = 1, 2, \dots, k ,$$

can be polynomially expressed by the k independent $\sigma_n^p(c_1, \dots, c_{n-k}, x_1, \dots, x_k)$, $p = 1, 2, \dots, k$, since, as is well-known /4/, the Newton sums are a generating system in the algebra of symmetric polynomials in $x_1, x_2, \dots, x_k$. Now, S_n is the sum of the matrix

$$X = \begin{pmatrix} 0 & & & & & 0 \\ & \ddots & & & & \\ & & 0 & & & \\ & & & x_1 & & \\ & & & & \ddots & \\ 0 & & & & & x_k \end{pmatrix}$$

and a constant upper triangular matrix C. The difference

$$e_n^+ S_n^p e_n - e_n^+ X^p e_n = e_n^+ (X + C)^p e_n - e_n^+ X^p e_n$$

is therefore a symmetric polynomial in $x_1, x_2, \dots, x_k$ with powers smaller than p. However, $e_n^+ X^p e_n = s_p$ and property 2 follows by induction.

Property 3. Let $c_1, c_2, \dots, c_n \in C$ be given. Then any solution $(x_1, x_2, \dots, x_k)$, $k \leq n$, of the system of k equations

$$\sigma_n^p(c_1, c_2, \dots, c_n) = \sigma_n^p(c_1, \dots, c_{n-k}, x_1, \dots, x_k) \qquad (10)$$
$$p = 1, 2, \dots, k$$

differs from $(c_{n-k+1}, \dots, c_n)$ by permutation of the components only.

In order to show property 3 we consider another generating system in the set of symmetric polynomials in k variables, the system of the so-called fundamental polynomials

$$\xi_p^{(k)}(x_1, \dots, x_k) = \sum_{(i_1, i_2, \dots, i_p)} x_{i_1} x_{i_2} \cdots x_{i_p} , \qquad (11)$$
$$p = 1, 2, \dots, k,$$

where the summation runs over all sequences $(i_1, i_2, \dots, i_p)$ with $1 \leq i_1 < i_2 < \dots < i_p \leq k$. As any polynomial $\sigma_n^p(c_1, \dots, c_{n-k}, x_1, \dots, x_k)$, $p = 1, 2, \dots, k$ can be expressed in terms of $\xi_1, \dots, \xi_p$ and vice versa, (Property 2), the system (10) is equivalent to the system

$$\xi_p(c_{n-k+1}, \dots, c_n) = \xi_p(x_1, \dots, x_k) , \quad p = 1, 2, \dots, k . \qquad (10')$$

Due to the well-known Vieta formula the sequence $(x_1,\dots,x_k)$ solves the system (10) if and only if $x_1,\dots,x_k$ will be the set of all roots of the k-th order equation

$$x^k + a_1 x^{k-1} + \dots + a_k = 0, \qquad (12)$$

where $a_p = \xi_p(c_{n-k+1},\dots,c_n)$. As coefficients a_p are the values of the polynomials ξ_p at the point $(c_{n-k+1},\dots,c_n) \in C^k$ the set of all roots consists just of $c_{n-k+1},\dots,c_n$. Therefore we obtain all solutions $(x_1,\dots,x_k)$ of the system (10′) by permutations of the components of the solution $(c_{n-k+1},\dots,c_n)$.

Property 4. The following reccurrent relations $(n \geq 2)$:

$$\sigma_n^p(x_1,\dots,x_n) = \sum_{u=0}^{p}\binom{p}{u}\left(\tfrac{1}{2}\right)^{p-u}\cdot\sigma_{n-1}^u(x_1,\dots,x_{n-1}) - \sum_{u=0}^{p-1}\sum_{v=0}^{p-1-u}\binom{p-1-u}{v}\cdot\sigma_{n-1}^v(x_1,\dots,x_{n-1})\left(\tfrac{1}{2}\right)^{p-1-u-v}\times y_n^u + y_n^p, \qquad (14)$$

where $y_n = x_n + \frac{n-1}{2}$, are valid.

The proof follows immediately from the recurrent relation for the p-th power of the matrix S_n

$$S_n^p = \left(\begin{array}{c|c} (S_{n-1}+\frac{1}{2}I)^p & -\sum_{u=0}^{p-1}(S_{n-1}+\frac{1}{2}I)^u\, y_n^{p-1-u}\cdot e_{n-1} \\ \hline 0 & y_n^p \end{array}\right)$$

Here I denotes the $(n-1)\times(n-1)$ unit matrix.

Property 5. Let $\beta_j^{(n)} = \frac{n+1}{2} - j$, $j = 1, 2, \dots, n$. Then

$$\sigma_n^p(\beta_1^{(n)} + x, \beta_2^{(n)} + x, \dots, \beta_n^{(n)} + x) = n\,x^p, \qquad (15)$$

for all $p = 0, 1, 2, \dots$ and any $x \in C$.

Proof: For $n = 1$ and all p we prove (15) directly from definition (9) because $\sigma_1^p(x_1) = (x_1)^p$ and $\beta_1^{(1)} = 0$. Assume now the validity of relation (15) for n and all p. It means that

$$\sigma_n^p(\beta_1^{(n+1)} + x, \dots, \beta_n^{(n+1)} + x) = \sigma_n^p\left(\beta_1^{(n)} + (x+\tfrac{1}{2}), \dots, \beta_n^{(n)} + (x+\tfrac{1}{2})\right) = n\left(x+\tfrac{1}{2}\right)^p, \qquad (16)$$

because $\beta_k^{(n+1)} = \beta_k^{(n)} + \frac{1}{2}$ for $k \leq n$. Substitution into the recurrent relation (14) gives immediately the desired result, i.e., the validity of relation (15) for $n+1$ and any p.

4. *Eigenvalues of the Casimir Operators*

Lemma 1. Let C_{n-1}^p, $n \geq 2$, be the Casimir operators in the canonical realization of $gl(n-1,R)$ (with generators $F_{\mu\nu}$). Then the Casimir operators C_n^p of the Lie algebra $gl(n,R)$ in the realization (1) are connected with C_{n-1}^p by the following relations

$$C_n^p = \sum_{u=0}^{p}\binom{p}{u} C_{n-1}^u\left(\tfrac{1}{2}\right)^{p-u} - \sum_{u=0}^{p-1}\sum_{v=0}^{p-1-u}\binom{p-1-u}{v}\left(\tfrac{1}{2}\right)^{p-1-u-v} C_{n-1}^v\, y_n^u + y_n^p, \qquad (17)$$

where $y_n = i\alpha + \frac{n-1}{2}$ and $p = 0, 1, 2, \dots$.

Proof: In the first part of this paper /1/ we have proved that in realization (1) the Casimir operators do not depend on the canonical pairs $q_1, p_1; \dots; q_{n-1+m}, p_{n-1+m}$ (proof of theorem 1). It is very economical to use this fact in the present proof in the following manner. Any element

x from the enveloping algebra $U[gl(n,R)]$ can be written in realization (1) in the form

$$x = \sum_{u,v} \alpha_{uv} q^u p^v ,$$

where

$$q^u p^v = q_1^{u_1} \dots q_{n-1}^{u_{n-1}} p_1^{v_1} \dots p_{n-1}^{v_{n-1}}$$

$\alpha_{uv} = \alpha_{uv}(F_{\mu\nu})$, and we define the "projection" $||x|| = \alpha_{00}$. From this definition there immediately follow some simple rules:

$$||q_i x|| = 0 , \quad ||x p_i|| = 0 ,$$

$$||\alpha x + \beta y|| = \alpha||x|| + \beta||y|| \text{ for } \alpha, \beta \in W_{2m}, \; i = 1, 2, \dots, n-1, \tag{18}$$

and any $x, y \in W_{2(n-1+m)}$ however, in general $||x q_i|| \neq 0$ $||p_i x|| \neq 0$ (e. g. $||p_1 q_1|| = ||q_1 p_1 + 1|| = 1$).
We have

$$C_n^p = ||C_n^p|| ,$$

as was mentioned above, however calculations with $||C_n^p||$ are much more simple than those with C_n^p alone.

We can easily see that the lemma is true for $p = 0$; therefore we assume $p \geq 1$.

Let us denote the first $n-1$ terms of $C_n^p = ||D_{ii}^p||$ by A^p,

$$A^p = ||D_{\mu\mu}^p|| \tag{19}$$

and the last term by B^p

$$B^p = ||D_{nn}^p|| . \tag{20}$$

Then

$$D_{ij}^p = E_{ik} D_{kj}^{p-1}$$

holds by definition of D_{ij}. Therefore A^p can be written as

$$A^p = ||E_{\mu\nu} D_{\nu\mu}^{p-1}|| + ||E_{\mu n} D_{n\mu}^{p-1}|| , \tag{21}$$

from which, using eqs. (1) and (18), we obtain:

$$A^p = (F_{\mu\nu} + \frac{1}{2}\delta_{\mu\nu}) ||D_{\nu\mu}^{p-1}|| ,$$

which simply leads to

$$A^p = \sum_{v=0}^{p} \binom{p}{v} (\frac{1}{2})^{p-v} \cdot C_{n-1}^{v} , \quad p = 0, 1, \dots . \tag{22}$$

We now consider

$$B^p = ||D_{nn}^p|| = ||E_{n\mu} D_{\mu n}^{p-1}|| + ||E_{nn} D_{nn}^{p-1}|| . \tag{23}$$

The second term in the r.h.s. equals $(i\alpha - \frac{n-1}{2})||D_{nn}^{p-1}||$.

Since $||D_{\mu n}^{p-1} E_{n\mu}|| = 0$ the first term can be expressed by the commutation relations (7)

$$||E_{n\mu} \cdot D_{\mu n}^{p-1}|| = ||[E_{n\mu}, D_{\mu n}^{p-1}]|| = \delta_{\mu\mu} ||D_{nn}^{p-1}|| - \delta_{nn} ||D_{\mu\mu}^{p-1}|| . \tag{24}$$

Thus the equation

$$B^p = (i\alpha + \frac{n-1}{2}) B^{p-1} - A^{p-1} , \quad p = 1, 2, \dots \tag{25}$$

holds which can be solved iteratively.
The solution is

$$B^p = - A^{p-1} - y_n A^{p-2} - \dots - y_n^{p-2} A - y_n^{p-1} A^0 + y_n^p , \tag{26}$$

where $y_n = (i\alpha + \frac{n-1}{2})$.

Due to

$$C^p_n = A^p + B^p$$

we get the relation (17) using (22) and (26)

L e m m a 2: Assume that

$C^p_{n-1} = \sigma^p_{n-1}(x_1, x_2, \dots, x_{n-1})$ for some $x_1, \dots, x_{n-1} \in C$ and all $p = 1, 2, \dots,$ then $C^p_n = \sigma^p_n(x_1, x_2, \dots, x_{n-1}, i\alpha_n)$.

This lemma follows immediately from lemma 1 and property 4 of the polynomials σ^p_n.

We are now ready to formulate our main theorem.

T h e o r e m 1: Let C^p_n be the Casimir operator defined by (6) of the realization $gl(n, R)$, $n \geq 2$, with signature $(d;\ 0, \dots, 0, \alpha_{n-d}, \dots, \alpha_n)$, then

$C^p_n = \sigma^p_n(y^{(n-d)}_1, \dots, y^{(n-d)}_{n-d}, i\alpha_{n-d+1}, \dots, i\alpha_n)$

for $p = 0, 1, 2, \dots,$ where

$$y^{(n-d)}_k = i\,\frac{\alpha_{n-d}}{n-d} + \left(\frac{n-d+1}{2} - k\right).$$

Proof: For $n = 2$ there is only one type of signature, namely $(1; \alpha_1, \alpha_2)$. The realization with this signature is given by formulae (1) for $F_{11} = i\alpha_1 I$ and $\alpha = \alpha_2$. F_{11} can be considered as realization of $gl(1, R)$. We have $C^p_1 = \sigma^p_1(i\alpha_1) = (i\alpha_1)^p$. Lemma 1 and property 4 yield $C^p_2 = \sigma^p_2(i\alpha_1, i\alpha_2)$. For induction from $n-1$ to n we must distinguish between the two cases $d = 1$ and $d > 1$. For $d > 1$ the theorem follows immediately from lemma 2. Let us consider the case $d = 1$. The realization with signature $(1; 0, \dots, 0, \alpha_{n-1}, \alpha_n)$ is the realization (1) with $F_{\mu\nu} = i\alpha_{n-1}\,\frac{\delta_{\mu\nu}}{n-1} \cdot I$ and $\alpha = \alpha_n$.

The Casimir operator C^p_{n-1}, $p = 0, 1, 2, \dots,$ of the realization of $gl(n-1, R)$ with the generators $F_{\mu\nu}$ is

$$C^p_{n-1} = (n-1)\left(\frac{i\alpha_{n-1}}{n-1}\right)^p.$$

If we show that C^p_{n-1} can be expressed as the value of polynomial σ^p_{n-1} at the point $(y^{(n-1)}_1, \dots, y^{(n-1)}_{n-1})$ then the theorem 1 follows from the lemma 2. But

$$\sigma^p_{n-1}(y^{(n-1)}_1, \dots, y^{(n-1)}_{n-1}) = (n-1)\left(\frac{i\alpha_{n-1}}{n-1}\right)^p$$

holds due to Property 5 of the polynomials σ^p_n.

Since different realizations may have the same eigenvalues of any Casimir operator C^p_n, the question arises about the classes of realizations uniquely characterized by these eigenvalues.

T h e o r e m 2: Let $[d; 0, \dots, \alpha_{n-d}, \dots, \alpha_n]$ be the class of all realizations with signatures

$(d; 0, \dots, (1 - \delta_{dn-1})\alpha_{n-d} + \delta_{dn-1}\alpha_{\pi(n-d)}, \alpha_{\pi(n-d+1)}, \dots, \alpha_{\pi(n)})$,

where π denotes a permutation of indices $1, 2, \dots, n$ or $n-d+1, \dots, n$ respectively.

(i) Then the Casimir operator C^p_n has the same eigenvalue for all realizations of the class $[d; 0, \dots, 0, \alpha_{n-d}, \dots, \alpha_n]$.

(ii) Let C^p_n and $\tilde{C}^p_n$ be the generating Casimir operators for two different classes of realizations of $gl(n, R)$. Then there exists at least one p_0 such that $C^{p_0}_n$ and $\tilde{C}^{p_0}_n$ have different eigenvalues.

P r o o f: Assertion (i) follows from theorem 1 and the symmetry property 1 of the polynomials σ^p_n. (ii) Assume $C^p_n = \tilde{C}^p_n$, i.e.,

$$C_n^p = \sigma_n^p(y_1^{(n-d)}, \dots, y_{n-d}^{(n-d)}, i\alpha_{n-d+1}, \dots, i\alpha_n) =$$
$$= \sigma_n^p(\tilde{y}_1^{(n-\tilde{d})}, \dots, \tilde{y}_{n-\tilde{d}}^{(n-\tilde{d})}, i\tilde{\alpha}_{n-\tilde{d}+1}, \dots, i\tilde{\alpha}_n) = \tilde{C}_n^p,$$

for all $p = 1, 2, \dots$
Due to Property 5 the sequences $(y_1^{(n-d)}, \dots, i\alpha_n)$ and $(\tilde{y}^{(n-\tilde{d})}, \dots, i\tilde{\alpha}_n)$ may be different at most by permutation of their components.
Thus:
a) $d = \tilde{d}$, because the number of components with a nonzero real part must be the same.
b) $\alpha_{n-d} = \tilde{\alpha}_{n-d}$ if $d < n-1$ because the complex components $(y_1^{(n-d)}, \dots, y_{n-d}^{(n-d)})$ can be permutation of the complex components $(\tilde{y}_1^{(n-d)}, \dots, \tilde{y}_{n-d}^{(n-d)})$ only.
c) $(\alpha_D, \dots, \alpha_n)$, $D = n - d + 1 - \delta_{n-1\ d}$ is a permutation of $(\tilde{\alpha}_D, \dots, \tilde{\alpha}_n)$ because the purely imaginary components must be permuted separately.

Therefore the realizations with the same eigenvalues of any Casimir operator lie in the same class and proof is completed.

5. *Concluding Remarks*

i. If we consider for d fixed the set of all signatures $(d; 0, \dots, 0, \alpha_{n-d}, \dots, \alpha_n)$ then the corresponding realizations have generating Casimir operators C_n^p whose eigenvalues are polynomials in the $d+1$ parameters $\alpha_{n-d}, \dots, \alpha_n$, thus we cannot expect n independent polynomials for $d < n-1$. Actually, the Property 2 implies that only the first $d+1$ operators $C_n^p = \sigma_n^p$, $p = 1, 2, \dots, d+1$ are independent and the remaining ones, $C_n^p = \sigma_n^p$, $p > d+1$ polynomially depend on $C_n^1, \dots, C_n^{d+1}$. Only if $d = n-1$, the eigenvalues of all generating Casimir operators C_n^p, $p = 1, \dots, n$ will be independent polynomials.

ii. If we substitute p_i, q_i in the considered realizations by some of their representations we obtain a representation of $gl(n, R)$ or $sl(n, R)$, respectively.

It may happen that two realizations characterized by different signatures lead to equivalent representations. If, however, realizations have signatures from different classes $[d; \dots]$ the corresponding representations cannot be equivalent as they differ in an eigenvalue of at least one Casimir operator. This illustrates the usefulness of Theorem 2.

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