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CONTINUOUS REPRESENTATION FOR SPIN 1/2, QUANTUM PROBABILITY THEORY AND BELL PARADOX

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I. Introduction. The probability calculus of the quantum mechanics differs essentially from the classical probability theory (first of all by its law of addition of probabilities)[1]. In what follows we shall consider the quantum mechanics of spin 1/2. A comparison of both the probability theories is difficult because the classical and quantum theories use different languages. To make it possible we shall translate the quantum mechanics of spin 1/2 into the continuous representation for spin 1/2 (the $\vec{S}$ -representation, or the coherent state representation[x]), some sort of the classical language.

In particular, the $\vec{S}$ -representation is of interest for analysing the Bell paradox[2,3] (for further information see refs.[4-20]): the Bell iequality

$$|c_{cl}(\vec{a},\vec{b})-c_{cl}(\vec{a},\vec{b}')|+|c_{cl}(\vec{a}',\vec{b})+c_{cl}(\vec{a}',\vec{b}')|\leq 2 \qquad (1)$$

obtained in the framework of the classical probability theory is violated by quantum correlators both theoretically and experimentally (see, however, [13,14]). What this means is still being discussed. Bell and his followers treat this fact as indication against locality and hidden variables in the quantum mechanics. Brody argues in favour of the irrelevance of the Bell inequality for these problems [15].

Here it will be shown in the classical terms adopted that the quantum correlator of two spins $\frac{1}{2}$ in the singlet state equals 9 times a classical one. Thus, it is simply illegitimate to put it into the Bell inequality (1). The problems of locality and hidden variables seem however to remain apart.

In the $\vec{S}$-representation the quantum equations of motion in the Schrödinger and Heisenberg pictures take a classical form, being the first order partial differential equations similar to the Liouville equation in classical mechanics. Their solutions can be expressed via characteristics subjected to the set of the first order ordinary differential equations relative to the classical Hamilton equations. However, the quantum theory still differs from the classical one in choice of probability densities and in construction of various quantities (e.g., the above correlator), of an analog of the Markov property, etc. These quantities and relations can be convertsd into

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[x] See refs.[21-25] for some information on the subject. However, our treatment will be self-contained.

their classical form (of the classical probability (stochastic process) theory) in the framework of quantum mechanics as well, but only in terms of modified "probability densities", which are not positive definite.

Besides the approach based on the sphere S^2, an approach is formulated using the sphere S^3.

<u>2. Continuous representation ($\vec{s}$ -representation) for spin 1/2.</u> Instead of the standard matrix formalism for spin 1/2 let us introduce an equivalent continuous representation. Let $|a\rangle = \xi$ be the complex two-component spinor

$$|a\rangle = U(a)|\;\rangle \equiv \xi \equiv \xi(a) = U(a)\binom{1}{0} \tag{2}$$

where $U(a)$ is a unitary SU(2) 2x2 matrix

$$U(a) = a_0 1 + i a_m \sigma_m = a_0 e_0 + a_m e_m \,. \tag{3}$$

Here σ_m are the Pauli 2x2 matrices, and the last expression is given in terms of the quaternions which absorb i . The variables a take all the values on the unit sphere S^3

$$\bar{\xi}\xi = a^2 = a_\mu a_\mu = 1 \,. \qquad (S^3) \tag{4}$$

There are two ways for the realization of the Hopf map (the Hopf fibre bundle) $S^3 \to S^2$ (fiber $S^1 = SO(2) = U(1)$):

Way a: $\quad \xi \to s_m = \bar{\xi}\sigma_m\xi \qquad (5)$

(where s_m are base variables, $\vec{s} \in S^2$, $\vec{s}^2 = 1$),

Way b: $\quad \xi \to |a\rangle\langle a| \equiv \xi_\alpha \bar{\xi}_\beta = \frac{1}{2}(1+\vec{s}\vec{\sigma})_{\alpha\beta}$ x $\qquad (6)$

($s_m = \bar{\xi}\sigma_m\xi$ again). This expression is a hermitian matrix (the use of the quaternions introduces i) and the idempotent

$|a\rangle\langle a|\cdot|a\rangle\langle a| = |a\rangle\langle a|$, or $\xi_\alpha\bar{\xi}_\beta\xi_\beta\bar{\xi}_\gamma = \xi_\alpha\bar{\xi}_\gamma$. With all possible $a \in S^3$ or $\vec{s} \in S^2$ we have the continuum of the idempotents.

As consequences of the matrix completeness relations

$$\frac{4}{\Omega_3}\int d^4a\,\delta(a^2-1)\,|U^+(a)|\otimes\|U(a)\| = \qquad (\Omega_3 = 2\pi^2) \tag{7.a}$$

$$= \frac{1}{2}\left(|\mathbb{1}|\otimes\|\mathbb{1}\| + |\sigma_m|\otimes\|\sigma_m\|\right) = |\mathbb{1}\|\otimes\|\mathbb{1}| \tag{7.b}$$

we obtain the following completeness relations

$$c\int d^3s\,\delta(\vec{s}^2-1)\,|a\rangle\langle a| = \mathbb{1} \qquad \left(c\int d^3s\,\delta(\vec{s}^2-1) = 2,\; c = \frac{4}{\Omega_2} = \frac{1}{\pi}\right)$$

$$c\int d^3s\,\delta(\vec{s}^2-1)\,\xi_\alpha\bar{\xi}_\beta = c\int d^3s\,\delta(\vec{s}^2-1)\,\frac{1}{2}(1+\vec{s}\vec{\sigma})_{\alpha\beta} = \delta_{\alpha\beta}\,, \tag{8}$$

x It is the identity (6) which serves as the main tool in what follows. A derivation from eq. (7.b) is immediate (see Appendix A).

$$3c\int d^3s\,\delta(\vec{s}^2-1)\,|a\rangle\langle a|\otimes\|a\rangle\langle a\| = |\mathbb{1}|\otimes\|\mathbb{1}\| + |\mathbb{1}\|\otimes\|\mathbb{1}|\,,$$

$$3c\int d^3s\,\delta(\vec{s}^2-1)\,\xi_\alpha\bar{\xi}_\beta\xi_\gamma\bar{\xi}_\delta =$$

$$= 3c\int d^3s\,\delta(\vec{s}^2-1)\,\frac{1}{2}(1+\vec{s}\vec{\sigma})_{\alpha\beta}\,\frac{1}{2}(1+\vec{s}\vec{\sigma})_{\gamma\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta}\;{}^{x\;xx} \tag{9}$$

The fiber variable (i.e. the general phase factor of the spinor ξ) has fallen out of these relations, and therefore, the integration over S^3 is reduced to the integration over S^2 . These relations have been obtained merely by integrating over $\vec{s}$ and by using additionally in eq. (9) the completeness relation (7.b). The sets of unitary SU(2) matrices $U(a)$ in eq. (7.a), the spinors in eq.(8) and the idempotents (orthogonal matrices) in eq. (9) are not only complete but are extremely overcomplete. The following two statements are valid.

<u>1) The reconstruction theorem</u>. From the completeness relation (9) it follows that

$$\hat{F}_{\alpha\beta} = -\delta_{\alpha\beta}\,\hat{F}_{\gamma\gamma} + 3c\int d^3s\,\delta(\vec{s}^2-1)\,\xi_\alpha\bar{\xi}_\beta\,(\bar{\xi}\hat{F}\xi)$$

$$= -\delta_{\alpha\beta}\,\hat{F}_{\gamma\gamma} + 3c\int d^3s\,\delta(\vec{s}^2-1)\,\frac{1}{2}(1+\vec{s}\vec{\sigma})_{\alpha\beta}\,(\bar{\xi}\hat{F}\xi)\,, \tag{10}$$

$$\mathrm{tr}\,\hat{F} = \hat{F}_{\gamma\gamma} = c\int d^3s\,\delta(\vec{s}^2-1)\,(\bar{\xi}\hat{F}\xi)\,. \tag{11}$$

That means that any operator $\hat{F} = f_0 1 + f_m\sigma_m$ is completely defined by its expectation values

$F(\vec{s}) = \bar{\xi}\hat{F}\xi = f_0 + f_m s_m$ (the operator $\hat{F}$ in the $\vec{s}$-representation)(12)

The representative of any operator is always a <u>linear function</u> of $\vec{s}$.

<u>2)</u> All matrix calculations in the theory of spin 1/2 can be converted to equivalent calculations in terms of the $\vec{s}$- representation, i.e. in terms of the functions of $\vec{s}$.

In particular, from eq. (9) or better from eq. (10) it follows immediately for the trace of two operators

$$\mathrm{tr}(\hat{\rho}\,\hat{F}) = -(\mathrm{tr}\,\hat{\rho})(\mathrm{tr}\,\hat{F}) + 3c\int d^3s\,\delta(\vec{s}^2-1)\,\rho(\vec{s})\,F(\vec{s})\,, \tag{13}$$

x The same way (using the identity (6)) produces further integrals with more spinors (see eqs. (E.3) and (E.4) of Appendix E).

xx For a discrete analog of eq. (9) see Appendix B.

where $\rho(\vec{s}) = \bar{\xi}\hat{\rho}\xi$,

$$tr\,\hat{\rho} = \hat{\rho}_{\alpha\alpha} = c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\rho(\vec{s}). \tag{14}$$

If $\hat{\rho}$ is the density matrix, $tr\,\hat{\rho}=1$ and eq. (13) gives the expectation value of the operator $\hat{\mathbb{F}}$.

3. Density matrices in terms of the $\vec{s}$-representation.

Translating into the $\vec{s}$-representation by the rule

$$\sigma_m \longrightarrow \bar{\xi}\,\sigma_m\,\xi = s_m \tag{15}$$

$$\hat{\mathbb{F}} = f_0 1 + f_m\sigma_m \longrightarrow \mathbb{F}(\vec{s}) = \bar{\xi}\hat{\mathbb{F}}\xi = f_0 + f_m s_m \tag{16}$$

we find for the density matrices the following probability densities in the $\vec{s}$-representation:

for the density matrix of the state with spin $+\frac{1}{2}$ along $\vec{s}_0$

$$\hat{\rho}_{\vec{s}_0} = |a^0\rangle\langle a^0| = \xi^0_\alpha\bar{\xi}^0_\beta = \frac{1}{2}(1+\vec{s}_0\vec{\sigma})_{\alpha\beta} \tag{17.a}$$

$$\longrightarrow \rho_{\vec{s}_0}(\vec{s}) = \bar{\xi}\hat{\rho}_{\vec{s}_0}\xi = \frac{1}{2}(1+\vec{s}_0\vec{s}) \tag{17.b}$$

and for the density matrix of the singlet state of two particles a and b of spin 1/2

$$\hat{\rho}^{\,singlet}_{\alpha\beta,\alpha'\beta'} = \frac{1}{4}\left[\delta_{\alpha\alpha'}\delta_{\beta\beta'} - (\sigma^a_m)_{\alpha\alpha'}(\sigma^b_m)_{\beta\beta'}\right] = \frac{1}{2}\left(\delta_{\alpha\alpha'}\delta_{\beta\beta'} - \delta_{\alpha\beta'}\delta_{\beta\alpha'}\right) = \frac{1}{2}\varepsilon_{\alpha\beta}\varepsilon_{\alpha'\beta'} \tag{18.a}$$

$$\longrightarrow \rho^{singlet}(\vec{s}^{\,a},\vec{s}^{\,b}) = \bar{\xi}^a_\alpha\bar{\xi}^b_\beta\,\rho^{singlet}_{\alpha\beta,\alpha'\beta'}\,\xi^a_{\alpha'}\xi^b_{\beta'} = \frac{1}{4}(1-\vec{s}^{\,a}\vec{s}^{\,b}). \tag{18.b}$$

4. Predictions of the quantum mechanics of spin 1/2 in terms of the $\vec{s}$-representation will be given below.

The normalization condition for the density matrix (17.a), the expectation value of the spin projection along $\vec{n}$ in this state and the probability to find the spin $+\frac{1}{2}$ along $\vec{n}$ in the state with the spin $+\frac{1}{2}$ along $\vec{m}$ are written as

$$tr\,\hat{\rho}_{\vec{s}_0} = c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\rho_{\vec{s}_0}(\vec{s}) = 1, \tag{19}$$

$$tr(\hat{\rho}_{\vec{s}_0}(\vec{n}\vec{\sigma})) = 3c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\rho_{\vec{s}_0}(\vec{s})\,\vec{n}\vec{s} = \vec{n}\vec{s}_0, \tag{20}$$

$$w(+\tfrac{1}{2}\text{ along }\vec{n}\,|\,+\tfrac{1}{2}\text{ along }\vec{m}) = tr(\hat{\rho}_f\hat{\rho}_i) = tr\left(\tfrac{1}{2}(1+\vec{n}\vec{\sigma})\tfrac{1}{2}(1+\vec{m}\vec{\sigma})\right) =$$

$$= -1 + 3c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\rho_f(\vec{s})\,\rho_i(\vec{s}) = \frac{1}{2}(1+\vec{n}\vec{m}) \tag{21}$$

where $\rho_i(\vec{s}) = \frac{1}{2}(1+\vec{m}\vec{s})$, $\rho_f(\vec{s}) = \frac{1}{2}(1+\vec{n}\vec{s})$

To obtain expressions (20) and (21), eq. (13) is used.

Let us now consider the singlet state of two spins 1/2. The normalization of the singlet state (18.b), the correlation of the spin projections of the particle a along $\vec{a}$ and of the particle b along $\vec{b}$ and the probability to find in the singlet state the spin $+\frac{1}{2}$ along $\vec{a}$ for the particle a and the spin $+\frac{1}{2}$ along $\vec{b}$ for the particle b are written as follows:

$$\rho^{singlet}_{\alpha\beta,\alpha\beta} = c^2\int d^3s^a\,d^3s^b\,\delta(\vec{s}^{\,a2}-1)\,\delta(\vec{s}^{\,b2}-1)\,\rho^{singlet}(\vec{s}^{\,a},\vec{s}^{\,b}) = 1, \tag{22}$$

$$c(\vec{a},\vec{b}) = \hat{\rho}^{\,singlet}_{\alpha\beta,\alpha'\beta'}(\vec{a}\vec{\sigma}^a)_{\alpha'\alpha}(\vec{b}\vec{\sigma}^b)_{\beta'\beta} =$$

$$= 3^2c^2\int d^3s^a\,d^3s^b\,\delta(\vec{s}^{\,a2}-1)\,\delta(\vec{s}^{\,b2}-1)\,\rho^{singlet}(\vec{s}^{\,a},\vec{s}^{\,b})(\vec{a}\vec{s}^{\,a})(\vec{b}\vec{s}^{\,b}) \tag{23}$$

$$= -\vec{a}\vec{b}$$

$$w(+\tfrac{1}{2}\text{ along }\vec{a}, +\tfrac{1}{2}\text{ along }\vec{b}\,|\,\text{singlet}) = \hat{\rho}^{\,singlet}_{\alpha\beta,\alpha'\beta'}(\hat{\rho}_{\vec{a}})_{\alpha'\alpha}(\hat{\rho}_{\vec{b}})_{\beta'\beta} =$$

$$= -2\,\hat{\rho}^{\,singlet}_{\alpha\beta,\alpha\beta}(\hat{\rho}_{\vec{a}})_{\gamma\gamma}(\hat{\rho}_{\vec{b}})_{\delta\delta} +$$

$$+ 3^2c^2\int d^3s^a\,d^3s^b\,\delta(\vec{s}^{\,a2}-1)\,\delta(\vec{s}^{\,b2}-1)\,\rho^{singlet}(\vec{s}^{\,a},\vec{s}^{\,b})\,\rho_{\vec{a}}(\vec{s}^{\,a})\,\rho_{\vec{b}}(\vec{s}^{\,b}) =$$

$$= \frac{1}{4}(1-\vec{a}\vec{b}) \tag{24}$$

where $\rho_a(\vec{s}^a) = \frac{1}{2}(1+\vec{a}\vec{s}^a)$, $\rho_{\vec{b}}(\vec{s}^b) = \frac{1}{2}(1+\vec{b}\vec{s}^b)$ and in the second expression of eq. (24) all the traces in the first term equal 1, and therefore this term equals -2.

5. A comparison with the classical probability theory. On the one hand, we observe some similarity of the predictions (19) and (20) or (22) and (23) with those for the same objects within the framework of the classical probability theory if one takes as probability densities the functions (17.b) or (18.b), respectively. On the other hand, the coefficients 3 for the expectation value of the spin projection and 3^2 for the spin correlator destroy the similarity. Other predictions also contain the main terms of the form similar to the classical probability theory but with "wrong" coefficients $3, 3^2$ (and so on) [x)] and some additional terms (as in eqs. (13), (21) and (24)). These coefficients and additional terms embody quantum effects. One can say that the quantum probability theory of spin 1/2 is a deformation of the mentioned underlying classical probability theory, i.e. it is constructed out of elements of the latter, but in fact goes beyond the latter, being radically different.

The above quantities do not contain the Planck constant $\hbar$ (apart from the general factors $\frac{\hbar}{2}$, $(\frac{\hbar}{2})^2$). The substitution $3 \to 1$ (in the sense $j(j+1) \to j^2$) and neglecting of the additional terms may be treated as a passage to the classical limit.

6. An inequality for the quantum correlators. Instead of the Bell inequality (I) we obtain for the quantum correlators (23) the inequality

$$|c(\vec{a},\vec{b}) - c(\vec{a},\vec{b}')| + |c(\vec{a}',\vec{b}) + c(\vec{a}',\vec{b}')| \leqslant 2\cdot 3^2 \qquad (25)$$

(if one wishes he can simply repeat the standard derivation, with the second expression (23)). However, this estimation is absolutely

x) The appearance of the quantum coefficients 3 is inevitable as it is seen from the comparison of the main operations in the matrix and $\vec{s}$ -representations

$$tr\, 1 = 2, \qquad tr\, \sigma_m = 0, \qquad tr\, \sigma_m\sigma_n = 2\delta_{mn}$$

$$\int d^3s\, \delta(\vec{s}^2-1)\{1, s_m, s_m s_n\} = \{1, 0, \tfrac{1}{3}\delta_{mn}\}\int d^3s\, \delta(\vec{s}^2-1)$$

unrestrictive since for the actual correlator $c(\vec{a},\vec{b}) = -\vec{a}\vec{b}$ the left-hand side does not exceed 4 (it is obvious) and even 3 (by the Schwartz inequality) and in fact $2\sqrt{2}$. In the classics with the probability density (18.b) we obtain

$$c_{cl}(\vec{a},\vec{b}) = -\frac{1}{9}(\vec{a}\vec{b}) \qquad (26)$$

and for it the Bell inequality (I) is unrestrictive too. The quantum correlator equals 9 times the classical one and it is illegitimate to put it into the Bell inequality (1). The problems of locality and hidden variables seem however to remain apart. Note that the variables s_m may possibly play the role of hidden variables.

In what follows we consider other aspects of the $\vec{s}$-representation.

7. A representative of the product of two operators. Let us express two operators $\hat{F}_1$ and $\hat{F}_2$ via their representatives by eqs. (10) and (11). Then, for the representative of their product, we find

$$\mathbb{F}(\vec{s}) = \mathbb{F}_1(\vec{s}) * \mathbb{F}_2(\vec{s}) \equiv \xi\, \hat{F}_1\hat{F}_2\, \xi =$$
$$= \hat{F}_{1\varkappa\varkappa}\hat{F}_{2\lambda\lambda} - 3c\hat{F}_{1\varkappa\varkappa}\int d^3s'\,\delta(\vec{s}'^2-1)\,\tfrac{1}{2}(1+\vec{s}\vec{s}')\mathbb{F}_2(\vec{s}') -$$
$$-3c\hat{F}_{2\varkappa\varkappa}\int d^3s'\,\delta(\vec{s}'^2-1)\,\tfrac{1}{2}(1+\vec{s}\vec{s}')\mathbb{F}_1(\vec{s}') +$$
$$+9c^2\int d^3s'\,\delta(\vec{s}'^2-1)\int d^3s''\,\delta(\vec{s}''^2-1)\,K(\vec{s},\vec{s}',\vec{s}'')\mathbb{F}_1(\vec{s}')\mathbb{F}_2(\vec{s}''), \qquad (27)$$

where $K(\vec{s},\vec{s}',\vec{s}'')$ is a kernel of the form

$$K(\vec{s},\vec{s}',\vec{s}'') = \frac{1}{4}\left[1 + \vec{s}'\vec{s}'' + \vec{s}'\vec{s} + \vec{s}''\vec{s} + i\varepsilon_{klm}s'_k s''_l s_m\right]. \qquad (28)$$

Operation (27) over the representatives $\mathbb{F}_1(\vec{s})$ and $\mathbb{F}_2(\vec{s})$ is denoted symbolically by the asterisk * ("multiplication"). With the representative of the product of two operators one can reconstruct the product $\hat{F}_1\hat{F}_2$ itself according to eq. (10).

For symmetrical and antisymmetrical parts of the product of two operators, we obtain the representatives

$$\mathbb{F}_s(\vec{s}) = \frac{1}{2}\left(\mathbb{F}_1(\vec{s}) * \mathbb{F}_2(\vec{s}) + \mathbb{F}_2(\vec{s}) * \mathbb{F}_1(\vec{s})\right) = \frac{1}{2}\xi(\hat{F}_1\hat{F}_2 + \hat{F}_2\hat{F}_1)\xi =$$
$$= \hat{F}_{1\varkappa\varkappa}\hat{F}_{2\lambda\lambda} - 3c\hat{F}_{1\varkappa\varkappa}\int d^3s'\,\delta(\vec{s}'^2-1)\,\tfrac{1}{2}(1+\vec{s}\vec{s}')\mathbb{F}_2(\vec{s}') -$$
$$-3c\hat{F}_{2\varkappa\varkappa}\int d^3s'\,\delta(\vec{s}'^2-1)\,\tfrac{1}{2}(1+\vec{s}\vec{s}')\mathbb{F}_1(\vec{s}') +$$
$$+9c^2\int d^3s'\,\delta(\vec{s}'^2-1)\int d^3s''\,\delta(\vec{s}''^2-1)\,K_s(\vec{s},\vec{s}',\vec{s}'')\mathbb{F}_1(\vec{s}')\mathbb{F}_2(\vec{s}''), \qquad (29)$$

where

$$K_s(\vec{s},\vec{s}',\vec{s}'') = \frac{1}{4}\left[1+\vec{s}'\vec{s}''+\vec{s}'\vec{s}+\vec{s}''\vec{s}\right] \tag{30}$$

$$\mathbb{F}_a(\vec{s}) = -\frac{i}{2}\left(\mathbb{F}_1(\vec{s})*\mathbb{F}_2(\vec{s})-\mathbb{F}_2(\vec{s})*\mathbb{F}_1(\vec{s})\right) = -\frac{i}{2}\bar{\xi}[\hat{\mathbb{F}}_1,\hat{\mathbb{F}}_2]\xi =$$

$$= 9c^2\int d^3s'\delta(\vec{s}'^2-1)\int d^3s''\delta(\vec{s}''^2-1)\,\varepsilon_{k\ell m}s'_k s''_\ell s_m\,\mathbb{F}_1(\vec{s}')\mathbb{F}_2(\vec{s}''). \tag{31}$$

8. Left and right operator representatives. The nonoperator representatives $\mathbb{F}(\vec{s}) = \bar{\xi}\hat{\mathbb{F}}\xi$ have been defined above. It is useful to introduce also operator repesentatives which act on the nonoperator ones. They can be defined as follows:

$$\bar{\xi}\hat{\mathbb{F}}\hat{\mathbb{G}}\xi = \mathbb{F}^\ell(\bar{\xi}\hat{\mathbb{G}}\xi) = \mathbb{F}^\ell\mathbb{G} = \mathbb{G}^r(\bar{\xi}\hat{\mathbb{F}}\xi) = \mathbb{G}^r\mathbb{F} \ . \tag{32}$$

Let us stress that the left and right representatives $\mathbb{F}^\ell$ and $\mathbb{G}^r$ are partial differential operators acting on $\bar{\xi}$ and ξ. The left and right representatives of the matrices σ_m are

$$\sigma^\ell_m = \bar{\xi}\sigma_m\left[\frac{\partial}{\partial\bar{\xi}} - \frac{1}{2}\xi\left(\xi\frac{\partial}{\partial\xi}+\bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right)\right] + \bar{\xi}\sigma_m\xi =$$

$$= \bar{\xi}\sigma_m\frac{\partial}{\partial\bar{\xi}} + s_m\left[1-\frac{1}{2}\left(\xi\frac{\partial}{\partial\xi}+\bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right)\right], \tag{33}$$

$$\sigma^r_m = \xi\sigma^T_m\left[\frac{\partial}{\partial\xi} - \frac{1}{2}\bar{\xi}\left(\xi\frac{\partial}{\partial\xi}+\bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right)\right] + \bar{\xi}\sigma_m\xi =$$

$$= \xi\sigma^T_m\frac{\partial}{\partial\xi} + s_m\left[1-\frac{1}{2}\left(\xi\frac{\partial}{\partial\xi}+\bar{\xi}\frac{\partial}{\partial\bar{\xi}}\right)\right], \tag{34}$$

where $\bar{\xi}\sigma_m\frac{\partial}{\partial\bar{\xi}} = \bar{\xi}_\alpha(\sigma_m)_{\alpha\beta}\frac{\partial}{\partial\bar{\xi}_\beta}$ and so on. One can easily verify that

$$\sigma^\ell_m(\bar{\xi}\hat{\mathbb{F}}\xi) = \bar{\xi}\sigma_m\hat{\mathbb{F}}\xi\,, \qquad \sigma^\ell_k\sigma^\ell_\ell = \delta_{k\ell}+i\varepsilon_{k\ell m}\sigma^\ell_m\,, \tag{35}$$

$$\sigma^r_m(\bar{\xi}\hat{\mathbb{F}}\xi) = \bar{\xi}\hat{\mathbb{F}}\sigma_m\xi\,, \qquad \sigma^r_k\sigma^r_\ell = \delta_{k\ell}-i\varepsilon_{k\ell m}\sigma^r_m\,. \tag{36}$$

With the operators σ^ℓ_m and σ^r_m we can construct left and right representatives for any operator $\hat{\mathbb{F}} = f_0 1 + f_m\sigma_m$ [x]

$$\mathbb{F}^\ell = f_0 + f_m\sigma^\ell_m\,, \tag{37}$$

$$\mathbb{F}^r = f_0 + f_m\sigma^r_m\,. \tag{38}$$

Note that all the terms of eqs. (33) and (34), except for the first

[x] $[\sigma^\ell_m,\sigma^r_n] = [\mathbb{F}^\ell,\mathbb{G}^r] = 0$ due to associativity. For other rules to handle the l,r-operators see refs./25,28-32/.

ones, are due to the normalization of the spinors ξ. One has merely

$$\sigma^\ell_m(\bar{\eta}\hat{\mathbb{F}}\eta) = \left(\bar{\eta}\sigma_m\frac{\partial}{\partial\bar{\eta}}\right)(\bar{\eta}\hat{\mathbb{F}}\eta) = \bar{\eta}\sigma_m\hat{\mathbb{F}}\eta \tag{39}$$

$$\sigma^r_m(\bar{\eta}\hat{\mathbb{F}}\eta) = \left(\eta\sigma^T_m\frac{\partial}{\partial\eta}\right)(\bar{\eta}\hat{\mathbb{F}}\eta) = \bar{\eta}\hat{\mathbb{F}}\sigma_m\eta \tag{40}$$

for unnormalized spinors η. We start with these obvious relations to obtain the above operators σ^ℓ_m and σ^r_m (see Appendix C).

Equations (32) supply us with two more expressions for the representative of the product of two operators in addition to eq. (27).

9. Equations of motion for the probability density and observables. We start with the standard equations of motion for the density matrix $\hat{\rho}$ and for any operator (observable) $\hat{\mathbb{F}}$ which does not depend explicitly on time, i.e. with the Neumann and Heisenberg-Born-Jordan-Dirac equations

$$\hbar\frac{d}{dt}\hat{\rho}(t) = -i[\hat{H},\hat{\rho}(t)]\,, \qquad \frac{d}{dt}\hat{\mathbb{F}} = 0\,, \tag{41}$$

$$\hbar\frac{d}{dt}\hat{\mathbb{F}}(t) = i[\hat{H},\hat{\mathbb{F}}(t)]\,, \qquad \frac{d}{dt}\hat{\rho} = 0 \tag{42}$$

in the Schrödinger and Heisenberg pictures, respectively. If the Hamiltonian $\hat{H}$ does not depend on time, formal solutions of these equations are

$$\hat{\rho}(t) = e^{-i\hbar^{-1}\hat{H}t}\hat{\rho}(0)\,e^{i\hbar^{-1}\hat{H}t}\,, \tag{43}$$

$$\hat{\mathbb{F}}(t) = e^{i\hbar^{-1}\hat{H}t}\hat{\mathbb{F}}(0)\,e^{-i\hbar^{-1}\hat{H}t}\,. \tag{44}$$

Between $\bar{\xi}$ and ξ eqs. (41) and (42) take a form of the Liouville equation (cf. refs./25,28-32/)

$$\frac{\partial}{\partial t}\rho(\vec{s},t) = -\mathcal{L}\,\rho(\vec{s},t)\,, \qquad \frac{\partial}{\partial t}\mathbb{F}(\vec{s}) = 0\,, \tag{45}$$

$$\frac{\partial}{\partial t}\mathbb{F}(\vec{s},t) = \mathcal{L}\,\mathbb{F}(\vec{s},t)\,, \qquad \frac{\partial}{\partial t}\rho(\vec{s}) = 0 \tag{46}$$

in the Schrödinger and Heisenberg pictures, respectively. Here $\rho(\vec{s},t) = \bar{\xi}\hat{\rho}(t)\xi$, $\mathbb{F}(\vec{s}) = \bar{\xi}\hat{\mathbb{F}}\xi$, $\mathbb{F}(\vec{s},t) = \bar{\xi}\hat{\mathbb{F}}(t)\xi$, $\rho(\vec{s}) = \bar{\xi}\hat{\rho}\xi$ and

$$\mathcal{L} = i\hbar^{-1}(H^\ell - H^r) \tag{47}$$

is a Liouvillian, a partial differential operator. As formal solutions of eqs. (45) and (46) we have

$$\rho(\vec{s},t) = e^{-\mathcal{L}t}\rho(\vec{s},0), \tag{48}$$

$$\mathbb{F}(\vec{s},t) = e^{\mathcal{L}t}\,\mathbb{F}(\vec{s},0), \tag{49}$$

supposing that the Hamiltonian, and therefore, the Liouvillian do not depend on time. These solutions follow also from eqs. (43) and (44).

The Hamiltonian and its nonoperator and operator representatives are written as follows

$$\hat{H} = -\frac{\hbar}{2}\vec{\omega}\vec{\sigma}, \qquad \vec{\omega} = \frac{e}{2mc}\vec{B}, \tag{50}$$

$$H(\vec{s}) = -\frac{\hbar}{2}\vec{\omega}\vec{s}, \quad H^{l} = -\frac{\hbar}{2}\vec{\omega}\vec{\sigma}^{l}, \quad H^{r} = -\frac{\hbar}{2}\vec{\omega}\vec{\sigma}^{r}, \tag{51}$$

and therefore, the Liouvillian is

$$\mathcal{L} = -\frac{i}{2}\vec{\omega}(\vec{\sigma}^{l} - \vec{\sigma}^{r}) = -\frac{i}{2}\,\omega_k\left(\bar{\xi}\sigma_k\frac{\partial}{\partial\bar{\xi}} - \xi\sigma_k^{T}\frac{\partial}{\partial\xi}\right) =$$

$$= -\omega_k\varepsilon_{klm}s_l\frac{\partial}{\partial s_m}, \tag{52}$$

where ε_{klm} is the totally antisymmetric tensor, $\varepsilon_{123} = 1$. Here the relation (see eq. (A.16) in Appendix A and Appendix C)

$$\bar{\xi}\sigma_k\frac{\partial}{\partial\bar{\xi}} - \xi\sigma_k^{T}\frac{\partial}{\partial\xi} = -2i\,\varepsilon_{klm}s_l\frac{\partial}{\partial s_m} \tag{53}$$

was used. Now eqs. (45) and (46) take the form

$$\frac{\partial}{\partial t}\rho(\vec{s},t) = \omega_k\varepsilon_{klm}s_l\frac{\partial}{\partial s_m}\rho(\vec{s},t), \qquad \frac{\partial}{\partial t}\mathbb{F}(\vec{s}) = 0, \tag{54}$$

$$\frac{\partial}{\partial t}\mathbb{F}(\vec{s},t) = -\omega_k\varepsilon_{klm}s_l\frac{\partial}{\partial s_m}\mathbb{F}(\vec{s},t), \qquad \frac{\partial}{\partial t}\rho(\vec{s}) = 0, \tag{55}$$

and the solutions (48) and (49) are

$$\rho(\vec{s},t) = \exp\left(\omega_k\varepsilon_{klm}s_l\frac{\partial}{\partial s_m}\right)\rho(\vec{s},0), \tag{56}$$

$$\mathbb{F}(\vec{s},t) = \exp\left(-\omega_k\varepsilon_{klm}s_l\frac{\partial}{\partial s_m}\right)\mathbb{F}(\vec{s},0). \tag{57}$$

One can also write eqs. (54) and (55) in terms of the nonoperator representative of the Hamiltonian

$$\frac{\partial}{\partial t}\rho(\vec{s},t) = 2\hbar^{-1}\varepsilon_{klm}s_k\frac{\partial H(\vec{s})}{\partial s_l}\,\frac{\partial\rho(\vec{s},t)}{\partial s_m}, \tag{58}$$

$$\frac{\partial}{\partial t}\mathbb{F}(\vec{s},t) = -2\hbar^{-1}\varepsilon_{klm}s_k\frac{\partial H(\vec{s})}{\partial s_l}\,\frac{\partial\mathbb{F}(\vec{s},t)}{\partial s_m}. \tag{59}$$

This form of the equations resembles the Liouville equation in the classical mechanics (see, e.g., refs. /29–32/)

$$\frac{\partial}{\partial t}\rho(x,p,t) = -\mathcal{L}\rho(x,p,t)\left(\text{or}\ \frac{\partial}{\partial t}\mathbb{F}(x,p,t) = \mathcal{L}\mathbb{F}(x,p,t)\right) \text{ with } \mathcal{L} = \sum_i\left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial p_i}\right).$$

The right-hand sides of eqs. (58) and (59) resemble the Poisson bracket. But here the variables s_m on the sphere $\vec{s}^{\,2}=1$ participate instead of the phase space variables x and p in the usual Poisson bracket.

One more way to write the Liouville equation via the nonoperator representatives is to use the multiplication operation (27)

$$\frac{\partial}{\partial t}\rho(\vec{s},t) = -i\hbar^{-1}\left(H(\vec{s}) * \rho(\vec{s},t) - \rho(\vec{s},t) * H(\vec{s})\right) =$$

$$= 2\hbar^{-1}(3c)^2\int d^3s'\,\delta(\vec{s}\,'^2-1)\int d^3s''\,\delta(\vec{s}\,''^2-1)\,\varepsilon_{klm}s_k s'_l s''_m H(\vec{s}\,')\rho(\vec{s}\,'',t) \tag{60}$$

and similarly for eq. (42). Equation (60) may be reduced to eqs. (54) or (58) by integrating over $\vec{s}\,'$ and $\vec{s}\,''$ in the r.h.s. of eq. (60).

One particular solution of eq. (55) is

$$s_m(t) = e^{\mathcal{L}\tau}s_m = e^{\mathcal{L}\tau}s_m e^{-\mathcal{L}\tau}, \quad (\tau = t - t_0), \qquad s_m(t_0) = s_m. \tag{61}$$

It is defined by its initial condition. One can easily check that these functions form the general solution of the set of ordinary differential equations of the first order: ˣ

$$\frac{d}{dt}s_m(t) = -\omega_k\varepsilon_{klm}s_l(t), \qquad s_m(t_0) = s_m \tag{62}$$

and thus, are characteristics of eq. (55). Due to that $\mathcal{L}$ is a partial differential operator of the first order, any solution of eq. (55) is expressed via these characteristics as follows:

$$\mathbb{F}(\vec{s},t) = e^{\mathcal{L}\tau}\mathbb{F}(\vec{s},t_0) = \mathbb{F}\left(e^{\mathcal{L}\tau}\vec{s}\,e^{-\mathcal{L}\tau},t_0\right) = \mathbb{F}(\vec{s}(t),t_0). \tag{63}$$

Analogously, one can express via the characterictics any solution of eq. (54)

$$\rho(\vec{s},t;\vec{s}_0,t_0) = e^{-\mathcal{L}\tau}\rho(\vec{s},t_0;\vec{s}_0,t_0) = \tag{64.a}$$

$$= \rho\left(e^{-\mathcal{L}\tau}\vec{s}\,e^{\mathcal{L}\tau},t_0;\vec{s}_0,t_0\right) = \rho\left(\vec{s}(2t_0-t),t_0;\vec{s}_0,t_0\right) = \tag{64.b}$$

$$= e^{\mathcal{L}^{0}\tau}\rho(\vec{s},t_0;\vec{s}_0,t_0) = \tag{64.c}$$

$$= \rho\left(\vec{s},t_0;e^{\mathcal{L}^{0}\tau}\vec{s}_0\,e^{-\mathcal{L}^{0}\tau},t_0\right) = \rho\left(\vec{s},t_0;\vec{s}_0(t),t_0\right). \tag{64.d}$$

Here $\mathcal{L}^{0}$ is the Liouvillian (52), acting on the vector $\vec{s}_0$, and

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ˣ Which resembles the classical Hamilton equations $\frac{d}{dt}s_m = -2\hbar^{-1}\varepsilon_{mkl}s_k\frac{\partial H}{\partial s_l}$.

$\vec{s}_0(t)$ are the functions (61) given by the initial condidion $\vec{s}_0(t_0) = \vec{s}_0$. The evolution in time of the probability density (17.b) is written in terms of the characteristics as follows:

$$\rho(\vec{s},t;\vec{s}_0,t_0) = |\xi\, e^{-i\hbar^{-1}\hat{H}(t-t_0)}\xi_0|^2 = \tag{65.a}$$

$$= \xi\, e^{-i\hbar^{-1}\hat{H}(t-t_0)}\xi_0 \otimes \bar{\xi}_0\, e^{i\hbar^{-1}\hat{H}(t-t_0)}\xi =$$

$$= e^{-\mathcal{L}(t-t_0)}\,|\xi\,\xi_0|^2 = \tag{65.b}$$

$$= \bar{\xi}_0\, e^{i\hbar^{-1}\hat{H}(t-t_0)}\xi \otimes \bar{\xi}\, e^{-i\hbar^{-1}\hat{H}(t-t_0)}\xi_0 =$$

$$= e^{\mathcal{L}^{\circ}(t-t_0)}\,|\xi\,\xi_0|^2 = \tag{65.c}$$

$$= \tfrac{1}{2}\left(1+\vec{s}\,\vec{s}_0(t)\right) = \tag{65.d}$$

$$= \tfrac{1}{2}\left(1+\vec{s}(2t_0-t)\vec{s}_0\right), \tag{65,e}$$

$$\rho(\vec{s},t_0;\vec{s}_0,t_0) = (\bar{\xi}\xi_0)(\bar{\xi}_0\xi) = \tfrac{1}{2}(1+\vec{s}\,\vec{s}_0). \tag{66}$$

The vector $\vec{s}_0(t)$ is the vector $\vec{s}_0$ rotated by means of an orthogonal SO(3) matrix, and therefore, the vector $\vec{s}(2t_0-t)$ is the vector $\vec{s}$ rotated by means of the inverse matrix.

If the magnetic field $\vec{B}$, and therefore, the Hamiltonian and the Liouvillian depend on time, then the exponents $e^{\mathcal{L}t}$ must be replaced by the T-exponents.

10. An analog of the Markov property. For complex amplitudes in the quantum mechanics the decompositions

$$\langle m|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|m_0\rangle = \sum_{m_{N-1}=-1,+1}\cdots\sum_{m_2=-1,+1}\ \sum_{m_1=-1,+1}$$
$$\langle m|e^{-i\hbar^{-1}\hat{H}(t-t_{N-1})}|m_{N-1}\rangle\cdots\langle m_2|e^{-i\hbar^{-1}\hat{H}(t_2-t_1)}|m_1\rangle\langle m_1|e^{-i\hbar^{-1}\hat{H}(t_1-t_0)}|m_0\rangle = \tag{67.a}$$

$$= c^{N-1}\int d^3s_1\,\delta(\vec{s}_1^{\,2}-1)\int d^3s_2\,\delta(\vec{s}_2^{\,2}-1)\cdots\int d^3s_{N-1}\,\delta(\vec{s}_{N-1}^{\,2}-1)\cdot$$
$$(\bar{\xi}\, e^{-i\hbar^{-1}\hat{H}(t-t_{N-1})}\xi_{N-1})\cdots(\bar{\xi}_2\, e^{-i\hbar^{-1}\hat{H}(t_2-t_1)}\xi_1)(\bar{\xi}_1 e^{-i\hbar^{-1}\hat{H}(t_1-t_0)}\xi_0) \tag{67.b}$$

are valid which resemble the Markov property in the classical dynamics. However for the probability densities an analog of the Markov property has another form

$$\rho(\vec{s},t;\vec{s}_0,t_0) = |\bar{\xi}\, e^{-i\hbar^{-1}\hat{H}(t-t_0)}\xi_0|^2 =$$

$$= -1+3c\int d^3s_1\,\delta(\vec{s}_1^{\,2}-1)|\bar{\xi}\, e^{-i\hbar^{-1}\hat{H}(t-t_1)}\xi_1|^2\,|\bar{\xi}_1 e^{-i\hbar^{-1}\hat{H}(t_1-t_0)}\xi_0|^2 \tag{68.a}$$

$$= -1-3-3^2-\ldots-3^{N-2}+(3c)^{N-1}\int d^3s_1\,\delta(\vec{s}_1^{\,2}-1)\int d^3s_2\,\delta(\vec{s}_2^{\,2}-1)\ldots\int d^3s_{N-1}\,\delta(\vec{s}_{N-1}^{\,2}-1)$$
$$|\bar{\xi}\, e^{-i\hbar^{-1}\hat{H}(t-t_{N-1})}\xi_{N-1}|^2\ldots|\bar{\xi}_2 e^{-i\hbar^{-1}\hat{H}(t_2-t_1)}\xi_1|^2\,|\bar{\xi}_1 e^{-i\hbar^{-1}\hat{H}(t_1-t_0)}\xi_0|^2 \tag{68.b}$$

It can be checked by using the completeness relation (9). Again there appear additional coefficients and terms,thus demonstrating distinction of the quantum probability theory from the classical one.

Note that if the Hamiltonian depends on time, decompositions (67) and (68.b) remain valid approximately for small intervals $\Delta t_i = t_i - t_{i-1}$ with their own constant Hamiltonian $\hat{H}_{i-1} = \hat{H}(t_{i-1})$ at each Δt_i.

11. Pseudoclassical formulations. Additional coefficients and terms can be hidden in such a manner that the above formulas take "pseudoclassical" form . To this end we convert the completeness relation (9) into

$$c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\tfrac{1}{2}(1+\vec{s}\vec{\sigma})_{\alpha\beta}\,\tfrac{1}{2}(1+3\vec{s}\vec{\sigma})_{\gamma\delta} =$$
$$= c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\tfrac{1}{2}(1+\sqrt{3}\,\vec{s}\vec{\sigma})_{\alpha\beta}\,\tfrac{1}{2}(1+\sqrt{3}\,\vec{s}\vec{\sigma})_{\gamma\delta} = \delta_{\alpha\delta}\delta_{\gamma\beta}\ ^{x} \tag{69}$$

Matrices $\frac{1}{2}(1+3\vec{s}\vec{\sigma})$ and $\frac{1}{2}(1+\sqrt{3}\,\vec{s}\vec{\sigma})$ are hermitian as before but are not idempotents and cannot be decomposed into spinors (unlike eq. (6)). Together with eq. (8) there are also valid the relations

$$c\int d^3s\,\delta(\vec{s}^{\,2}-1)\tfrac{1}{2}(1+\sqrt{3}\,\vec{s}\vec{\sigma})_{\alpha\beta} = c\int d^3s\,\delta(\vec{s}^{\,2}-1)\tfrac{1}{2}(1+3\vec{s}\vec{\sigma})_{\alpha\beta} = \delta_{\alpha\beta}. \tag{70}$$

Now using eqs. (69) we can represent any operator $\hat{F}$ in the following three ways (instead of eqs. (10),(11))

$$\hat{F}_{\alpha\beta} = c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\tfrac{1}{2}(1+3\vec{s}\vec{\sigma})_{\alpha\beta}F(\vec{s}) = \tag{71.a}$$

$$= c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\tfrac{1}{2}(1+\sqrt{3}\,\vec{s}\vec{\sigma})_{\alpha\beta}\tilde{F}(\vec{s}) = \tag{71.b}$$

$$= c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\tfrac{1}{2}(1+\vec{s}\vec{\sigma})_{\alpha\beta}\tilde{\tilde{F}}(\vec{s}), \tag{71.c}$$

where

$$F(\vec{s}) = \bar{\xi}\hat{F}\xi,\quad \tilde{F}(\vec{s}) = F(\sqrt{3}\,\vec{s}),\quad \tilde{\tilde{F}}(\vec{s}) = F(3\vec{s}). \tag{72}$$

x For discrete analogs of eqs. (69) see Appendix E.

Instead of eq. (13) for the trace $tr(\hat{\rho}\hat{\mathbb{F}})$ we find

$$tr(\hat{\rho}\hat{\mathbb{F}}) = c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\tilde{\tilde{\rho}}(\vec{s})\,\mathbb{F}(\vec{s}) = \tag{73.a}$$

$$= c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\tilde{\rho}(\vec{s})\,\tilde{\mathbb{F}}(\vec{s}) = \tag{73.b}$$

$$= c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\rho(\vec{s})\,\tilde{\tilde{\mathbb{F}}}(\vec{s})\,. \tag{73.c}$$

Under $\hat{\rho}$ one can think any operator. If $\hat{\rho}$ is the density matrix (17.a), then

$$\rho(\vec{s}) = \bar{\xi}\,\hat{\rho}\,\xi = \frac{1}{2}(1+\vec{s}\vec{s}_0)\,, \tag{74.a}$$

$$\tilde{\rho}(\vec{s}) = \rho(\sqrt{3}\,\vec{s}) = \frac{1}{2}(1+\sqrt{3}\,\vec{s}\vec{s}_0)\,, \tag{74.b}$$

$$\tilde{\tilde{\rho}}(\vec{s}) = \rho(3\vec{s}) = \frac{1}{2}(1+3\vec{s}\vec{s}_0)\,. \tag{74.c}$$

The densities $\tilde{\rho}$ and $\tilde{\tilde{\rho}}$ are not positive definite (unlike ρ). Using them we allow negative probabilities in the intermediate steps of calculations[x] . Such a situation is known in phase space representations (see refs.[/29-32/], ρ , $\tilde{\rho}$ and $\tilde{\tilde{\rho}}$ correspond to PSR-1,2 and 3, respectively).

In the case of $tr(\hat{\rho}\hat{\mathbb{F}})$ we can avoid the negative probabilities using expression (73.c) where only the representative $\tilde{\tilde{\mathbb{F}}}$ is modified. However, we cannot avoid the negative probabilities everywhere. For example, expressions (21) and (24) can be written as

$$w\left(+\tfrac{1}{2}\text{ along }\vec{n}\,;\,+\tfrac{1}{2}\text{ along }\vec{m}\right) = c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\rho_f(\vec{s})\,\tilde{\tilde{\rho}}_i(\vec{s}) = \tag{75.a}$$

$$= c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\tilde{\rho}_f(\vec{s})\,\tilde{\rho}_i(\vec{s}) = \tag{75.b}$$

$$= c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\tilde{\tilde{\rho}}_f(\vec{s})\,\rho_i(\vec{s}) \tag{75.c}$$

$$w\left(+\tfrac{1}{2}\text{ along }\vec{a}\,,\,+\tfrac{1}{2}\text{ along }\vec{b}\,;\text{ singlet}\right) =$$

$$= c^2\int d^3s^a\,\delta(\vec{s}^{a2}-1)\int d^3s^b\,\delta(\vec{s}^{b2}-1)\,\rho^{\text{singlet}}(\vec{s}^a,\vec{s}^b)\,\tilde{\tilde{\rho}}_{\vec{a}}(\vec{s}^a)\,\tilde{\tilde{\rho}}_{\vec{b}}(\vec{s}^b) \tag{76.a}$$

$$= c^2\int d^3s^a\,\delta(\vec{s}^{a2}-1)\int d^3s^b\,\delta(\vec{s}^{b2}-1)\,\tilde{\rho}^{\text{singlet}}(\vec{s}^a,\vec{s}^b)\,\tilde{\rho}_{\vec{a}}(\vec{s}^a)\,\tilde{\rho}_{\vec{b}}(\vec{s}^b) \tag{76.b}$$

$$= c^2\int d^3s^a\,\delta(\vec{s}^{a2}-1)\int d^3s^b\,\delta(\vec{s}^{b2}-1)\,\tilde{\tilde{\rho}}^{\text{singlet}}(\vec{s}^a,\vec{s}^b)\,\rho_{\vec{a}}(\vec{s}^a)\,\rho_{\vec{b}}(\vec{s}^b), \tag{76.c}$$

[x] R.P.Feynman argues that there are no obstacles to use them for simplification of calculations provided that the final answer is positive and correct (in conformity to the exceedingly fruitful mathematical idea of negative numbers, in general).

where

$$\tilde{\rho}^{\text{singlet}}(\vec{s}^a,\vec{s}^b) = \rho^{\text{singlet}}(\sqrt{3}\,\vec{s}^a,\sqrt{3}\,\vec{s}^b) = \frac{1}{4}(1-3\vec{s}^a\vec{s}^b), \tag{77.a}$$

$$\tilde{\tilde{\rho}}^{\text{singlet}}(\vec{s}^a,\vec{s}^b) = \rho^{\text{singlet}}(3\vec{s}^a,3\vec{s}^b) = \frac{1}{4}(1-9\vec{s}^a\vec{s}^b). \tag{77.b}$$

Obviously, other modifications of the latter density are possible.

From eq. (70) there follow the normalization conditions

$$c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\rho(\vec{s}) = c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\tilde{\rho}(\vec{s}) = c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\tilde{\tilde{\rho}}(\vec{s}) = 1, \tag{78}$$

$$c^2\int d^3s^a\,\delta(\vec{s}^{a2}-1)\int d^3s^b\,\delta(\vec{s}^{b2}-1)\,\rho^{\text{singlet}}(\vec{s}^a,\vec{s}^b) =$$

$$= c^2\int d^3s^a\,\delta(\vec{s}^{a2}-1)\int d^3s^b\,\delta(\vec{s}^{b2}-1)\,\tilde{\rho}^{\text{singlet}}(\vec{s}^a,\vec{s}^b) =$$

$$= c^2\int d^3s^a\,\delta(\vec{s}^{a2}-1)\int d^3s^b\,\delta(\vec{s}^{b2}-1)\,\tilde{\tilde{\rho}}^{\text{singlet}}(\vec{s}^a,\vec{s}^b) = 1\,. \tag{79}$$

Expressions (20) and (23) for the expectation value of the spin projection and the correlator can now be written

$$tr(\hat{\rho}_{\vec{s}_0}(\vec{n}\vec{\sigma})) = c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\tilde{\rho}_{\vec{s}_0}(\vec{s})\,\sqrt{3}\,(\vec{n}\vec{s}) = \tag{80.a}$$

$$= c\int d^3s\,\delta(\vec{s}^{\,2}-1)\,\tilde{\tilde{\rho}}_{\vec{s}_0}(\vec{s})\,(\vec{n}\vec{s}), \tag{80.b}$$

$$c(\vec{a},\vec{b}) =$$

$$= 3c^2\int d^3s^a\,\delta(\vec{s}^{a2}-1)\int d^3s^b\,\delta(\vec{s}^{b2}-1)\,\tilde{\rho}^{\text{singlet}}(\vec{s}^a\,\vec{s}^b)(\vec{a}\vec{s}^a)(\vec{b}\vec{s}^b) = \tag{81.a}$$

$$= c^2\int d^3s^a\,\delta(\vec{s}^{a2}-1)\int d^3s^b\,\delta(\vec{s}^{b2}-1)\,\tilde{\tilde{\rho}}^{\text{singlet}}(\vec{s}^a,\vec{s}^b)(\vec{a}\vec{s}^a)(\vec{b}\vec{s}^b)\,. \tag{81.b}$$

At last, reformulate the analog (68) of the Markov property. Let us introduce

$$\rho(\vec{s}_i,t_i;\vec{s}_{i-1},t_{i-1}) = \left|\bar{\xi}_i\,e^{-i\hbar^{-1}\hat{H}(t_i-t_{i-1})}\,\xi_{i-1}\right|^2 =$$

$$= tr\left(\xi_i\otimes\bar{\xi}_i\,e^{-i\hbar^{-1}\hat{H}(t_i-t_{i-1})}\,\xi_{i-1}\otimes\bar{\xi}_{i-1}\,e^{i\hbar^{-1}\hat{H}(t_i-t_{i-1})}\right) =$$

$$= tr\left(\hat{\rho}_i\,e^{-i\hbar^{-1}\hat{H}(t_i-t_{i-1})}\,\hat{\rho}_{i-1}\,e^{i\hbar^{-1}\hat{H}(t_i-t_{i-1})}\right), \tag{82}$$

$$\hat{\rho}_i = \frac{1}{2}(1+\vec{s}_i\vec{\sigma}),$$

$$\tilde{\rho}(\vec{s}_i,t_i;\vec{s}_{i-1},t_{i-1}) = \rho(\sqrt{3}\,\vec{s}_i,t_i;\sqrt{3}\,\vec{s}_{i-1},t_{i-1}). \tag{83}$$

Then, eq. (68.b) can be written in the pseudoclassical manner

$$\tilde{\rho}\,(\vec{s},t\,;\,\vec{s}_0,t_0)=c^{N-1}\int d^3s_1\delta(\vec{s}_1^{\,2}-1)\int d^3s_2\delta(\vec{s}_2^{\,2}-1)\cdots\int d^3s_{N-1}\delta(\vec{s}_{N-1}^{\,2}-1)$$

$$\cdot\,\tilde{\rho}(\vec{s},t\,;\vec{s}_{N-1},t_{N-1})\cdots\tilde{\rho}(\vec{s}_2,t_2\,;\vec{s}_1,t_1)\,\tilde{\rho}(\vec{s}_1,t_1;\vec{s}_0,t_0). \qquad (84)$$

Positive definiteness for all the densities here is not guaranteed. One obtains the positive definite density $\rho(\vec{s},t\,;\vec{s}_0,t_0)$ as follows:

$$\rho\,(\vec{s},t\,;\vec{s}_0,t_0)=c^{N-1}\int d^3s_1\delta(\vec{s}_1^{\,2}-1)\int d^3s_2\delta(\vec{s}_2^{\,2}-1)\cdots\int d^3s_{N-1}\delta(\vec{s}_{N-1}^{\,2}-1)$$

$$\cdot\,\tilde{\rho}'(\vec{s},t;\vec{s}_{N-1},t_{N-1})\cdots\tilde{\rho}(\vec{s}_2,t_2;\vec{s}_1,t_1)\,\tilde{\rho}'(\vec{s}_1,t_1;\vec{s}_0,t_0), \qquad (85)$$

where the first and last $\tilde{\rho}$ are defined otherwise:

$$\tilde{\rho}'(\vec{s},t\,;\vec{s}_{N-1},t_{N-1})=\rho\,(\vec{s},t\,;\sqrt{3}\,\vec{s}_{N-1},t_{N-1}) \qquad (86.a)$$

$$\tilde{\rho}'(\vec{s}_1,t_1;\vec{s}_0,t_0)=\rho(\sqrt{3}\,\vec{s}_1,t_1;\vec{s}_0,t_0). \qquad (86.b)$$

At the first glance the above ways of the pseudoclassical formulations seem artificial. Nevertheless, they are similar to phase space representations in quantum mechanics and quantum field theory (the coherent state representation, the Wigner one and others) and serve as interesting simple models for these representations.

Let us point out one more possibility of the pseudoclassical formulation. We now modify the measure "without" change of the representatives $F(\vec{s})$ and $\rho(\vec{s})$. It is clear that the completeness relations (8) and (9) (or eqs. (70) and (69)) can be rewritten as follows:

$$\frac{c}{\sqrt{3}}\int d^3s\,\delta(\vec{s}^{\,2}-3)\,\tfrac{1}{2}(1+\vec{s}\vec{\sigma})_{\alpha\beta}=\delta_{\alpha\beta}\,, \qquad (87)$$

$$\frac{c}{\sqrt{3}}\int d^3s\,\delta(\vec{s}^{\,2}-3)\,\tfrac{1}{2}(1+\vec{s}\vec{\sigma})_{\alpha\beta}\,\tfrac{1}{2}(1+\vec{s}\vec{\sigma})_{\gamma\delta}=\delta_{\alpha\delta}\,\delta_{\beta\gamma}, \qquad (88)$$

where the integration is performed in "unphysical" region, i.e. over the sphere of the radius $|\vec{s}|=\sqrt{3}$ [x] instead of $|\vec{s}|=1$. All the formulas can be written as above in this sec., but omitting the tildes$\sim$ and $\approx$ and with the new measure ($\sim\delta(\vec{s}^{\,2}-3)$) including the coefficient $c/\sqrt{3}$ instead of c . For example,

$$\mathrm{tr}\,\hat{\rho}_{\vec{s}_0}=\frac{c}{\sqrt{3}}\int d^3s\,\delta(\vec{s}^{\,2}-3)\,\rho_{\vec{s}_0}(\vec{s})=1, \qquad (89)$$

[x] This integration levels up the coefficients:

$$\int d^3s\,\delta(\vec{s}^{\,2}-3)\{1,s_m,s_ms_n\}=\{1,0,\delta_{mn}\}\int d^3s\,\delta(\vec{s}^{\,2}-3).$$

$$\mathrm{tr}\,(\hat{\rho}_{\vec{s}_0}(\vec{n}\vec{\sigma}))=\frac{c}{\sqrt{3}}\int d^3s\,\delta(\vec{s}^{\,2}-3)\,\rho_{\vec{s}_0}(\vec{s})(\vec{n}\vec{s}), \qquad (90)$$

$$\rho^{\text{singlet}}_{\alpha\beta,\alpha\beta}=\frac{c^2}{3}\int d^3s^a\,\delta(\vec{s}^{\,a2}-3)\int d^3s^b\,\delta(\vec{s}^{\,b2}-3)\,\rho^{\text{singlet}}(\vec{s}^{\,a},\vec{s}^{\,b})=1, \qquad (91)$$

$$c(\vec{a},\vec{b})=$$

$$=\frac{c^2}{3}\int d^3s^a\,\delta(\vec{s}^{\,a2}-3)\int d^3s^b\,\delta(\vec{s}^{\,b2}-3)\,\rho^{\text{singlet}}(\vec{s}^{\,a},\vec{s}^{\,b})(\vec{a}\vec{s}^{\,a})(\vec{b}\vec{s}^{\,b}) \qquad (92)$$

$$\rho(\vec{s},t;\vec{s}_0,t_0)=\left(\frac{c}{\sqrt{3}}\right)^{N-1}\int d^3s_1\delta(\vec{s}_1^{\,2}-3)\int d^3s_2\delta(\vec{s}_2^{\,2}-3)\cdots\int d^3s_{N-1}\delta(\vec{s}_{N-1}^{\,2}-3)$$

$$\rho(\vec{s},t\,;\vec{s}_{N-1},t_{N-1})\cdots\rho(\vec{s}_2,t_2;\vec{s}_1,t_1)\,\rho(\vec{s}_1,t_1;\vec{s}_0,t_0) \qquad (93)$$

One can go to the classical limit in eqs. (89)-(92) by replacing $3\to1$ (in the sense $j(j+1)\to j^2$). However, the equation thus obtained from eq. (93) is not satisfied by the density. (82). This means that the quantum density (82) is unacceptable as a classical density. At the same time, the equation of motion (54) is classical in form and has certainly the classical solution

$$\rho_{cl}(\vec{s},t\,;\vec{s}_0,t_0)=\delta^2(\vec{s},\vec{s}_0(t))=\delta^2(\vec{s}(2t_0-t),\vec{s}_0), \qquad (94)$$

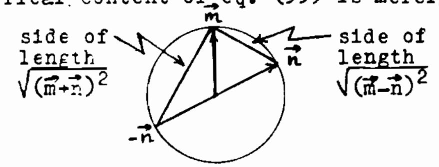
where δ^2 is the delta-function on the sphere S^2. It satisfies eq. (93) with 1 substituted for all 3's. However it is a nonlinear function of $\vec{s}$ contrary to the quantum rules of the game.

12. The spin projections $-\frac{1}{2}$. The above probabilities (probability densities) for the spin projection $+\frac{1}{2}$ along some direction (say, $\vec{n}$) can be transformed into ones for the spin projection $-\frac{1}{2}$ simply by the reflection of the direction ($\vec{n}\to-\vec{n}$). The sums of these probabilities equal unity:

$$w(+\tfrac{1}{2}\text{ along }\vec{n}\,;+\tfrac{1}{2}\text{ along }\vec{m})+w(-\tfrac{1}{2}\text{ along }\vec{n}\,;+\tfrac{1}{2}\text{ along }\vec{m})=1, \qquad (95)$$

$$\sum_{m=-\frac{1}{2},+\frac{1}{2}}\;\sum_{n=-\frac{1}{2},+\frac{1}{2}} w(m\text{ along }\vec{a},\,n\text{ along }\vec{b}\,;\text{ singlet})=1, \qquad (96)$$

being in fact the sums over two opposite values of vectors (e.g., $\vec{n}$ and $-\vec{n}$ in eq. (95)). The geometrical content of eq. (95) is merely the Pythagoras theorem for the right-angled triangle $\vec{m}$, $\vec{n}$, $-\vec{n}$ (the points of the unit sphere S^2):



$$(\vec m + \vec n)^2 + (\vec m - \vec n)^2 = 4\,, \tag{97}$$

$$w(\pm \tfrac{1}{2} \text{ along } \vec n;\ + \tfrac{1}{2} \text{ along } \vec m) = \tfrac{1}{4}(\vec m \pm \vec n)^2 \tag{98}$$

In terms of the spinors the densities for the spin projections $+\frac{1}{2}$ and $-\frac{1}{2}$ along a direction $\vec s_0$ are written as follows:

$$\rho_{\vec s_0}(\vec s) = (\bar\xi\xi_0)(\bar\xi_0\xi) = \tfrac{1}{2}(1+\vec s\vec s_0), \tag{99.a}$$

$$\rho_{\vec s_0^R}(\vec s) = (\bar\xi\xi_0^R)(\bar\xi_0^R\xi) = (\bar\xi^R\xi_0)(\bar\xi_0\xi^R) = (\xi\sigma_2^T\xi_0)(\bar\xi\sigma_2\bar\xi_0) = \tfrac{1}{2}(1-\vec s\vec s_0), \tag{99.b}$$

where R denotes the operation

$$\xi^R = \sigma_2\bar\xi \tag{100}$$

which transforms

$$s_m = \bar\xi\sigma_m\xi \quad \text{into} \quad s^R = \bar\xi^R\sigma_m\xi^R = -s_m\,, \tag{101}$$

the antipodal map of the sphere S^2 onto itself. For the probabilities w the same expressions are valid:

$$w(+\tfrac{1}{2} \text{ along } \vec n;\ +\tfrac{1}{2} \text{ along } \vec m) = (\bar\xi\xi_0)(\bar\xi_0\xi) = \tfrac{1}{2}(1+\vec n\vec m), \tag{101.a}$$

$$w(-\tfrac{1}{2} \text{ along } \vec n;\ +\tfrac{1}{2} \text{ along } \vec m) = (\bar\xi^R\xi_0)(\bar\xi_0\xi^R) = (\xi\sigma_2^T\xi_0)(\bar\xi\sigma_2\bar\xi_0) = \tfrac{1}{2}(1-\vec n\vec m). \tag{101.b}$$

The sum of the probabilities equals unity due to the identity

$$(\bar\xi\xi_0)(\bar\xi_0\xi) + (\bar\xi^R\xi_0)(\bar\xi_0\xi^R) = (\bar\xi\xi_0)(\bar\xi_0\xi) + (\xi\sigma_2^T\xi_0)(\bar\xi\sigma_2\bar\xi_0) = (\bar\xi\xi)(\bar\xi_0\xi_0) \tag{102}$$

valid for any (in general unnormalized) spinors.

The probabilities $w(+\frac{1}{2}$ along $\vec n$; $+\frac{1}{2}$ along $\vec m)$ and $w(+\frac{1}{2}$along $\vec a$, $+\frac{1}{2}$ along $\vec b$; singlet) satisfy also the conditions (19), (78), (89) and (22), (79), (91) with the integration over all the directions of the vectors (e.g. of the vector $\vec n$).

13. Formalism of real amplitudes based on the sphere S^3. Let us introduce instead of the complex amplitudes $\bar\xi\xi_0$ and $\xi\sigma_2^T\xi_0$, the real amplitudes and the corresponding probability densities:

$$a a^0 = a_\mu a^0_\mu \quad (\mu=1,2,3,4)\,, \qquad \rho_{a^0}(a) = (a a_0)^2\,, \tag{103.a}$$

$$a a^{0R} \equiv a_\mu a^{0R}_\mu = a^R a^0\,, \qquad \rho_{a^{0R}}(a) = (a a^{0R})^2\,, \tag{103.b}$$

where a, a^0, and a^R are points on the unit sphere S^3. The vector a^R

corresponds to the antipodal map (100), (101) and can easily be found from eq. (100) (see Appendix D).

The probability densities introduced are normalised as follows:

$$C\int d^4a\,\delta(a^2-1)\rho_{a^0}(a) = C\int d^4a\,\delta(a^2-1)\rho_{a^0R}(a) = 1 \quad \left(C=\frac{8}{\Omega_3},\ \Omega_3=2\pi^2\right) \tag{104}$$

(see Appendix E, eq. (E.2)). The amplitudes and probability densities (103) depend on the fiber variables of the fiber bundle $S^3 \longrightarrow S^2$ (fiber $S^1=SO(2)=U(1)$): one enters through a and the other through a^0. They fall out from the densities (99.a) (or (17.b)) and (99.b). Integrating over the fibre variable (it is enough to integrate over that of one of the 4-vectors a or a^0), we go back to the probability densities (99.a) and (99.b):

$$\frac{2}{\pi}\int\limits_{\text{fiber}} d^2n_a\,\delta(\vec n_a^2-1)(a a^0)^2 = (\bar\xi\xi_0)(\bar\xi_0\xi) = \tfrac{1}{2}(1+\vec s\vec s_0), \tag{105.a}$$

$$\frac{2}{\pi}\int\limits_{\text{fiber}} d^2n_a\,\delta(\vec n_a^2-1)(a a^{0R})^2 = (\bar\xi\xi_0^R)(\bar\xi_0^R\xi) = (\xi\sigma_2^T\xi_0)(\bar\xi\sigma_2\bar\xi_0) = \tfrac{1}{2}(1-\vec s\vec s_0) \tag{105.b}$$

(see Appendix D).

Using eq. (E.2) of Appendix E we can deduce

$$3C\int d^4a\,\delta(a^2-1)\rho_{a^0}(a)\,s_m = s_m^0\,, \tag{106.a}$$

$$3C\int d^4a\,\delta(a^2-1)\rho_{a^0R}(a)\,s_m = -s_m^0\,, \tag{106.b}$$

where $s_m = s_m(a) = \bar\xi(a)\sigma_m\xi(a)$ are bilinear forms of a (see Appendix A), $s_m^0 = s_m(a^0) = \bar\xi_0\sigma_m\xi_0$. These forms do not depend on the fiber variables. This fact permits us to perform in eqs. (106) integration over the fiber by eqs. (105) after the change of variables

$$d^4a = \frac{1}{16|\vec s|}\,d\alpha\,d^3s = \frac{1}{8|\vec s|}\,d\lambda\,d^3s, \tag{107}$$

$$\int d^4a\,\delta(a^2-1)\cdots = \int\frac{d^3s}{16|\vec s|}\,2\delta(\vec s^2-1)\int\limits_{\substack{0\\ \text{fiber}}}^{4\pi} d\alpha\cdots = \int\frac{d^3s}{8|\vec s|}\,2\delta(\vec s^2-1)\int\limits_{\substack{0\\ \text{fiber}}}^{2\pi} d\lambda\cdots$$

$$= \int\frac{d^3s}{8|\vec s|}\,2\delta(\vec s^2-1)\,2\int\limits_{\text{fiber}} d^2n_a\,\delta(\vec n_a^2-1)\cdots \tag{108}$$

where the fibre variables α or $\lambda=\frac{\alpha}{2}$ or $\vec n=(n_0,n_1)=(\cos\lambda,\sin\lambda)$ are defined by eqs. (A.2), (A.5) and (D.3), respectively. This way reduces eqs. (106) to the $\vec s$-representation formulas

$$3c\int d^3s\,\delta(\vec s^2-1)\,\rho_{\vec s_0}(\vec s)\,s_m = s_m^0\,, \tag{109.a}$$

$$3c\int d^3s\,\delta(\vec s^2-1)\,\rho_{\vec s_0R}(\vec s)\,s_m = -s_m^0 \qquad (\vec s_0{}^R = -\vec s\,) \tag{109.b}$$

With the reference to eq. (106.a) we have

$$\mathrm{tr}(\hat\rho\hat{\mathbb F}) = -\frac12 C\int d^4a'\,\delta(a'^2-1)\rho_{a^0}(a')\,C\int d^4a\,\delta(a^2-1)\mathbb F(\vec s)+3C\int d^4a\,\delta(a^2-1)\rho_{a^0}(a)\mathbb F(a) \qquad (110)$$

($\mathbb F(\vec s) = f_0 + f_m s_m$, as before). However, with $\mathbb G(\vec s)$ instead of $\rho_{a^0}(a)$ the right-hand side of eq. (110) produces $2\mathrm{tr}(\hat{\mathbb G}\hat{\mathbb F})$.

Let us express some quantities of interest in terms of the formalism based on S^3. The expectation value of the spin projection along $\vec n$ equals

$$3C\int d^4a\,\delta(a^2-1)\rho_{a^0}(a)(\vec n\vec s) = \vec n\vec s_0 . \qquad (111)$$

The singlet state density

$$\rho^{\text{singlet}}(\vec s^a,\vec s^b) = \frac12(\xi_a\sigma_2^T\xi_b)(\bar\xi_a\sigma_2\bar\xi_b) = \frac12(\bar\xi_a\xi_b^R)(\bar\xi_b^R\xi_a) = \frac12(\bar\xi_a^R\xi_b)(\bar\xi_b\xi_a^R) = \frac14(1-\vec s^a\vec s^b) \qquad (112)$$

(the first expression follows directly from the last expression (18.a)) can be replaced by the following S^3-density

$$\rho^{\text{singlet}}(a,b) = (a^R b)^2 = (a b^R)^2 \qquad (113)$$

with the normalization condition

$$C^2\int d^4a\,\delta(a^2-1)\int d^4b\,\delta(b^2-1)\rho^{\text{singlet}}(a,b) = 1 . \qquad (114)$$

The correlator is defined as

$$c(\vec m,\vec n) = (3C)^2\int d^4a\,\delta(a^2-1)\int d^4b\,\delta(b^2-1)\rho^{\text{singlet}}(a,b)(\vec m\vec s^a)(\vec n\vec s^b) = -\vec m\vec n . \qquad (115)$$

The amplitude (103.a) is divisible as follows

$$a a_0 = C^{N-1}\int d^4a_1\delta(a_1^2-1)\int d^4a_2\delta(a_2^2-1)\dots\int d^4a_{N-1}\delta(a_{N-1}^2-1)$$
$$(a a_{N-1})(a_{N-1}a_{N-2})\cdots(a_2a_1)(a_1a_0), \qquad (116)$$

whereas for the density (103.a) we have

$$(a a_0)^2 = -\frac12(1+3+3^2+\dots+3^{N-2}) + (3C)^{N-1}\int d^4a_1\delta(a_1^2-1)\dots\int d^4a_{N-1}\delta(a_{N-1}^2-1)$$
$$(a a_{N-1})^2(a_{N-1}a_{N-2})^2\cdots(a_2a_1)^2(a_1a_0)^2 \qquad (117)$$

Integrating over the fiber (see eqs. (108), (105) and Appendix D) one can reduce these relations to the following ones:

$$\bar\xi\xi_0 = c^{N-1}\int d^3s_1\delta(\vec s_1^2-1)\int d^3s_2\delta(\vec s_2^2-1)\cdots\int d^3s_{N-1}\delta(\vec s_{N-1}^2-1)$$
$$(\bar\xi\xi_{N-1})(\bar\xi_{N-1}\xi_{N-2})\dots(\bar\xi_2\xi_1)(\bar\xi_1\xi_0) \qquad (118)$$

$$|(\bar\xi\xi_0)|^2 = -1-3-3^2-\dots-3^{N-2}+(3c)^{N-1}\int d^3s_1\delta(\vec s_1^2-1)\dots\int d^3s_{N-1}\delta(\vec s_{N-1}^2-1)$$
$$|(\bar\xi\xi_{N-1})|^2|(\bar\xi_{N-1}\xi_{N-2})|^2\cdots|(\bar\xi_2\xi_1)|^2|(\bar\xi_1\xi_0)|^2 , \qquad (119)$$

which underlie the quantum Markov properties (67.b) and (68.b), (84), (85), (93).

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Appendix A. The Hopf fiber bundle $S^3\to S^2$ in terms of of the spinors.

Let us consider the map $R^4\to R^3$ /26,27/

$$\eta \to x_m = \bar\eta\sigma_m\eta \quad (m=1,2,3), \qquad r\equiv|\vec x| = \bar\eta\eta=\rho^2 . \qquad (A.1)$$

where σ_m are the Pauli matrices, and η is the unnormalized (in general) complex 2-component spinor. If one parametrizes the spinor as follows

$$\eta = \begin{bmatrix} a_0+ia_1\\ a_0'+ia_1'\end{bmatrix} = \begin{bmatrix}u_2+iu_4\\ u_1+iu_2\end{bmatrix} = \begin{bmatrix}\sqrt r\cos\frac{\theta}{2}e^{\frac i2(\alpha-\varphi)}\\ \sqrt r\sin\frac{\theta}{2}e^{\frac i2(\alpha+\varphi)}\end{bmatrix} , \qquad (A.2)$$

then $0\le r\le\infty$, $0\le\theta\le\pi$, $0\le\varphi\le2\pi$, $0\le\alpha\le4\pi$,

$$x_1 = \bar\eta\sigma_1\eta = 2(a_0a_0'+a_1a_1') = r\sin\theta\cos\varphi ,$$
$$x_2 = \bar\eta\sigma_2\eta = 2(a_0a_1'-a_1a_0') = r\sin\theta\sin\varphi ,$$
$$x_3 = \bar\eta\sigma_3\eta = a_0^2+a_1^2-a_0'^2-a_1'^2 = r\cos\theta ,$$
$$r = \bar\eta\eta = a_0^2+a_1^2+a_0'^2+a_1'^2 = u_\mu u_\mu = \rho^2 . \qquad (A.3)$$

Thus, the parametrization of η by the Euler angles (the last expression of eq. (A.2)) leads to the parametrization of vector $\vec x$ by the usual spherical angles. For each fixed $r=\rho^2$ eqs. (A.1) embody the Hopf fiber bundle

$$S^3_\rho \to CP^1\to S^2_r \qquad (\text{ fiber is } S^1 = SO(2) = U(1)). \qquad (A.4)$$

The sphere S^2_r with the coordinates x_m (A.1) is a base, and the transformation group U(1) on a fiber

$$\eta\to\hat\eta = z\eta \qquad |z|^2=1 , \qquad z=e^{i\lambda} \qquad (A.5)$$

generates from one value of η the whole fiber ($e^{i\lambda}\eta$ with all $0\le\lambda\le2\pi$) which is a great circle on S^3_ρ . Each circle (fiber) is mapped into the only point of the base. The fiber variable α falls out. Let us stress that each component $x_m=\bar\eta\sigma_m\eta$ is separately invariant under the group U(1) (A.5) acting on the fiber.

Note that due to eq. (A.1) η is a "square root" of $\vec x$ and due to $U^{-1}(a)\,\sigma_\ell\, U(a)=r_{\ell m}\sigma_m$ the SU(2) matrix $U(a)$ (3) is a "square root" of the orthogonal SO(3) matrix $\|r_{\ell m}\|$ $(r_{\ell m}=(a_0^2-\vec a^2)\delta_{\ell m}+a_0\varepsilon_{\ell m n}a_n+2a_\ell a_m)$

$$\eta'=U(a)\eta\,,\qquad x'_\ell=\bar\eta'\sigma_\ell\eta'=r_{\ell m}x_m\,.$$

From the completeness relation (7.b), i.e.

$$\delta_{\alpha\beta}\delta_{\gamma\delta}+\sum_{m=1}^{3}(\sigma_m)_{\alpha\beta}(\sigma_m)_{\gamma\delta}=2\,\delta_{\alpha\delta}\,\delta_{\gamma\beta} \tag{A.6}$$

contracting it with $\bar\eta_\alpha\eta_\beta$ and with $\bar\eta_\alpha\eta_\beta\bar\eta_\gamma\eta_\delta$ we obtain the identities

$$\eta_\gamma\bar\eta_\delta=\frac12(r\,1+\vec x\vec\sigma)_{\gamma\delta}\,,\qquad r=|\vec x|\,, \tag{A.7}$$

$$(\bar\eta\eta_0)(\bar\eta_0\eta)=\frac12(r\,r_0+\vec x\vec x_0)\,,\qquad r_0=|\vec x_0|\,, \tag{A.8}$$

which play the main role in the text.

From eqs. (A.1) we obtain

$$\frac{\partial}{\partial\eta_\alpha}=\frac{\partial x_m}{\partial\eta_\alpha}\frac{\partial}{\partial x_m}=(\bar\eta\sigma_m)_\alpha\frac{\partial}{\partial x_m}\,, \tag{A.9}$$

$$\frac{\partial}{\partial\bar\eta_\alpha}=\frac{\partial x_m}{\partial\bar\eta_\alpha}\frac{\partial}{\partial x_m}=(\sigma_m\eta)_\alpha\frac{\partial}{\partial x_m} \tag{A.10}$$

in application to functions independent of fiber variable. From eqs. (A.9) and (A.10) there follow

$$\eta\frac{\partial}{\partial\eta}=(\bar\eta\sigma_m\eta)\frac{\partial}{\partial x_m}=x_m\frac{\partial}{\partial x_m}\,,\qquad \bar\eta\frac{\partial}{\partial\bar\eta}=(\bar\eta\sigma_m\eta)\frac{\partial}{\partial x_m}=x_m\frac{\partial}{\partial x_m}\,, \tag{A.11}$$

$$\eta\sigma_\ell^T\frac{\partial}{\partial\eta}=\bar\eta\sigma_m\sigma_\ell\eta\frac{\partial}{\partial x_m}\,,\qquad \bar\eta\sigma_\ell\frac{\partial}{\partial\bar\eta}=\bar\eta\sigma_\ell\sigma_m\eta\frac{\partial}{\partial x_m} \tag{A.12}$$

and hence

$$\eta\frac{\partial}{\partial\eta}+\bar\eta\frac{\partial}{\partial\bar\eta}=2x_m\frac{\partial}{\partial x_m}\,, \tag{A.13}$$

$$\eta\frac{\partial}{\partial\eta}-\bar\eta\frac{\partial}{\partial\bar\eta}=0\,, \tag{A.14}$$

$$\eta\sigma_\ell^T\frac{\partial}{\partial\eta}+\bar\eta\sigma_\ell\frac{\partial}{\partial\bar\eta}=\bar\eta(\sigma_m\sigma_\ell+\sigma_\ell\sigma_m)\eta\frac{\partial}{\partial x_m}=2r\frac{\partial}{\partial x_\ell}\,, \tag{A.15}$$

$$\eta\sigma_\ell^T\frac{\partial}{\partial\eta}-\bar\eta\sigma_\ell\frac{\partial}{\partial\bar\eta}=\bar\eta(\sigma_m\sigma_\ell-\sigma_\ell\sigma_m)\eta\frac{\partial}{\partial x_m}=2i\,\varepsilon_{\ell m n}x_m\frac{\partial}{\partial x_n}\,. \tag{A.16}$$

The relation (A.14) is the compatibility condition, i.e. the requirement to a function to be independent of the fiber variable α (see below eq. (A.18.c)).

Let us give also the SO(4) algebra (in fact, two its subalgebras SO(3)) in terms of the spinor

$$\frac12\left[u_\ell\frac{\partial}{\partial u_4}-u_4\frac{\partial}{\partial u_\ell}-\varepsilon_{\ell m n}u_m\frac{\partial}{\partial u_n}\right]=\frac{i}{2}\left(\eta\sigma_\ell^T\frac{\partial}{\partial\eta}-\bar\eta\sigma_\ell\frac{\partial}{\partial\bar\eta}\right), \tag{A.17}$$

$$\frac12\left[u_1\frac{\partial}{\partial u_4}-u_4\frac{\partial}{\partial u_1}+\varepsilon_{1mn}u_m\frac{\partial}{\partial u_n}\right]=\frac12\left(\bar\eta\sigma_2\frac{\partial}{\partial\eta}+\eta\sigma_2^T\frac{\partial}{\partial\bar\eta}\right), \tag{A.18.a}$$

$$\frac12\left[u_2\frac{\partial}{\partial u_4}-u_4\frac{\partial}{\partial u_2}+\varepsilon_{2mn}u_m\frac{\partial}{\partial u_n}\right]=\frac{i}{2}\left(\bar\eta\sigma_2\frac{\partial}{\partial\eta}-\eta\sigma_2^T\frac{\partial}{\partial\bar\eta}\right), \tag{A.18.b}$$

$$\frac12\left[u_3\frac{\partial}{\partial u_4}-u_4\frac{\partial}{\partial u_3}+\varepsilon_{3mn}u_m\frac{\partial}{\partial u_n}\right]=\frac{i}{2}\left(\eta\frac{\partial}{\partial\eta}-\bar\eta\frac{\partial}{\partial\bar\eta}\right)=\frac{\partial}{\partial\alpha} \tag{A.18.c}$$

In application to functions independent of the fiber variable α the generators (A.17) are reduced to the 3-dimensional angular momentum in terms of $\vec x$ (by eq. (A.16)) and the generator (A.18.c) vanishes (cf. eq. (A.14)).

In the $\vec s$-representation $\rho=\rho_0=r=r_0=1\,,\quad \vec x=\vec s\,.$

Appendix B. Discrete analogs of the completeness relations (9) and (69). The idempotents $\xi_\alpha\bar\xi_\beta$ entering into eq. (9) form an (infinitely) overcomplete set of 2x2 matrices. One can however restrict the set to six idempotents, e.g., to

$$\xi_\alpha\bar\xi_\beta=\frac12(1\pm\sigma_1)\,,\ \frac12(1\pm\sigma_2)\,,\ \frac12(1\pm\sigma_3)\,. \tag{B.1}$$

It is overcomplete too, unlike the set of four matrices $1,\sigma_1,\sigma_2,\sigma_3$. The completeness relation (7.b) can be converted into the forms

$$\sum_{m=1,2,3}\left\{\frac12(1+\sigma_m)_{\alpha\beta}\frac12(1+\sigma_m)_{\gamma\delta}+\frac12(1-\sigma_m)_{\alpha\beta}\frac12(1-\sigma_m)_{\gamma\delta}\right\}=\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\delta}\delta_{\gamma\beta}\,, \tag{B.2}$$

$$\begin{aligned}&\frac13\sum_{m=1,2,3}\left\{\frac12(1+\sigma_m)_{\alpha\beta}\frac12(1+3\sigma_m)_{\gamma\delta}+\frac12(1-\sigma_m)_{\alpha\beta}\frac12(1-3\sigma_m)_{\gamma\delta}\right\}=\\&=\frac13\sum_{m=1,2,3}\left\{\frac12(1+\sqrt3\sigma_m)_{\alpha\beta}(1+\sqrt3\sigma_m)_{\gamma\delta}+\frac12(1-\sqrt3\sigma_m)_{\alpha\beta}(1-\sqrt3\sigma_m)_{\gamma\delta}=\delta_{\alpha\delta}\delta_{\gamma\beta}\right.\end{aligned} \tag{B.3}$$

(cf. eqs. (9) and (69)). Let us introduce

$$\rho(k,m;k_0,m_0)=\frac1{12}(1+k\sigma_m)_{\alpha\beta}(1+k_0\sigma_{m_0})_{\beta\alpha}=\frac16(1+kk_0\delta_{mm_0})\quad(k,k_0=\pm1) \tag{B.4}$$

$$\tilde\rho(k,m;k_0,m_0)=\frac1{12}(1+k\sqrt3\sigma_m)_{\alpha\beta}(1+k_0\sqrt3\sigma_{m_0})_{\beta\alpha}=\frac16(1+3kk_0\delta_{mm_0}) \tag{B.5}$$

with the normalization condition

$$\sum_{m=1,2,3}\ \sum_{k=\pm1}\rho(k,m;k_0,m_0)=\sum_{m=1,2,3}\ \sum_{k=\pm1}\tilde\rho(k,m;k_0,m_0)=1\,. \tag{B.6}$$

Then from eqs. (B.2) and (B.3) there follow the divisibility properties

$$\rho(k,m;k_0,m_0) = -\frac{1}{3}\left(1+3+\cdots+3^{N-2}\right)+3^{N-1}\sum_{m_1=1,2,3}\sum_{k_1=\pm1}\cdots\sum_{m_{N-1}=1,2,3}\sum_{k_{N-1}=\pm1}$$

$$\rho(k,m;k_{N-1},m_{N-1})\,\rho(k_{N-1},m_{N-1};k_{N-2},m_{N-2})\cdots\rho(k_2,m_2;k_1,m_1)\,\rho(k_1,m_1;k_0,m_0)$$

$$\tilde\rho(k,m;k_0,m_0) = \sum_{m_1=1,2,3}\sum_{k_1=\pm1}\sum_{m_2=1,2,3}\sum_{k_2=\pm1}\cdots\sum_{m_{N-1}=1,2,3}\sum_{k_{N-1}=\pm1} \tag{B.7}$$

$$\tilde\rho(k,m;k_{N-1},m_{N-1})\cdots\tilde\rho(k_2,m_2;k_1,m_1)\,\tilde\rho(k_1,m_1;k_0,m_0). \tag{B.8}$$

Appendix C. Derivation of the representatives σ_m^l and σ_m^r .

Starting with eqs. (39) and (40) for the unnormalized spinors η , we obtain

$$\frac{\bar\eta\sigma_m\hat{F}\eta}{\bar\eta\eta} = \left(\bar\eta\sigma_m\frac{\partial}{\partial\bar\eta} + \frac{\bar\eta\sigma_m\eta}{\bar\eta\eta}\right)\frac{\bar\eta\hat{F}\eta}{\bar\eta\eta}, \tag{C.1}$$

$$\frac{\bar\eta\hat{F}\sigma_m\eta}{\bar\eta\eta} = \left(\eta\sigma_m^T\frac{\partial}{\partial\eta} + \frac{\bar\eta\sigma_m\eta}{\bar\eta\eta}\right)\frac{\bar\eta\hat{F}\eta}{\bar\eta\eta}. \tag{C.2}$$

Here the unnormalized spinors remain only in terms with the derivatives. The following identity

$$\frac{\partial}{\partial u_\mu} = n_\mu\frac{\partial}{\partial\rho} + \frac{1}{\rho}n_\lambda L_{\lambda\mu} \tag{C.3}$$

is valid in any dimension. Here $\rho=\sqrt{u_\mu u_\mu}$, $n_\mu=\frac{u_\mu}{\rho}$, and

$$L_{\lambda\mu} = u_\lambda\frac{\partial}{\partial u_\mu} - u_\mu\frac{\partial}{\partial u_\lambda} = n_\lambda\frac{\partial}{\partial n_\mu} - n_\mu\frac{\partial}{\partial n_\lambda}. \tag{C.4}$$

In our case $\mu=1,2,3,4$, $u=(u_1,u_2,u_3,u_4)$, $u_\mu u_\mu=\bar\eta\eta=\rho^2$, $n_\mu n_\mu=1$. In eq. (C.3) the derivative with respect to ρ is separated from those with respect to angular variables since the angular momentum $L_{\lambda\mu}$ is an inner differential operator on S^3: it commutes with ρ and acts only on angular variables, i.e. on the unit vector n_μ :

$$[L_{\lambda\mu}, n_\nu] = \delta_{\mu\nu}n_\lambda - \delta_{\lambda\nu}n_\mu . \tag{C.5}$$

The angular variables can be introduced in different ways, e.g., as in eq. (A.2). Using eq. (C.3) we obtain

$$\frac{\partial}{\partial\eta_1} = \frac{1}{2}\left(\frac{\partial}{\partial u_3} - i\frac{\partial}{\partial u_4}\right) = \frac{1}{2}(n_3-in_4)\frac{\partial}{\partial\rho} + \frac{1}{2}\frac{n_\lambda}{\rho}\left(L_{\lambda3}-iL_{\lambda4}\right) =$$

$$= \frac{1}{2\rho}\left(\frac{\partial}{\partial n_3} - i\frac{\partial}{\partial n_4}\right) + \frac{1}{2}(n_3-in_4)\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}n_\lambda\frac{\partial}{\partial n_\lambda}\right) =$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\xi_1} + \frac{1}{2}\bar\xi_1\left[\frac{\partial}{\partial\rho} - \frac{1}{\rho}\left(\xi\frac{\partial}{\partial\xi} + \bar\xi\frac{\partial}{\partial\bar\xi}\right)\right], \tag{C.6}$$

$$\frac{\partial}{\partial\eta_2} = \frac{1}{2}\left(\frac{\partial}{\partial u_1} - i\frac{\partial}{\partial u_2}\right) = \frac{1}{2}(n_1-in_2)\frac{\partial}{\partial\rho} + \frac{1}{2}\frac{n_\lambda}{\rho}\left(L_{\lambda1}-iL_{\lambda2}\right) =$$

$$= \frac{1}{2\rho}\left(\frac{\partial}{\partial n_1} - i\frac{\partial}{\partial n_2}\right) + \frac{1}{2}(n_1-in_2)\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}n_\lambda\frac{\partial}{\partial n_\lambda}\right) =$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\xi_2} + \frac{1}{2}\bar\xi_2\left[\frac{\partial}{\partial\rho} - \frac{1}{\rho}\left(\xi\frac{\partial}{\partial\xi} + \bar\xi\frac{\partial}{\partial\bar\xi}\right)\right], \tag{C.7}$$

$$\frac{\partial}{\partial\bar\eta_1} = \frac{1}{\rho}\frac{\partial}{\partial\bar\xi_1} + \frac{1}{2}\xi_1\left[\frac{\partial}{\partial\rho} - \frac{1}{\rho}\left(\xi\frac{\partial}{\partial\xi} + \bar\xi\frac{\partial}{\partial\bar\xi}\right)\right], \tag{C.8}$$

$$\frac{\partial}{\partial\bar\eta_2} = \frac{1}{\rho}\frac{\partial}{\partial\bar\xi_2} + \frac{1}{2}\xi_2\left[\frac{\partial}{\partial\rho} - \frac{1}{\rho}\left(\xi\frac{\partial}{\partial\xi} + \bar\xi\frac{\partial}{\partial\bar\xi}\right)\right], \tag{C.9}$$

where the notation

$$\frac{\partial}{\partial\xi_1} = \frac{1}{2}\left(\frac{\partial}{\partial n_3} - i\frac{\partial}{\partial n_4}\right), \qquad \frac{\partial}{\partial\bar\xi_1} = \frac{1}{2}\left(\frac{\partial}{\partial n_3} + i\frac{\partial}{\partial n_4}\right), \tag{C.10}$$

$$\frac{\partial}{\partial\xi_2} = \frac{1}{2}\left(\frac{\partial}{\partial n_1} - i\frac{\partial}{\partial n_2}\right), \qquad \frac{\partial}{\partial\bar\xi_2} = \frac{1}{2}\left(\frac{\partial}{\partial n_1} + i\frac{\partial}{\partial n_2}\right) \tag{C.11}$$

is adopted and the identity

$$n_\lambda\frac{\partial}{\partial n_\lambda} = \xi\frac{\partial}{\partial\xi} + \bar\xi\frac{\partial}{\partial\bar\xi} \tag{C.12}$$

is taken into account. Here ξ denotes the normalized spinor ($\bar\xi\xi=1$). Putting eqs. (C.6)-(C.9) into eqs. (C.1) and (C.2), we obtain for the normalized spinors

$$\bar\xi\sigma_m\hat{F}\xi = \left\{\bar\xi\sigma_m\left[\frac{\partial}{\partial\bar\xi} - \frac{1}{2}\xi\left(\xi\frac{\partial}{\partial\xi} + \bar\xi\frac{\partial}{\partial\bar\xi}\right)\right] + \bar\xi\sigma_m\xi\right\}\bar\xi\hat{F}\xi, \tag{C.13}$$

$$\bar\xi\hat{F}\sigma_m\xi = \left\{\xi\sigma_m^T\left[\frac{\partial}{\partial\xi} - \frac{1}{2}\bar\xi\left(\xi\frac{\partial}{\partial\xi} + \bar\xi\frac{\partial}{\partial\bar\xi}\right)\right] + \bar\xi\sigma_m\xi\right\}\bar\xi\hat{F}\xi \tag{C.14}$$

and hence the representatives (33) and (34). The derivative $\partial/\partial\rho$ falls out in application to the spinor ξ , which depends only on n_μ (on angular variables), but not on ρ .

Equations (A.16), (A.17) and (A.18) remain valid in terms of ξ .

Appendix D. Integration over the fiber in eqs. (105). The vector a^R can easily be found from eq. (100). Namely, in terms of the first and second parametrizations (A.2) for

$$a=(a_0,a_1,a_0',a_1')=(u_3,u_4,u_1,u_2), \quad a_\mu a_\mu = u_\mu u_\mu = 1, \tag{D.1}$$

$$a^R=(-a_1',-a_0',a_1,a_0)=(-u_2,-u_1,u_4,u_3). \tag{D.2}$$

The transformation U(1) on the fiber can be written as

$$a_0=n_0\alpha_0+n_1\alpha_1, \qquad a_1=n_0\alpha_1-n_1\alpha_0,$$
$$a_0'=n_0\alpha_0'+n_1\alpha_1', \qquad a_1'=n_0\alpha_1'-n_1\alpha_0' \qquad (n_0^2+n_1^2=1), \tag{D.3}$$

where $\alpha=(\alpha_0,\alpha_1,\alpha'_0,\alpha'_1)$ is some fixed vector (a fixed gauge)^{x}. When the 2-vector $\vec{n}=(n_0,n_1)$ runs the whole sphere S^1, the vector a runs the whole fiber. In these terms

$$ab = n_0(\alpha_0 b_0+\alpha_1 b_1+\alpha'_0 b'_0+\alpha'_1 b'_1)+n_1(\alpha_1 b_0-\alpha_0 b_1+\alpha'_1 b'_0-\alpha'_0 b'_1), \quad (D.4)$$

$$a^R b = n_0(-\alpha'_1 b_0-\alpha'_0 b_1+\alpha_1 b'_0+\alpha_0 b'_1)+n_1(\alpha'_0 b_0-\alpha'_1 b_1-\alpha_0 b'_0+\alpha_1 b'_1) \quad (D.5)$$

and we can easily perform in eqs. (105) the integration over the fiber:

$$\frac{2}{\pi}\int_{\text{fiber}} d^2 n_a\,\delta(\vec{n}_a^2-1)(ab)^2=(\alpha_0 b_0+\alpha_1 b_1+\alpha'_0 b'_0+\alpha'_1 b'_1)^2+$$
$$+(\alpha_1 b_0-\alpha_0 b_1+\alpha'_1 b'_0-\alpha'_0 b'_1)^2=(\bar{\xi}\xi_0)(\bar{\xi}_0\xi) \quad (D.6)$$

$$\frac{2}{\pi}\int_{\text{fiber}} d^2 n_a\,\delta(\vec{n}_a^2-1)(a^R b)^2=(-\alpha'_1 b_0-\alpha'_0 b_1+\alpha_1 b'_0+\alpha_0 b'_1)^2+$$
$$+(\alpha'_0 b_0-\alpha'_1 b_1-\alpha_0 b'_0+\alpha_1 b'_1)^2=(\bar{\xi}^R\xi_0)(\bar{\xi}_0\xi^R), \quad (D.7)$$

since

$$\bar{\xi}\xi_0=\alpha_0 b_0+\alpha_1 b_1+\alpha'_0 b'_0+\alpha'_1 b'_1-i(\alpha_1 b_0-\alpha_0 b_1+\alpha'_1 b'_0-\alpha'_0 b'_1), \quad (D.8)$$

$$\xi\sigma_2\xi_0=-\alpha'_1 b_0-\alpha'_0 b_1+\alpha_1 b'_0+\alpha_0 b'_1+i(\alpha'_0 b_0-\alpha'_1 b_1-\alpha_0 b'_0+\alpha_1 b'_1) \quad (D.9)$$

where ξ and ξ_0 are parametrized by $\alpha=(\alpha_0,\alpha_1,\alpha'_0,\alpha'_1)$ and $b=(b_0,b_1,b'_0,b'_1)$, respectively. Equations (105) are proved. The final expressions (D.6) and (D.7) are invariant under the group U(1) on the fiber, i.e. are independent of the fiber variable. The vector $\alpha=(\alpha_0,\alpha_1,\alpha'_0,\alpha'_1)$ can be replaced by any vector $a=(a_0,a_1,a'_0,a'_1)$ on the same fiber.

For passage from eq. (116) to eq. (118) we need also the following integral over the fiber

$$\frac{2}{\pi}\int_{\text{fiber}} d^2 n_a\,\delta(\vec{n}_a^2-1)(n_0^a-in_1^a)(ab)=$$
$$=(\alpha_0 b_0+\alpha_1 b_1+\alpha'_0 b'_0+\alpha'_1 b'_1)-i(\alpha_1 b_0-\alpha_0\beta_1+\alpha'_1 b'_0-\alpha'_0 b'_1)=$$
$$=\bar{\xi}^\alpha\xi_0^b=\bar{\xi}^\alpha\xi^\beta(n_0^b-in_1^b), \quad (D.10)$$

where b and β are related in the same manner as a and α in eq. (D.3). Using eq. (D.10) we go from eq. (116) to eq. (118) as follows:

$$\frac{2}{\pi}\int d^2 n_a\,\delta(\vec{n}_a^2-1)(n_0^a-in_1^a)\,C\int d^4 b\,\delta(b^2-1)(ab)(bc)=$$
$$=C\int d^4 b\,\delta(b^2-1)(\bar{\xi}^\alpha\xi^b)(bc)=C\int d^4 b\,\delta(b^2-1)(\bar{\xi}^\alpha\xi^\beta)(n_0^b-in_1^b)(bc)=$$
$$=c\int d^3 s_b\,\delta(\vec{s}_b^{\,2}-1)(\bar{\xi}^\alpha\xi^\beta)\frac{2}{\pi}\int_{\text{fiber}} d^2 n_b\,\delta(\vec{n}_b^2-1)(n_0^b-in_1^b)(bc)=$$
$$=c\int d^3 s^b\,\delta(\vec{s}_b^{\,2}-1)(\bar{\xi}^\alpha\xi^\beta)(\bar{\xi}^\beta\xi^c)=c\int d^3 s_b\,\delta(\vec{s}_b^{\,2}-1)(\bar{\xi}^\alpha\xi^b)(\bar{\xi}^b\xi^c) \quad (D.11)$$

^{x} For example, one can always choose $\alpha=(\alpha_0,\alpha_1,\alpha'_0,0)$.

The last expression implies that we can replace the vector $\beta=(\beta_0,\beta_1,\beta'_0,\beta'_1)$ by any vector $b=(b_0,b_1,b'_0,b'_1)$ on the same fiber.

Appendix E. List of some relevant integrals.

$$c\int d^3 s\,\delta(\vec{s}^{\,2}-1)\{1,s_m s_n,s_k s_l s_m s_n\}=2\{1,\tfrac{1}{3}\delta_{mn},\tfrac{1}{15}(\delta_{kl}\delta_{mn}+\delta_{km}\delta_{ln}+\delta_{kn}\delta_{ml})\}, \quad (E.1)$$

$$C\int d^4 a\,\delta(a^2-1)\{1,a_\mu a_\nu,a_\mu a_\nu a_\lambda a_\rho\}=4\{1,\tfrac{1}{4}\delta_{\mu\nu},\tfrac{1}{24}(\delta_{\mu\nu}\delta_{\lambda\rho}+\delta_{\mu\lambda}\delta_{\nu\rho}+\delta_{\mu\rho}\delta_{\lambda\nu})\}. \quad (E.2)$$

In addition to eqs. (8) and (9) we find

$$c\int d^3 s\,\delta(\vec{s}^{\,2}-1)\xi_{\alpha_1}\bar{\xi}_{\beta_1}\xi_{\alpha_2}\bar{\xi}_{\beta_2}\xi_{\alpha_3}\bar{\xi}_{\beta_3}=\frac{1}{6}(\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_3}\delta_{\alpha_3\beta_2}+\delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1}\delta_{\alpha_3\beta_3}+\delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_1}), \quad (E.3)$$

$$c\int d^3 s\,\delta(\vec{s}^{\,2}-1)\xi_{\alpha_1}\bar{\xi}_{\beta_1}\xi_{\alpha_2}\bar{\xi}_{\beta_2}\xi_{\alpha_3}\bar{\xi}_{\beta_3}\xi_{\alpha_4}\bar{\xi}_{\beta_4}=-\frac{1}{10}\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}\delta_{\alpha_4\beta_4}+$$
$$+\frac{1}{15}(\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_4}\delta_{\alpha_4\beta_3}+\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_3}\delta_{\alpha_3\beta_2}\delta_{\alpha_4\beta_4}+\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_4}\delta_{\alpha_3\beta_3}\delta_{\alpha_4\beta_2}+$$
$$+\delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1}\delta_{\alpha_3\beta_3}\delta_{\alpha_4\beta_4}+\delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_1}\delta_{\alpha_4\beta_4}+\delta_{\alpha_1\beta_4}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}\delta_{\alpha_4\beta_1})+$$
$$+\frac{1}{30}(\delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1}\delta_{\alpha_3\beta_4}\delta_{\alpha_4\beta_3}+\delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_4}\delta_{\alpha_3\beta_1}\delta_{\alpha_4\beta_2}+\delta_{\alpha_1\beta_4}\delta_{\alpha_2\beta_3}\delta_{\alpha_3\beta_2}\delta_{\alpha_4\beta_1}). \quad (E.4)$$

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Полубаринов И.В. [illegible] Е2-88-80
Непрерывное пред[illegible]
вероятностей и [illegible]

Квантовая [illegible] представления /"представления когер[illegible] терминах. Это позволяет сравнить [illegible] классической. Анализируется парадокс Белла. [illegible] для двух спинов 1/2 в синглетном состо[illegible] классическому, умноженному на 9, и потому [illegible] Белла [illegible]. Истинное квантовое неравенство [illegible]. В [illegible] представлении квантовые уравнения [illegible] вид классических уравнений, подобных уравнению Лиувилля в [illegible]. Их решения можно выразить через характеристики, которые [illegible] системе уравнений, родственных классическим уравнениям Гамильтона. Однако квантовая теория отличается от классической выбором плотностей вероятности, определением различных величин /например, коррелятора/. Иную конструкцию имеет аналог марковского свойства и другие соотношения. Этим величинам и соотношениям и в рамках квантовой механики можно придать их классический вид, но только ценой введения модифицированных "плотностей вероятности", не обладающих положительной определенностью.

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Polubarinov I.V. E2-88-80
Continuous Representation for Spin 1/2, Quantum Probability Theory and Bell Paradox

Quantum mechanics of spin 1/2 is translated into a classical language namely into a continuous representation ("coherent state representation"). This permits us to compare the quantum probability theory with the classical one. The Bell paradox is analysed. It is shown that the quantum correlator of two spins 1/2 in the singlet state turns out to be equal 9 times a classical one, and therefore, it is illegitimate to put it in the Bell inequality. The true quantum inequality is absolutely unrestrictive. In the continuous representation, equations of motion take a classical form similar to the Liouville equation in classical mechanics. Their solutions can be expressed via characteristics, subjected to equations relative to the Hamilton ones. However quantum theory still differs from classical one in choice of probability densities and in construction of correlators, of other quantities, of an analog of the Markov property, etc. These quantities and relations can be converted into their classical form in the framework of the quantum mechanics as well, but only in terms of modified "probability densities", which cannot be possible definite.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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