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исследований
дубна

E2-88-931

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THE GAUGE FIXING EXTENSION OF THE KRICHEVER-NOVIKOV ALGEBRA IN THE CLOSED STRING THEORY

Submitted to "Physics Letters B"

1988

By examining the light-cone formulation of the string theory it was found [1] that the symmetry of the quantum system is larger than the usual BRST-invariance [2,3]. This is the group of the global OSP(1.1|2) symmetry. Its existence may be related to the extended algebra of constraints Φ_i and subsidiary conditions Ψ^j [4,5],

$$\{\Phi_i, \Phi_j\} = \sum_k U^k_{ij} \Phi_k \quad , \tag{1}$$

$$\{\Phi_i, \Psi^j\} = B^j_i - \sum_k T^j_{ik} \Psi^k \quad , \tag{2}$$

$$\{\Psi^i, \Psi^j\} = 0. \tag{3}$$

The braces mean the canonical Poisson brackets. The constants B^j_i and T^k_{ij} satisfy the relations which follow from the Jacobi identities. If the constraint algebra (1) is the classical Virasoro algebra, then $B^j_i = \delta^j_i$.

The study of the questions connected with string interactions [6] as well as the investigations in the two-dimensional conformal field theories [7] lead to the problems which are formulated on Riemann surfaces of arbitrary genus. In papers [8,9] Krichever and Novikov introduced a basis on these surfaces in the space of vector fields holomorphic out of two distinguished points $P_\pm$, and studied the tensor objects arising here. The basic vector fields e_i form the Krichever-Novikov algebra (KN algebra) wich reduces to the Virasoro algebra in the g = 0 case,

$$[e_i, e_j] = \sum_{s=-g_0}^{g_0} C^s_{ij} e_{i+j-s} \, . \tag{4}$$

Index i takes integral values for even g and half-integral values for odd g, and $g_0 = \frac{3}{2}g$.

Algebra (4) is interesting from the physical point of view.

Its realization in terms of Virasoro-type operators obtained in paper [9] is a constraints algebra for the closed interacting string. In this case the question of its extension to the algebra of (1)-(3) kind arises. To answer it, it is natural to make clear the existence of the corresponding geometrical construction on the given Riemann surface, and then try to realize it by means of dynamical variables of the closed string. **This is what the present note deals with.**

Let us consider a family of tensor fields $f_j^{(\lambda)}$ on Riemann surface Σ of genus g, parametrized by real numbers λ (conformal weight) [8]. They are holomorphic everywhere on Σ except possibly the poles in $P_\pm$. The forms $f_j^{(\lambda)}$ have the following behaviour near the punctures:

$$f_j^{(\lambda)} = a_j^{(\lambda)\pm} (Z_\pm)^{\pm j - S(\lambda)} [1+O(Z_\pm)](dZ_\pm)^\lambda, \qquad (5)$$

where $S(\lambda)=\frac{g}{2}-\lambda(g-1)$. For $\lambda=0,1$ and $|j|\le\frac{g}{2}$ the definition of $f_j^{(\lambda)}$ is slightly different [8,9]. Formula (5), as a consequence of the Riemann-Roch theorem, uniquely determines the forms $f_j^{(\lambda)}$ up to a constant on whole surface Σ. We normalize them by setting $a_j^{(\lambda)+}=1$. Let us also introduce convenient notations

$$e_j = f_j^{(-1)}, \qquad A_j = f_j^{(o)}, \qquad \omega^j = f_{-j}^{(1)}, \qquad \Omega^j = f_{-j}^{(2)}. \qquad (6)$$

The duality relations

$$\frac{1}{2\pi i}\oint f_i^{(\lambda)} f_{-j}^{(1-\lambda)} = \delta_{ij} \qquad (7)$$

hold for all values of i and j. Here and further, if there is no special notice, the integration is over a nonselfcrossing contour, which divides Σ into two parts $\Sigma^\pm$, so that $P_\pm \subset \Sigma^\pm$. So far as all these contours are homologous and integrands are holomorphic out

of $P_\pm$, then the integral does not depend on the choice of contour.

On the tensor fields $f_j^{(\lambda)}$ the representations of KN-algebra are realized.

$$\mathcal{L}_{e_i} f_j^{(\lambda)} = e_i \nabla f_j^{(\lambda)} + \lambda f_j^{(\lambda)} \nabla e_i = \sum_k R_{ijk}^{(\lambda)} f_k^{(\lambda)}. \qquad (8)$$

The structure constants $R_{ijk}^{(\lambda)}$ ($R_{ijk}^{(-1)} = C_{ij}^{i+j-k}$) are equal to

$$R_{ijk}^{(\lambda)} = \frac{1}{2\pi i}\oint f_{-k}^{(1-\lambda)} [e_i \nabla f_j^{(\lambda)} + \lambda f_j^{(\lambda)} \nabla e_i] = -R_{i,-k,-j}^{(1-\lambda)}. \qquad (9)$$

The covariant holomophic differential ∇ in the local coordinate system has the form $\nabla = dZ \otimes \nabla_{\partial/\partial Z}$, and its action on the basis $(dZ)^\lambda$ is defined from the formula

$$\nabla_{\partial/\partial Z} (dZ)^\lambda = -\lambda\, \Gamma(Z)(dZ)^\lambda. \qquad (10)$$

The transformation properties of the connection $\Gamma(Z)$ in going from a patch U_α with the complex coordinates Z_α to another patch U_β with the coordinates $Z_\beta = f(Z_\alpha)$ are $\Gamma_\beta(Z_\beta) f' = \Gamma_\alpha(Z_\alpha) - f''/f'$, where $f' = \partial Z_\beta / \partial Z_\alpha$. Expressions (8) and (9) are independent of the connection.

Assume that the algebra (1)-(3) is amongst the set of infinite dimensional algebras characterized by λ:

$$\begin{aligned} [L_i, L_j] &= \sum_k R_{ijk}^{(-1)} L_k, \\ [L_i, N_j^{(\lambda)}] &= \sum_k R_{ijk}^{(1-\lambda)} N_k^{(\lambda)} + \xi_{ij}^{(\lambda)}, \\ [N_i^{(\lambda)}, N_j^{(\lambda)}] &= 0. \end{aligned} \qquad (11)$$

Here the brackets are the abstract commutators. To find possible central extensions $\xi_{ij}^{(\lambda)}$ of the algebra (11), let us make additional transformations. For this, according to the paper [8], consider the set of contours C_τ on the surface Σ, defined as level curves of the function $\mathrm{Re}\, p(Q)$, where $p(Q)=\int_{Q_o}^{Q}\omega$, Q_o is an arbitrary fixed point, and ω is the unique meromorphic differential of the third kind, which has simple poles at $P_\pm$ with residues equal to (± 1) and purely imaginary periods over all cycles.

On the contour C_τ we introduce the delta-functions

$$\Delta_\tau^{(\lambda)}(Q,Q') = \sum_i f_i^{(\lambda)}(Q)\, f_{-i}^{(1-\lambda)}(Q') = \Delta_\tau^{(1-\lambda)}(Q',Q), \tag{12}$$

and the tensor fields, denoted by

$$N^{(\lambda)}(Q) = \sum_i N_i^{(\lambda)} f_{-i}^{(\lambda)}(Q). \tag{13}$$

Using these notations as well as duality relations (7), the algebra (11) can be written as:

$$\begin{aligned}
&[T(Q),T(Q')] = \nabla T(Q)\Delta(Q,Q') + 2T(Q)\nabla\Delta(Q,Q'),\\
&[N^{(\lambda)}(Q),T(Q')] = \nabla N^{(\lambda)}(Q)\Delta(Q,Q') + \lambda N^{(\lambda)}(Q)\nabla\Delta(Q,Q') - \xi^{(\lambda)}(Q,Q'),\\
&[N^{(\lambda)}(Q),N^{(\lambda)}(Q')] = 0,
\end{aligned} \tag{14}$$

where $T(Q)\equiv N^{(2)}(Q)$, $\Delta(Q,Q')\equiv \Delta_\tau^{(-1)}(Q,Q')$, and $Q,Q'\in C_\tau$.

We look for the solution $\xi^{(\lambda)}(Q,Q')$ of the form

$$\xi^{(\lambda)}(Q,Q') = \sum_{n\geq 0} \xi_n^{(\lambda)}(Q)\ \nabla^n \Delta_\tau^{(-1)}(Q,Q'). \tag{15}$$

Theorem: The nontrivial central extensions of the algebra (14) exist only for the tensor fields $N^{(\lambda)}(Q)$ at $\lambda=0,1,2$ and are given by

a. $\xi^{(0)}(Q,Q') = \sigma(Q)\Delta(Q,Q') + c_1\nabla\Delta(Q,Q')$,

b. $\xi^{(1)}(Q,Q') = c_2\nabla^2\Delta(Q,Q')$,

c. $\xi^{(2)}(Q,Q') = c_3\nabla^3\Delta(Q,Q')$,

where c_1,c_2,c_3=const., and, $\sigma(Q)$ **is the form nonexact on C_τ.**

To prove this statement we shall use the constraint derived from the Jacobi identity with three non-vanishing double commutators

which involve cyclic permutations of the operators $N^{(\lambda)}(Q)$, $T(Q')$, $T(Q'')$.

$$\begin{aligned}
&\Delta(Q,Q')\nabla\xi^{(\lambda)}(Q,Q'') + \lambda\xi^{(\lambda)}(Q,Q'')\nabla\Delta(Q,Q') - (Q'\leftrightarrow Q'') =\\
&\qquad = \xi^{(\lambda)}(Q,Q')\nabla'\Delta(Q',Q'') - (Q'\leftrightarrow Q'').
\end{aligned}$$

Inserting here expression (15) and picking terms of the same powers of $\nabla^n\Delta(Q,Q'')$, we obtain

$$\begin{aligned}
&\Delta(Q,Q'')\ \{[\lambda\xi_o^{(\lambda)}(Q)-\nabla\xi_1^{(\lambda)}(Q)]\nabla\Delta(Q,Q') - \sum_{n\geq 2}\nabla\xi_n^{(\lambda)}(Q)\nabla^n\Delta(Q,Q')\} -\\
&-\nabla\Delta(Q,Q'')\ \{[\lambda\xi_o^{(\lambda)}(Q)-\nabla\xi_1^{(\lambda)}(Q)]\Delta(Q,Q') + \sum_{n\geq 2}\xi_n^{(\lambda)}(Q)(\lambda+1-n)\nabla^n\Delta(Q,Q')\} +\\
&+\sum_{n\geq 2}\nabla^n\Delta(Q,Q'')\ \{\Delta(Q,Q')\nabla\xi_n^{(\lambda)}(Q) + (\lambda+1-n)\xi_n^{(\lambda)}(Q)\nabla\Delta(Q,Q') +\\
&+\sum_{m\geq n+1}\xi_m^{(\lambda)}(Q)\ (C_m^{n-1} - C_m^{n})\nabla^{m-n+1}\Delta(Q,Q')\} = 0,
\end{aligned} \tag{16}$$

where C_m^n -are the binomial coefficients. Equation (16) leads to the constraints on functions $\xi_m^{(\lambda)}(Q)$:

$$\begin{aligned}
&\lambda\xi_o^{(\lambda)}(Q) = \nabla\xi_1^{(\lambda)}(Q),\\
&\nabla\xi_n^{(\lambda)}(Q) = 0, \qquad \xi_n^{(\lambda)}(Q)(\lambda+1-n) = 0, \qquad n\geq 2.\\
&\xi_n^{(\lambda)}(Q) = 0, \qquad n\geq 4.
\end{aligned} \tag{17}$$

The theorem assertion is now obvious.

Amongst the above obtained algebras there is one which can be considered as an extended algebra of constraints and subsidiary conditions for the interacting string. Indeed, suppose that $\lambda=0$ in (14) and choose $\xi^{(0)}(Q,Q') = \sigma(Q)\Delta(Q,Q')$. Then using duality relations (7) and denoting $N_i^{(2)}\equiv L_i$, we obtain the algebra of (11) type, where

$$\xi_{i,-j}^{(0)} = \frac{1}{2\pi i}\oint \omega^j\, e_i\, \sigma. \tag{18}$$

In a particular case, the third kind differential ω may be considered as σ. Then the central element $\xi_{i,-j}^{(0)}$ will satisfy the local condition: $\xi^{(0)}_{i,-j} = 0$ at $|i-j|>g$. At $g=0$ it is equal to $\xi^{(0)}_{i,-j}|_{g=o,\sigma=\omega} = \delta^j_i$, and the whole construction coincides with the well-known extended algebra of constraints and subsidiary conditions for the free bosonic string [4,5].

Let us describe now the realization of algebra (11) at $\lambda=0$, in terms of dynamical variables of the closed string. Note that the sets of basic functions $\{f^{(\lambda)}_j\}$ are full on the contour C_τ at each λ. Therefore the dynamical variables of the string may be expanded as

$$X_\mu(Q)=X^n_\mu\ A_n(Q), \qquad P_\mu(Q)=P_{\mu n}\omega^n(Q). \tag{19}$$

The expansion coefficients obey the Poisson brackets

$$[X^n_\mu, X^m_\nu]=0, \qquad [X^n_\mu, P_{\nu m}]=\eta_{\mu\nu}\delta^n_m, \qquad [P^\mu_n, P^\nu_m]=0. \tag{20}$$

The expressions for the operators L_k were obtained in [9]

$$L_k = 1/2\ \ell^{mn}_k\ \alpha^\mu_m\ \alpha^\mu_n, \qquad a \qquad \ell^{mn}_k = \frac{1}{2\pi i}\oint_{C_\tau} \omega^m\ \omega^n\ e_k. \tag{21}$$

Unlike the variables X^n_μ and P^μ_m, which depend on τ, the coefficients $\sqrt{2}\alpha^\mu_n = (P_{\mu n}+i\gamma_{nm}X^m_\mu)$ are τ-independent. This follows from the **Stokes-theorem**. The constants γ_{nm} are equal to

$$\gamma_{nm} = \frac{1}{2\pi i}\oint_{C_\tau} A_n\ dA_m\ . \tag{22}$$

To solve the second equation in (11) and thus to find the functions $\Psi^i \equiv N^{(0)}_{-i}$, it is useful to transform this equation into

$$[\Psi(Q), T(Q')] = [d\Psi(Q)-\sigma(Q)]\Delta(Q,Q'), \tag{23}$$

where $\Psi(Q)= \Psi^n A_n(Q)$. The solution is given by

$$\Psi(Q) = X(Q_o) + \int_{Q_o}^{Q} (\pi + \sigma), \qquad Q,Q_o \in C_\tau\ . \tag{24}$$

Here $X(Q_o)\equiv X_\mu(Q_o)k^\mu$ is the integration constant, k_μ is the constant Lorentz vector, $\pi(Q)\equiv k_\mu\pi_\mu(Q)$, $\pi_\mu(Q)=\frac{1}{\sqrt{2}}\alpha_{\mu n}\omega^n(Q)$. The differential $\sigma(Q)$ is defined from the agreement condition

$$\oint_{C^i_\tau} [\pi(Q)+\sigma(Q)] = 0, \tag{25}$$

where C^i_τ are the connected components of the contour C_τ. The arbitrary light-like vector k_μ breaks the manifest Lorentz invariance of the theory by picking out a preferred direction in space-time. Therefore it is merely an auxiliary quantity. (The fulfilment of the last equation in (11) will be guaranteed by $k^2=0$). The problem of k_μ-elimination has been analyzed in papers [10,11].

The general solution of (23) is the sum of the partial solution and the solution of the homogeneous equation. The constant $X(Q_o)$ in (24), except α^μ_n, obviously depends on $\bar{\alpha}^\mu_n= \frac{1}{\sqrt{2}}(P_{\mu n}-i\gamma_{nm}X^m_\mu)$. The latter operator commutes with α^μ_n, and hence with $T(Q)$. To eliminate this dependence we have to separate X^n_μ as $X^n_\mu=x^n_\mu(\alpha)+\bar{x}^n_\mu(\bar{\alpha})$, and reject the second term. In general it is difficult, because we don't know the inverse matrix to γ_{nm}. However, in each concrete case, i.e. when the genus of the surface Σ is fixed, and the matrix elements γ_{nm} are known, the given procedure may be realized.

In conclusion we point out that all the above results may be similarly applied to the conjugate sector of the closed string. We hope that the extention of the KN-algebra obtained here will be useful in calculations of the g-loop string diagrams and for applying operator methods to the problems of the conformal field theory on the genus g Riemann surfaces.

The authors are grateful to Professor A.T.Filippov for interesting discussions.

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Received by Publishing Department
on December 30, 1988.

Кашаев Р.М., Осипов А.А. E2-88-931

Расширение алгебры Кричевера-Новикова, фиксирующее калибровку в теории замкнутой струны

Обсуждаются возможные специальные расширения алгебры Кричевера-Новикова. Среди них имеется такое, которое можно трактовать как замкнутую алгебру связей и дополнительных условий в теории бозонной струны с мировой поверхностью фиксированной топологии. Получена реализация данной алгебры в терминах струнных переменных. Отсюда делается вывод о том, что симметрия изучаемой квантовой системы шире, чем обычная BRST-инвариантность.

Работа выполнена в Лаборатории ядерных проблем ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1988

Kashaev R.M., Osipov A.A. E2-88-931

The Gauge Fixing Extension of the Krichever-Novikov Algebra in the Closed String Theory

Possible special extensions of the Krichever-Novikov algebra are discussed. Among them there is one which can be interpreted as the closed algebra of constraints and subsidiary conditions in the theory of the boson string with the fixed topology world-sheet. Realization of the given algebra is obtained in terms of string variables. The conclusion is drawn that the symmetry of the quantum system studied is wider than the usual BRST-invariance.

The investigation has been performed at the Laboratory of Nuclear Problems, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1988