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HAMILTON-DIRAC METHOD AND EVOLUTION OPERATOR IN INTERACTION REPRESENTATION

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In 1946 S.Tomonaga proposed a relativistic-invariant formulation for the quantum field theory. According to it, the state amplitude $\Phi(\sigma)$ is a functional of a space-like surface σ. The surface acts as time, and evolution of the system is determined by the covariant generalization of the Schrödinger equation in the interaction representation [1,2]

$$i\frac{\delta\Phi(\sigma)}{\delta\sigma(x)} = H_I(x,\sigma)\Phi(\sigma), \qquad (1)$$

where $H_I(x,\sigma)$ is the density of the interaction Hamiltonian constructed of field operators complying with free equations of motion. If Φ_0 is the state vector determining the initial physical state on the surface σ_0, the solution of equation (I) can be written as

$$\Phi(\sigma) = U(\sigma,\sigma_0)\Phi_0, \qquad (2)$$

where $U(\sigma,\sigma_0)$ is the evolution operator.

F.Dyson [3] obtained a representation for the operator $U(\sigma,\sigma_0)$ relating state vectors on arbitrary space-like, surfaces in the form of a series of the perturbation theory in powers of the interaction constant

$$U(\sigma,\sigma_0) = T\exp(-i\int_{\sigma_0}^{\sigma} d^4x H(x)), \qquad (3)$$

T is the symbol of chronological ordering, and the integration region in (3) is between the surfaces σ_0 and σ.

A well-known example of another form of $U(\sigma,\sigma_0)$ with no relation to the perturbation theory is the representation for the evolution operator, connecting states on flat surfaces x_0 = const

$$U(t,0) = e^{iH_0t}\, e^{-iHt}, \qquad (4)$$

where H_0, H are the Hamiltonians of the free and interacting systems, respectively. Another example is the representation for the evolution operator describing evolution of the system from the surface $x_0 = 0$ to the surface $\lambda x = 0$; $\lambda^2 = 1$, $\lambda_0 > 0$

$$U(\lambda x = 0,\ x_0 = 0) = e^{-i\mathring{M}_{0j}\omega^j}\, e^{iM_{0j}\omega^j}, \qquad (5)$$

$M_{\mu\nu}$, $\mathring{M}_{\mu\nu}$ are generators of the Lorentz group with and without allowance for the interaction [4,5] $\omega^j = (\lambda^j/|\vec{\lambda}|)\,\mathrm{arsh}|\vec{\lambda}|$.

Note that operators (4) and (5) are related to transformations from the Poincare group through time translations and pure Lorentz transforms. If, however, there is another space-time symmetry in theory, the respective generators C and C_0 (in the free case) may be regarded as evolution operators. Similarly to (4) and (5), we have [6]

$$U(\sigma,\ x_0 = 0) = \exp(-i\alpha C_0)\exp(i\alpha C). \qquad (6)$$

Here σ is the surface obtained from $x_0 = 0$ by a symmetry transformation, α is the transformation parameter *.

Examples (4)-(6) indicate a possibility of writing the evolution operator as

$$U(\sigma_2,\sigma_1) = \mathring{T}(\sigma_2,\sigma_1)T^{+}(\sigma_2,\sigma_1), \qquad (7)$$

where $T(\sigma_2,\sigma_1)$ is the operator constructed of interacting fields, $\mathring{T}(\sigma_2,\sigma_1)$ is the same operator when there is no interaction connecting the field operators defined on the surface σ_1 and σ_2:

$$q(x)|_{x\in\sigma_2} = T^{+}(\sigma_2,\sigma_1)\, q(x)|_{x\in\sigma_1} T(\sigma_2,\sigma_1). \qquad (8)$$

Note that, as in (4), an operator of three-dimensional translations $\exp(i\vec{p}\cdot\vec{a})$ can be added to the operator $\exp(iHt)$, $T(\sigma_2,\sigma$ is defined ambiguously. The ambiguity is caused by trasnformations which do not change the form of the surface, only changing its parametrization.

Note that representations (4)-(6) are valid in the instan dynamics form with the initial condition surface $x_0 = 0$.

In this paper we shall show that representation (7) is valid for arbitrary space-like surfaces, the explicit way to construct $T(\sigma_2,\sigma_1)$ being presented. We shall also show that the Dyson formula is universal, i.e. expansion (3) does not depend on the way of dividing the evolution operator into a "free" part and an "interacting" one.

To prove formula (7), we shall use P.A.M.Dirac's approach to quantization of relativistic systems, based on Hamilton's method [7,8].

I. RELATIVISTIC THEORY OF INTERACTING FIELDS A LA DIRAC

To obtain the relativistic quantum theory, according to Dirac[9], one must start with the classical Lagrange relativistic field theory, go to the Hamilton form and proceed to the quantum one following definite rules. A desire to retain the explicit relativistic invariance within the Hamilton formalism makes one introduce additional degrees of freedom related to arbitrariness of the surface σ, where the physical conditions are fixed. In other words, there must be equations of motion whose solutions contain arbitrary functions, i.e. the Hamilton theory must be a theory with first-class constraints. For this purpose, for any initial Hamiltonian it is enough to use variables defining the surface itself, as additional dynamical variables and transform the theory into a form where the Hamiltonian is equal to zero in a weak sense[9]. To do this, a common procedure is used, which will first be exemplified within quantum mechanics.

a. Quantum Mechanics without Absolute Time

Lagrange equations of motion follow from the stationary character of the action integral

$$S[q] = \int_{t_1}^{t_2} dt\, \mathcal{L}(q_i(t), \dot{q}_i(t)); \quad i = 1,..., n. \tag{9}$$

One can go from the Lagrangian $\mathcal{L}(q,\dot{q})$ related to an absolute time t to another Lagrangian $\mathcal{L}^*$, regarding time as a new dynamic coordinate $q^\circ(\tau)$ depending on a parameter τ:

$$\frac{dq^\circ}{d\tau}\, \mathcal{L}\left(q^i, \frac{dq^i/d\tau}{dq^\circ/d\tau}\right) = \mathcal{L}^*\left(q^k, \frac{dq^k}{d\tau}\right), \quad k = 0, 1, ..., n. \tag{10}$$

In this case we have for the action integral:

$$\int_{t_1}^{t_2} dt\mathcal{L} = \int_{\tau_1}^{\tau_2} d\tau\mathcal{L}^*. \tag{11}$$

The Lagrangian $\mathcal{L}^*$ contains one degree of freedom more than the initial one and is a homogeneous zero-order function of velocities. So, the theory described by the Lagrangian $\mathcal{L}^*$ is reparametrization-invariant; consequently, the existing canonical Hamiltonian is equal to zero in a weak sense. Thus, in the Hamilton theory based on $\mathcal{L}^*$ there is a constraint[x]

$$\phi = p^\circ + H \approx 0, \tag{12}$$

where H is the Hamiltonian of the initial theory, p° is the momentum conjugated to q_0. Equations of motion allow arbitrariness associated with the absence of absolute time

$$\frac{dg}{d\tau} = \{g, H^*_T\} \tag{13}$$

since the total Hamiltonian is of the form

$$H^*_T = a(\tau)(p^\circ + H) \tag{14}$$

where $a(\tau)$ is an arbitrary function of τ.

Besides, constraint (12) is a generator of gauge transformations

$$\delta q^k = \{q^k, \epsilon\phi\}, \quad \delta p^k = \{p^k, \epsilon\phi\}$$

with an infinitely small transformation parameter $\epsilon(\tau)$ complying with the boundary conditions $\epsilon(\tau_1) = \epsilon(\tau_2) = 0$.

According to Dirac's scheme, the corresponding quantum theory is obtained in the following way. We regard our variables q and p as operators complying with the transposition relations corresponding to the Poisson brackets in the classical theory

$$[\hat{q}_k, \hat{p}^\ell] = i\delta^\ell_k \tag{15}$$

and demand that the constraint on the state vector $|\Psi\rangle$ should hold true:

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[x] *We think that the Lagrangian $\mathcal{L}$ is not singular.*

$$H^*_T | \Psi > = 0. \tag{16}$$

In the configuration representation $\hat{q}|\Psi> = q|\Psi>$ constraint (16) is the Schrödinger equation

$$i \frac{d}{d\tau} |\Psi> = a(\tau) H |\Psi> \tag{17}$$

in the absence of absolute time which coincides with the ordinary one in the gauge $\chi \equiv q^\circ - \tau \approx 0$.

In the next section we obtain a generalization of equation (17) to the quantum field theory case; its formal solution will be determined by the operator $T(\sigma_2, \sigma_1)$.

b. Quantum Field Theory on Curved Surfaces

We follow the classical action

$$S[q] = \int d^4x \, \mathcal{L}(q^i(x), \frac{\partial q^i(x)}{\partial x^a})$$

for the system of fields $q^i(x)$ $(i = 1,\dots,N)$ determined in Minkowski's four-dimensional space x^a $(a = 0,1,2,3)$. Since in the relativistic theory the role of time is played by the three-dimensional surface, changing its characteristic variables into dynamic ones allows covariant writing of Hamilton equations of motion.

Let us introduce curvilinear coordinates y (y_1, y_2, y_3) on the surface σ. Functions $x^a(y)$ will characterize the surface σ in space-time and the way of its parametrization. Similarly to $q^0(\tau)$ in section a, we shall consider the functions $x^a(y)$ as dynamic variables. Now, we determine the parameter τ which changes when passing from one surface to another.

A new Lagrangian $\mathcal{L}^*$ is to be constructed similarly to (10):

$$\mathcal{L}^*(q^i, \frac{\partial q^i}{\partial y^a}, \frac{\partial x^a}{\partial y^b}) \equiv \det||I||\mathcal{L}(q^i, \frac{\partial q^i}{\partial y^a}(I^{-1})^a_b), \tag{18}$$

$$I^a_b = \frac{\partial x^a}{\partial y^b}, \quad y^\circ = \tau$$

considering that

$$\int \mathcal{L} \, d^4x = \int \mathcal{L}^* d^4y .$$

A theory with additional field functions $x^a(y)$ turns out reparametrization-invariant with regard to the substitution $y^{a'} = f^a(y)$; consequently, the total Hamiltonian is expressed as a linear combination of constraints with arbitrary functions $c^a(\vec{y}, \tau)$

$$H^*_T = \int d^3y \, c^a(y, \tau)(p_a + K_a),$$

where p_a are the momentum variables conjugated to x_a, K_a is determined through the energy-momentum of the initial theory $T^b{}_a$

$$K_a = T^b{}_a \, e^\circ_b$$

(e°_b is the cofactor of the corresponding element of matrix I).

A standard way is used to go to the quantum theory. We state that the field quantities $q_i(y)$, $p^i(y)$ and $x_a(y)$, $p^a(y)$ are operators with the corresponding algebra

$$[\hat{q}_i(y), \hat{p}^j(y')] = i\delta^j_i \, \delta(y, y'),$$

$$[\hat{x}_a(y), \hat{p}^b(y')] = i\delta^b_a \, \delta(y, y'), \tag{19}$$

$$[\hat{q}_i(y), \hat{q}_j(y')] = [\hat{q}_i(y), \hat{x}_a(y')] = [\hat{p}^i(y), \hat{p}^j(y')] = [\hat{p}^a(y), \hat{p}^b(y')] = 0$$

and demand that the constraint on the state vector should hold true:

$$H^*_T |\Psi> = 0. \tag{20}$$

Note, however, that in vitrue of the equations of motion $i\frac{dx^a}{d\tau} = [H^*_T, x^a] = c^a(y, \tau)$ the variables x^a for the fields remain c-number functions after quantization as well.

Considering that the field functions $q^i(y)$ and $x^a(y)$ are specified on a fixed space-like surface σ_0 $q^i(y) = q^i|_{\tau=\tau_0}$, $x^a(y) = x^a|_{\tau=\tau_0}$ and the state vector depends on the evolution parameter τ, we obtain the Schrödinger scheme. Evolution of the system is determined by equation (20) which has the following form in the coordinate representation

$$i\frac{d}{d\tau}|\Psi> = (\int d^3y \, c^a(y, \tau) K_a) \, |\Psi>. \tag{21}$$

Eq. (21) is a relativistic equation written unlike Tomonaga's equation (I), in Schrödinger's representation.

Taking the space-like surface in the form of a plane $x^0 = $ const, i.e. fixing the gauge $\chi \equiv x^0 - \tau \approx 0$, we obtain an ordinary Schrödinger equation.

A formal solution of eq. (21) can be written as

$$|\Psi>_\tau = \exp(-i \int_{\tau_0}^{\tau} d\tau' d^3y c^a K_a) |\Psi>_{\tau=\tau_0} , \tag{22}$$

where we have made use of the fact that K_a is explicitly independent of τ owing to translation invariance. Note that the operator on the right-hand side of relation (22) will connect field operators determined on different surfaces

$$q^i(y)|_\tau = \exp(-iF) q^i(y)|_{\tau_0} \exp(iF) ,$$
$$F \equiv \int_{\tau_0}^{\tau} d\tau' d^3y c^a(\tau', y) K_a . \tag{23}$$

2. EVOLUTION OPERATOR IN INTERACTION REPRESENTATION

Let us analyse the relation between the Tomonaga equation and Eq. (21). To do this, we explain the meaning of the operators in the right-hand part of Eq. (21) /9/. Let us introduce a local basis on the surface σ with unit vectors

$$(e^a_k = \frac{\partial x^a}{\partial y^k}, n^a); \; n_a n^a = 1, \; n_a e^a_k = 0,$$

and locally expand K_a in the normal $K_\perp = K_a n^a$ and tangent $K_r = K_a e^a_r$ components.

Assume that $\dot{x}_\perp = 0$, i.e. the surface does not change in the direction perpendicular to it, but only surface coordinates change. In this case a non-zero contribution comes from components K_r which transform field functions, leaving them on the fixed surface. It means that they can be determined on the geometrical basis, and they do not depend on the interaction. As to $K_\perp$, it corresponds to the surface motion normal to itself and is responsible for dynamics. So, operator (22), describing dynamics in Schrödinger's scheme, is used to obtain the evolution operator $U(\sigma, \sigma)$ in the interaction representation only after the effect of tangent components K is neutralised.

Let us introduce an operator

$$U(\tau, \tau_0) = \exp(-i \int_{\tau_0}^{\tau} d\tau' d^3y c^a \overset{\circ}{K}_a) \exp(i \int_{\tau_0}^{\tau} d\tau' d^3y c^a K_a) , \tag{24}$$

where $\overset{\circ}{K}_a$ are the corresponding operators in the absence of interaction. Assume that the following conditions are fulfilled on the surface $x^a(y)|_{\tau=\tau_0}$

$$q^i(y)|_{\tau_0} = q^i_f(y)|_{\tau_0} \; ; \quad p^i(y)|_{\tau_0} = p^i_f(y)|_{\tau_0} .$$

Here $q^i_f(y)$, $p^i_f(y)$ are the conjugated dynamic variables of the corresponding free theory. If these conditions are fulfilled, the operator $U(\tau, \tau_0)$ will be determined only through the components $\overset{\circ}{K}_\perp$ and $K_\perp$. To be sure, write an equation for $U(\tau, \tau_0)$

$$i \frac{\partial U(\tau, \tau_0)}{\partial \tau} = K_I(\tau) U(\tau, \tau_0), \tag{25}$$

where

$$K_I(\tau) = \exp(-i\overset{\circ}{F}) \int d^3y c^a(\tau, y)(K_a - \overset{\circ}{K}_a) \exp(i\overset{\circ}{F}) =$$
$$= \int d^3y c^a(y, \tau) K^I_a(y, \tau) ,$$
$$\overset{\circ}{F} = \int_{\tau_0}^{\tau} d\tau' d^3y c^a(\tau', y) \overset{\circ}{K}_a .$$

In the expression $c^a K^I_a$ only the normal components $c^\perp K^I_\perp$ make a contribution, the field operators $q^i(y)$ in K^I_a satisfying free equations of motion in virtue of (23).

Write down the formal solution of (25) as an expansion in powers of the interaction constant:

$$U(\tau, \tau_0) = T \exp(i \int_{\tau_0}^{\tau} d\tau' d^3y c^a(\tau', y) K^I_a(\tau', y))$$

or, with allowance for initial conditions,

$$K_a - \overset{\circ}{K}_a = (T^b{}_a - \overset{\circ}{T}{}^b{}_a) e^{\circ}_b = g_{ab} e^b_0 \mathcal{L}_I$$

($\mathcal{L}$ is the interaction Lagrangian) we have

$$U(\tau, \tau_0) = T \exp(i \int_\Omega d^4x \mathcal{L}_I(x)) ,$$

where Ω is the integration region between the surfaces with $\tau' = \tau_0$ and $\tau' = \tau$. Comparing (27) with Dyson's solution (3) makes us sure that representation (7) with

$$T(\sigma_2,\sigma_1) = \exp(-i \int_{\tau_1}^{\tau_2} d^3y\, d\tau'\, c^a(\tau', y)\, K_a)$$

holds true. Note that for the case of flat hypersurfaces

$$x^a(y, \tau) = A^a(\tau) + B^a_i(\tau) y^i$$

it follows from (28) that, fixing a certain gauge, one can obtain representations (4) and (5). For the former case we take $\chi = x^0 - \tau \approx 0$ and for the latter

$$\chi = \mathrm{th}\,\tau - x_0 / \frac{\vec{\lambda}\cdot\vec{x}}{|\vec{\lambda}|} \approx 0 .$$

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Хведелидзе А.М., Санадзе В.В. E2-88-768

Гамильтонов формализм Дирака и оператор эволюции в представлении взаимодействия

В рамках обобщенной гамильтоновой теории Дирака рассматривается оператор эволюции состояния между двумя произвольными пространственно-подобными поверхностями. Оператор эволюции строился с помощью набора первичных связей первого рода, генерирующих произвольные деформации поверхности. В картине взаимодействия получено представление для оператора эволюции в виде хронологически упорядоченной экспоненты Дайсона.

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Hamilton-Dirac Method and Evolution Operator in Interaction Representation

Based on the Hamilton theory extended by Dirac, an evolution operator between states on two arbitrary space-like surfaces is examined. The evolution operator is constructed with the use of the first class primary constraints generating arbitrary deformations of a surface. In the interaction representation the evolution operator is represented in the form of the chronological ordering of the Dyson exponential.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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