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ONE MORE THEOREM ON THE SHORT-TIME REGENERATION RATE

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I. INTRODUCTION

The semigroup property of reduced evolution represents a useful tool in the quantum theory of unstable systems[1]. It is well known that it cannot hold exactly, because otherwise the corresponding total Hamiltonian H should contain the whole real axis in its spectrum[1-4]. Hence various estimates of its violation become important.

It is further known that validity of the semigroup condition is equivalent to the absence of the decayed-state regeneration : the reduced evolution operator $V_t := E_u U_t E_u$ fulfils

$$V_{t+s} - V_t V_s = E_u U_t E_d U_s E_u \qquad (1.1)$$

for any $t,s \ge 0$, where E_u, E_d are the projections to the state subspace of the unstable system and its orthogonal complement, respectively, and $U_t = e^{-iHt}$ is the total evolution operator (we employ the notations used in Ref.1). The regeneration rate as a function of t,s is subjected to various restrictions. For example, Sinha has demonstrated[4] that regeneration cannot cease after a finite time : if there is a non-negative T_r such that

$$E_u U_t E_d U_s E_u = 0 \qquad (1.2)$$

for all $t \ge 0$ and $s \ge T_r$, then $\sigma(H) = \mathbb{R}$.

Another restriction concerns the regeneration rate at short times which must not be too slow unless the reduced evolution is an exact semigroup. Misra and Sinha have proven[5] the following assertion : suppose that for every ψ of some dense set D in the subspace $\mathcal{K}_u \equiv E_u \mathcal{K}$ of the unstable system[6] there is a non-negative C_ψ such that

$$\|(V_t V_s - V_{t+s})\psi\| \le C_\psi t^\alpha s^\alpha \tag{1.3}$$

with some $\alpha > 1$ holds for all $t,s\ge 0$, then $\{ V_t : t\ge 0 \}$ is a strongly continuous contractive semigroup, $V_t V_s = V_{t+s}$ for all $t,s\ge 0$.

This result has been recently generalized by Nishioka[7] who has shown that the conclusion is preserved if one replaces the bound (1.3) by

$$\|(V_t V_s - V_{t+s})\psi\| \le C_\psi t^\alpha s^\alpha (t+s)^\beta \tag{1.4}$$

for some $\alpha>1$, $2\alpha+\beta>0$ and $\alpha+\beta+1\ne 0$. The aim of the present paper is to derive another extension of the Misra-Sinha theorem.

II. THE MAIN RESULT

We are going to prove the following assertion.

Theorem : Let $\{ F(t) : t\ge 0 \}$ be a weakly continuous contractive family with $F(0)=I$ on a Hilbert space $\mathcal{K}$. Suppose there is a dense set D in $\mathcal{K}$ and a function $g : \{(t,s) : 0\le t\le s\} \to \mathbb{R}_+$ with the following properties

(i) $g(0,s)=0$,

(ii) there is a positive t_0 such that $g(t,s_1) \le g(t,s_2)$ for a fixed $t\le t_0$ and all $s_1\le s_2$,

(iii) there is a function $G \in L_{loc}(\mathbb{R}_+)$ such that $|t^{-1}g(t,s)| \le$ $\le G(s)$ holds for all sufficiently small t ,

(iv) the one-sided derivative $h(s) := \partial g(t,s)/\partial t\,|_{t=0+}$ exists and equals zero for all $s>0$,

such that

$$\|[F(t)F(s)-F(t+s)]\psi\| \le C_\psi g(t,s) \tag{2.1}$$

holds for every $\psi \in D$ and all $s\ge t\ge 0$, then $\{ F(t) : t\ge 0 \}$ is a strongly continuous contractive semigroup on $\mathcal{K}$.

Proof follows the same line as in Refs.5 and 7 . We take a sequence $\{\tau_i\}_{i=1}^n$ of positive numbers ; using the condition (2.1) repeatedly in combination with triangle inequality and contractivity of the family $\{F(t):t\ge 0\}$, we get the estimate

$$\| F(\sum_{i=1}^n \tau_i)\psi - F(\tau_1)\dots F(\tau_n)\psi \| \le C_\psi \sum_{j=2}^n g\Big(\tau_{j-1}, \sum_{i=j}^n \tau_i\Big) .$$

Substituting $\tau_i = t/n$, we obtain

$$\| F(t)\psi - F(t/n)^n\psi \| \le C_\psi \sum_{k=1}^{n-1} g\Big(\frac{t}{n}, \frac{t}{n}k\Big) .$$

For a given t , we choose n so large that $t/n \le t_0$; using then the assumption (ii), we can estimate the *rhs* as follows

$$\| F(t)\psi - F(t/n)^n\psi \| \le C_\psi \frac{n}{t} \int_{t/n}^{t} g\Big(\frac{t}{n}, s\Big) ds .$$

Next we employ the assumptions (i) and (iii) ; the last one allows us to use the dominated convergence theorem which yields

$$\lim_{n\to\infty} \int_{t/n}^{t} \frac{n}{t}\, g\left(\frac{t}{n}, s\right) ds =$$

$$= \lim_{n\to\infty} \int_{0}^{t} \frac{g\left(\frac{t}{n}, s\right) - g(0,s)}{\frac{t}{n}}\, \chi_{[t/n,t]}(s)\, ds = \int_{0}^{t} h(s)\, ds\,.$$

In view of (iv), we get finally

$$\lim_{n\to\infty} F(t/n)^n \psi = F(t)\psi \qquad (2.2)$$

for all $\psi \in D$, and since the family $\{F(t): t\geq 0\}$ is uniformly bounded, this conclusion extends to all $\psi\in\mathcal{H}$. The relation (2.2) yields easily the semigroup property[1,8] and the weak continuity implies the strong one[9]. ∎

For the regeneration rate $R_\psi(t,s) := \|(V_t V_s - V_{t+s})\psi\|^2$ which means the probability that the unstable system starting at $t=0$ in the state ψ and found decayed at s will be found undecayed again at a later instant $t+s$ we get the following

Corollary: If there is a dense set $D \subset \mathcal{H}_u$ and a function g with the properties listed in the theorem such that

$$R_\psi(t,s)^{1/2} \leq C_\psi g(t,s) \qquad (2.3)$$

holds for every $\psi \in D$ and all $s\geq t\geq 0$, then the reduced evolution $\{ V_t : t\geq 0 \}$ on $\mathcal{H}_u$ is a strongly continuous semigroup.

III. CONCLUDING REMARKS

Let us notice first that physically it is difficult to observe the regeneration, in particular, at short times. The reason is the same as in the case of short-time violations of the decay-law exponentiality[1,10] : the dynamics of the known decay processes is such that the interesting time region is inaccessible experimentally. Nevertheless, one cannot exclude discovery of other unstable systems (particles, nuclei, etc.) for which the semigroup approximation will not work so good, and furthermore, the Misra-Sinha theorem and its generalizations represent themselves interesting mathematical results.

Let us turn now to discussion of our hypotheses. First of all, the regeneration rate need not be estimated symmetrically in t,s ; in fact, one has to know the function g in an octant of the (t,s)-plane only. The assumption (i) is a weak one ; it should be fulfilled for every reasonable estimate. As for (ii), we shall comment on it a little later, while (iii) represents a not very strong regularity requirement. The assumption (iv) is essential ; it shows that *regeneration is excluded if only it starts at every instant s slowly enough.*

Our theorem generalizes the Misra-Sinha theorem ; one can check easily that the bound (1.3) fulfils the hypotheses. Let us compare it further to the Nishioka's result. The function g : $g(t,s) = t^\alpha s^\alpha (t+s)^\beta$ with the above indicated values of α,β fulfils the assumptions (i),(ii) and (iv), while (ii) is valid for $\alpha+\beta \geq 0$ only. Let us concentrate on the interesting case $\beta<0$. It was shown in Ref.7 that the original MS-theorem combined with the Schwarz inequality ensures the semigroup property for $2\alpha+\beta > 2$.

Our theorem yields a stronger result for $\beta \geq -2$. This case is covered by the Nishioka theorem as well as the region $\alpha > 1$, $-\beta/2 < \alpha < \beta$ (with exception of the halfline $\alpha+\beta+1 = 0$). In the last named case, however, the obtained sufficient condition represents a much weaker assertion : while for $\alpha+\beta > 0$ the estimate (1.4) is a restriction actually for short times only due to the contractivity, in the other case one must check it for all times which is considerably more difficult. Needless to say, the sufficient condition (2.1) covers a broader class of estimates than those of Refs.5 and 7 .

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6 One can demand, of course, the condition (1.3) to be valid for a dense set in the total Hilbert space $\mathscr{H}$ as in Ref.7 . It is clear, however, that the bound in $\mathscr{H}_u^{\perp}$ is trivial, and moreover, that $\{ V_t : t \geq 0 \}$ is a semigroup (fulfilling $\text{s-lim}_{t\to 0+} V_t = I_u$) on the subspace $\mathscr{H}_u$ only.

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Еще одна теорема о скорости регенерации при малых временах

Рассмотрим нестабильную квантовую систему, которая была найдена нераспавшейся в момент времени s, и обозначим через R(t,s) вероятность ее регенерации в начальное (нераспавшееся) состояние в более поздний момент t+s. Докажем, что приведенный оператор эволюции удовлетворяет полугрупповому условию, т.е. что вообще нет регенерации, если $R(t,s)^{1/2}$ может быть ограничено достаточно регулярной функцией, не растущей по s и с нулевой производной по t в точке t=0 для каждого s. Это обобщает теорему Мисры – Синхи в другом направлении, чем недавняя работа Нишиоки.

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One More Theorem on the Short-Time Regeneration Rate

Consider an unstable quantum system which has been found undecayed at an instant s and denote by R(t,s) the rate of its regeneration into an original (undecayed) state at a later instant t+s. We prove that the reduced evolution is a semigroup, i.e., there is no regeneration at all, provided $R(t,s)^{1/2}$ can be estimated by a sufficiently regular function which is non-decreasing in s and has zero derivative with respect to t at t=0 for every s. This generalizes the theorem of Mirsa and Sinha in a different direction than a recent paper by Nishioka.

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