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# ASYMPTOTICS OF THE ALTARELLI-PARISI-LIPATOV EVOLUTION KERNELS OF PARTON DISTRIBUTIONS

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## 1. Introduction

In recent years the problem of summation of large perturbative QCD corrections that occur in hard hadron-hadron scattering cross sections has been stressed /1/. Important ingredients of the analysis of these processes /2/ are the properties of parton distributions $f_{a/A}(x)$. These functions admit a simple probabilistic interpretation. The distribution function $f_{a/A}(x)$ may be thought of as the probability to find a parton $a$ in a hadron $A$ with a specified fraction $x$ of its longitudinal momentum. It has been shown /2/ that the origin of large perturbative corrections is closely related to the properties of the parton distributions as $x \to 1$ and, in particular, to the behaviour of the evolution kernels $P_{ab}(z)$ of these functions as $z \to 1$ /3,4/. The evolution kernels govern the dependence of $f_{a/A}(x)$ on the renormalization parameter $\mu$ /5/. It is our goal in the present paper to determine the asymptotics of the evolution kernels $P_{ab}(z)$ as $z \to 1$.

The paper is organized as follows. In section 2, the definitions of the parton distribution functions are given and their main properties are formulated. The asymptotics of the quark distribution function is studied in section 3. In this section, the factorized expression for the quark distribution is derived, whose properties established in section 4 play an important part in the further analysis. The asymptotics of the kernels governing the evolution of the quark distribution function is found in section 5. The gluon distributions and their evolution kernels are investigated in section 6.

## 2. Definitions of the parton distribution functions

Let us consider a spin-averaged hadron $A$ having a mass M and moving in the z-direction with a momentum $P_\mu = (P_+, P_-, P_T)$ where $P_\pm = (P_0 \pm P_3)/\sqrt{2}$ and $P_T = (P_1, P_2) = 0$, $P_+ >> P_- = M^2/2P_+$. The quark distribution in the hadron $A$ is defined as the following hadron expectation value /6,7/:

$$f_{q/A}(x) = \frac{1}{4\pi}\int dy_-\; e^{-ixP_+y_-}$$

$$\langle P|\bar{\Psi}(0,y_-,0_T)\gamma^+ \mathrm{P}\exp\left[ig\int_0^{y_-} ds\; A_+^a(0,s,0_T)t_a\right]\Psi(0)|P\rangle, \qquad (1)$$

where $\Psi(y_+,y_-,y_T)$ is the renormalized field operator of a quark of a certain flavour, $t_a$ are generators of the fundamental representation of SU(3) and P denotes the path-ordering of $t_a$. The product of operators in this definition is singular and it requires a subtraction of ultraviolet divergences /8/. In the following all the divergences are regularized by the dimensional regularization method combined with the minimal subtraction prescription ($\overline{\mathrm{MS}}$-scheme).

The antiquark distribution in the hadron $A$ is defined analogously to (1) /6,7/:

$$f_{\bar{q}/A}(x) = \frac{1}{4\pi}\int dy_-\; e^{-ixP_+y_-}$$

$$\mathrm{Tr}\left\{\gamma^+\langle P|\Psi(0,y_-,0_T)\mathrm{P}\exp\left[-ig\int_0^{y_-} ds\; A_+^a(0,s,0_T)(t_a)^T\right]\bar{\Psi}(0)|P\rangle\right\}$$

and for the gluon distribution we have /6/:

$$f_{g/A}(x) = \frac{1}{2\pi x P_+}\int dy_- e^{-ixP_+y_-}$$

$$\langle P|G^b_{+\nu}(0,y_-,0_T)\gamma^+ \mathrm{P}\exp\left[ig\int_0^{y_-} ds\; A_+^a(0,s,0_T)\sigma_a\right]G^b_{\nu-}(0)|P\rangle, \qquad (2)$$

where $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc}A^b_\mu A^c_\nu$ and $(\sigma_a)_{bc} = -if_{abc}$ are generators of the adjoint representation of SU(3).

Parton distribution functions thus defined are gauge invariant. They are normalized so that for free quarks and gluons with $A$ replaced by a parton state

$$f_{a/b}(x) = \delta_{ab}\,\delta(1-x).$$

The following properties of the parton distributions will be important for us.

(a) The parton distribution functions have the spectral properties /6,9/:

$$f_{q/A}(x) = -f_{\bar{q}/A}(-x),\quad f_{g/A}(x) = -f_{g/A}(-x),$$

$$f_{a/A}(x) = 0\;,\ \text{if } |x|>1\;,\ a = q,\bar{q},g$$



(b) The moments of the parton distributions, defined by

$$M_q(N) = \int_0^1 dx\; x^{N-1}f_{q/A}(x) + (-1)^N\int_0^1 dx\; x^{N-1}f_{\bar{q}/A}(x)$$

$$M_g(N) = (1+(-1)^N)\int_0^1 dx\; x^{N-1}f_{g/A}(x),$$

are related to matrix elements of the familiar twist-two operators that appear in the operator product expansion of two currents:

$$O_q^{\mu_1\ldots\mu_n} = \frac{1}{2}\bar{\Psi}(0)\left\{\gamma^{\mu_1}iD^{\mu_2}\ldots iD^{\mu_n}\right\}_{TS}\Psi(0)$$

$$O_g^{\mu_1\ldots\mu_n} = \left\{G^{\mu_1\nu}(0)\,iD^{\mu_2}\ldots iD^{\mu_{n-1}}\,G^{\nu\mu_n}(0)\right\}_{TS}$$

( here TS denotes the traceless symmetric part of the tensor ) by a simple equation /6/:

$$M_a(N) = (P_+)^{-N}\langle P|\,O_a^{+\ldots+}\,|P\rangle,\quad a = q,g.$$

(c) The dependence of the parton distributions on the unit of mass in the dimensional regularization $\mu$ is governed by the Altarelli-Parisi-Lipatov (APL) equations /5/:

$$D\,f_{a/A}(x) = \sum_b\int_x^1\frac{dy}{y}\,P_{ab}\left[\frac{x}{y}\right]f_{b/A}(y)\;,\quad a,b = q,\bar{q},g, \qquad (3)$$

where $D = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}$ + (mass terms). The APL equations may be rewritten in the form of equations for the moments of the parton distributions /10/:

$$D\,M_a(N) = \sum_b\gamma_{ab}(N)\,M_b(N)\quad a,b = q,g,$$

where the matrix of the anomalous dimensions

$$\gamma_{ab}(N) = \int_{-1}^1 dz\; z^{N-1}P_{ab}(z)$$

coincides, in accordance with the previous property (b), with the anomalous dimensions of the twist-two operators.

It follows from the APL equations (3) that the renormalization properties of the parton distribution functions $f_{a/A}(x)$ as $x\to 1$ are controlled by the behaviour of the evolution kernels $P_{ab}(z)$ as $z\to 1$ or, equivalently, by the asymptotics of the anomalous dimensions $\gamma_{ab}(N)$ at large N. Our further consideration is based on the investigation of the parton distributions as $x \to 1$. Their knowledge will enable us to find the asymptotics of the evolution kernels $P_{ab}(z)$ as $z \to 1$ with the use of the APL equation.

The properties of the anomalous dimensions at large numbers have been studied early in QED /3/ in connection with calculations of the Sudakov effects. It has been established that the anomalous dimension $\gamma(N)$ of the twist-two operator obtained from $O_q^{\mu_1 \dots \mu_n}$ by a mere redefinition of the fields has a single-logarithmic asymptotics at large N to all orders of perturbation theory (PT) /3/:

$$\gamma(N) \propto \log N + O(N^0). \tag{4}$$

Nowadays in QCD, the results of calculation of the evolution kernels and the matrix of the anomalous dimensions are known to the lowest orders of PT /7,11/. Using them we may conclude that up to the two-loop order when $z \to 1$ /12/:

$$P_{qq}(z) = 2 A_q \frac{1}{(1-z)_+} - 2 B_q \delta(1-z) + O((1-z)^0)^{1}$$

$$P_{gg}(z) = 2 A_g \frac{1}{(1-z)_+} - 2 B_g \delta(1-z) + O((1-z)^0)$$

$$P_{gq}(z) = O((1-z)^0) \quad P_{qg}(z) = O((1-z)^0),$$

where $\int_0^1 dx\, \varphi(x) \frac{1}{(1-x)_+} = \int_0^1 dx\, (\varphi(x) - \varphi(0)) \frac{1}{(1-x)}$ and

$$A_q = \frac{\alpha_s}{\pi} C_F + \left(\frac{\alpha_s}{\pi}\right)^2 C_F \left\{ C_A \left[\frac{67}{36} - \frac{\pi^2}{12}\right] - \frac{5}{9} T_F \right\}$$

$$A_g = \frac{C_A}{C_F} A_q \tag{5a}$$

$$B_q = -\frac{3}{4} \frac{\alpha_s}{\pi} C_F + \left(\frac{\alpha_s}{\pi}\right)^2 C_F \left\{ C_A \left[-\frac{17}{96} - \frac{11}{72}\pi^2 + \frac{3}{4}\zeta(3)\right] + T_F \left[\frac{1}{24} + \frac{1}{18}\pi^2\right] + C_F \left[-\frac{3}{32} + \frac{1}{8}\pi^2 - \frac{3}{2}\zeta(3)\right] \right\}$$

$$B_g = \frac{\alpha_s}{\pi} \left[-\frac{11}{12} C_A + \frac{1}{3} T_F\right] + \left(\frac{\alpha_s}{\pi}\right)^2 \left\{ -C_A^2 \left[\frac{2}{3} + \frac{3}{4}\zeta(3)\right] + \frac{1}{3} C_A T_F + \frac{1}{4} C_F T_F \right\}$$

[1] Hereafter $O((1-z)^0)$ denotes terms those moments $\int_0^1 dz\, z^{N-1} O((1-z)^0)$ tend to zero as $N \to \infty$.



$\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$, $T_F = \frac{1}{2} n_F$ for $n_F$ quark flavours and the Casimir operators of $SU(N_c)$ group equal to $C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}$, $C_A = N_c = 3$ at $N_c = 3$. For the matrix of the anomalous dimensions for $N \gg 1$ we have in the same approximation

$$\gamma_{qq}(N) = -2 \log(N e^{\mathbb{C}}) A_q - 2 B_q + o(N^0)$$

$$\gamma_{gg}(N) = -2 \log(N e^{\mathbb{C}}) A_g - 2 B_g + o(N^0) \tag{5b}$$

$$\gamma_{gq}(N) = o(N^0) \qquad \gamma_{qg}(N) = o(N^0),$$

where $\mathbb{C}$ is the Euler constant. Thus, we observe the important property: the two-loop corrections do not change the one-loop asymptotics of the evolution kernels $P_{ab}(z)$ as $z \to 1$ and the anomalous dimensions $\gamma_{ab}(N)$ for $N \gg 1$ /12/. It is just natural to suppose, following the analogy with QED - relation (4), that this property is retained in QCD to all orders of PT. The supposition is quite untrivial since some individual diagrams contributing to the two-loop $P_{ab}(z)$ and $\gamma_{ab}(N)$ possess more powerful term, viz. $\left[\frac{\log^2(1-x)}{1-x}\right]_+$ and $\log^3 N$, that, however, cancel out exactly in the sum of all diagrams.

## 3. The quark distribution as $x \to 1$

The evolution kernels $P_{ab}(z)$ are gauge invariant and do not depend on a particular form of the state entering into the definitions (1) and (2). Therefore, let us replace for simplicity $A$ by a spin-averaged state of a quark with a mass m and momentum $p_\mu$ ( $p_T = 0$, $p_+ \gg p_- = m^2/2p_+$). Then we fix the axial gauge of the gluon field $(nA^a(x)) = 0$ where $n_T = 0$, $n_- \gg n_+$ and introduce the following function:

$$\phi(x) = \frac{1}{4\pi} \int dy_-\, e^{-ixp_+y_-} \langle p | \bar{\Psi}(0, y_-, 0_T)\, \gamma^+\, \Psi(0) | p \rangle \tag{6}$$

differing from the definition of $f_{q/q}(x)$ by the absence of the path-ordered exponential. $\phi(x)$ is gauge variant and depends on the gauge fixing vector $n_\mu$. It may be easily noticed that in a special case of the light-like axial gauge $A_+^a(x) = 0$, i.e., $n_T = n_+ = 0$:

$$\phi(x) = f_{q/q}(x). \tag{7}$$

Our strategy will be to investigate the properties of $f_{q/q}(x)$ by studying the behaviour of $\phi(x)$ as $x \to 1$ and performing the limit to the axial gauge $A_+^a(x) = 0$.

The only source of the dependence of $\phi(x)$ on the unit mass $\mu$ as $n^2 \neq 0$ is the renormalization of the quark fields in (6). Therefore $\phi(x)$ obeys in that case the renormalization group equation:

$$D\,\phi(x) = 2\,\gamma_q(g)\,\phi(x). \tag{8}$$

where $\gamma_q$ is the anomalous dimension of the quark field in the axial gauge. However, if we put $n^2 = 0$, the additional ultraviolet divergences appear in $\phi(x)$ /6,7/ thus violating equation (8) and leading with the relation (7) in hand to the APL equation (3).

Let us study the dependence of $\phi(x)$ as $x \to 1$ on the vector $n_\mu$ and then examine the limit $n^2 \to 0$. The general form of the Feynman diagrams arising in the perturbative expansion of $\phi(x)$ when $x > 0$ is pictured in fig.1(a) /2/ where the dashed line represents the unitary cutoff transforming the virtual cut lines into the real ones.

The one-loop calculation of $\phi(x)$ for $n^2 \neq 0$ gives the result:

$$\phi(x)=\delta(1-x)\left[1-\frac{3}{4}\frac{\alpha_s}{\pi}C_F\log\frac{m^2}{\mu^2}\right]+\frac{1}{(1-x)_+}\frac{\alpha_s}{\pi}C_F\left[\log\frac{4(pn)^2}{m^2n^2}-2\right]+O((1-x)^0) \tag{9}$$

satisfying equation (8) but at $n^2 = 0$ we get the expression for $\phi(x)$:

$$\phi(x) = \delta(1-x)\left[1-\frac{3}{4}\frac{\alpha_s}{\pi}C_F\log\frac{m^2}{\mu^2}\right] + \frac{1}{(1-x)_+}\frac{\alpha_s}{\pi}C_F\left[\log\frac{\mu^2}{m^2}-1\right] - 2\frac{\alpha_s}{\pi}C_F\left[\frac{\log(1-x)}{(1-x)}\right]_+ + O((1-x)^0) \tag{10}$$

which obeys the APL equation (3).

The analysis of the multiloop properties of the diagrams of fig.1(a) has shown /2,13,14/ that the leading contribution to the function $\phi(x)$ as $x \to 1$ and $n^2 \neq 0$ (that is, the singular one is not suppressed by powers of $(1-x)$) comes only from diagrams with the structure shown in fig.1(b). These diagrams contain three subgraphs, i.e., hard, collinear and soft ones. The momenta $l_\mu$ of particles (quarks and gluons) belonging to the subgraphs are the



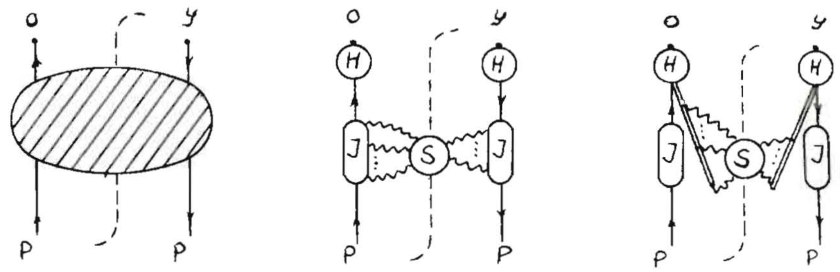


Fig.1(a) The general structure of the Feynman diagrams arising in the perturbative expasion of $\phi(x)$, defined in (6). The dashed line represents the unitart cutoff. The blob denotes an arbitrary subgraph;(b) The diagrams determining the leading contribution to $\phi(x)$;(c) The diagrams of fig.1(b) after summing over the soft gluons.

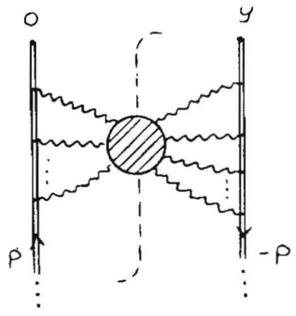


Fig.2. The diagrams arising in the expansion of the contour functional. The double line denotes the contour C in the Minkowski space.

following /14/:

$$\begin{aligned} &\text{(a) hard subgraph H:} && |\, l_+,\ l_-,\ l_T \,| = O(Q) \\ &\text{(b) collinear subgraph J:} && l_+ = O(Q),\ l_- = O(M^2/Q),\ l_T = O(M) \\ &\text{(c) soft subgraph S:} && |\, l_+,\ l_-,\ l_T \,| = O(M) \end{aligned} \tag{11}$$

where $Q^2 = (pn)^2/n^2$, $M = (1-x)p_+$.

It follows from the definition (6) that the integration regions over momenta of all the particles in fig.1(b) are restricted by the only condition: the total + component of momenta of all real particles equals to $(1-x)p_+$ and vanishes as $x \to 1$. Among quarks and gluons with momenta (11) only gluons belonging to a soft subgraph have a small + component of momenta. (Contribution of the quarks with small momenta to $\phi(x)$ is suppressed by powers of $(1-x)$). That is why all the real particles of the diagrams in fig.1(b) are soft gluons.

A subsequent transformation of the diagrams, fig.1(b), is associated with the factorization of an individual graph, fig.1(b), into a hard part and a collinear part. The presence of the soft gluon subgraph in fig.1(b) upsets the desired factorization. Nevertheless, the factorization is restored in the sum of the diagrams. If we sum over attachments of the soft gluons to the collinear subgraphs in fig.1(b), two factors accumulating all the soft gluon effects are factorized from the contribution of J /14,15/. These factors are denoted by double lines in fig.1(c) and they will be defined below in (15). The diagrams of fig.1(b) after summing over the soft gluons are pictured in fig.1(c).

There is no interaction between the subgraphs of the diagram fig.1(c), and its contribution to the function $\phi(x)$ may be expressed as follows:

$$\phi(x) = (HJ)^2 \frac{p_+}{2\pi} \int dy_- e^{iy_- p_+ (1-x)} S + O((1-x)^0), \tag{12}$$

where H,J and S denote the contributions of the corresponding subgraphs. A hard subgraph H describes interactions of quarks and gluons at short (as compared with $(pn)^2/n^2$) distances, and therefore H does not depend on the quark mass m and its behaviour



as $x \to 1$ is regular. The one-loop value of H is [2]:

$$H = H\left[\frac{(pn)^2}{n^2\mu^2}\right] = 1 + \frac{\alpha_s}{4\pi} C_F\left[ -\frac{1}{2} \log^2\left[\frac{4(pn)^2}{n^2\mu^2}\right] + \log\left[\frac{4(pn)^2}{n^2\mu^2}\right]\right]. \tag{13}$$

The collinear subgraph J describes the propagation of the jet-like particles in the direction of the quark momentum $p_\mu$. Its contribution to $\phi(x)$ depends in general on the variables $m^2$, $\mu^2$ and $(pn)^2/n^2$. However, a more detailed study of the collinear subgraph allows us to state that J does not really depend on $(pn)^2/n^2$ /16/. Performing a one-loop calculation we get:

$$J = J\left[\frac{m^2}{\mu^2}\right] = 1 + \frac{\alpha_s}{4\pi} C_F\left[ \frac{1}{2} \log^2 \frac{m^2}{\mu^2} + \frac{1}{2} \log\frac{m^2}{\mu^2}\right]. \tag{14}$$

The soft subgraph takes into account all the effects caused by soft gluons. Its contribution to the function $\phi(x)$ as $x \to 1$ may be represented as /15/:

$$S = \langle 0|\, E_{-p}(y,\infty)\, \{E_{-p}(0,\infty)\}^+ |0\rangle, \quad y_\mu = (0, y_-, 0_T) \tag{15}$$

with the following notation for the path-ordered exponentials:

$$E_{-p}(y,\infty) = P \exp\left[-ig\int_0^\infty ds\, e^{-\varepsilon s}\, p^\mu A_\mu^a(-ps+y)\, t_a\right], \quad \varepsilon \to 0.$$

The path-ordered exponentials entering into the relation (15) are pictured by double lines in fig.1(c). All the diagrams arising in the perturbative expansion of (15) possess infrared divergences that cancel out exactly in the r.h.s. of (15) /6,14/. S depends on the dimensionless product $(y_-\mu)$ and on the directions in Minkowski space indicated by the vectors $n_\mu$ and $p_\mu$, i.e., on $\frac{n_+}{n_-}$ and $\frac{p_+}{p_-}$. The invariance of S under the boosts in the z-direction uniquely fixes the functional dependence of S:

$$S = S\left[\frac{1}{2}y_-^2 \mu^2 \frac{p_+}{p_-}, \frac{p_+ n_-}{p_- n_+}\right] = S\left[(y_- p_+)^2 \frac{\mu^2}{m^2}, \frac{4(pn)^2}{m^2 n^2}\right].$$

The one-loop calculation gives:

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[2] Hereafter the dependence of H,J and S on the renormalized coupling constant $g = g(\mu)$ is implied.

$$S = 1 + \frac{\alpha_s}{2\pi} C_F \left[ \frac{1}{2} \log^2\left(\frac{4(pn)^2}{m^2 n^2}\right) - \log\left(\frac{\mu^2}{m^2} Y^2\right) \log\left(\frac{4(pn)^2}{m^2 n^2}\right) + 2 \log\left(\frac{\mu^2}{m^2} Y^2\right) - \log\left(\frac{4(pn)^2}{m^2 n^2}\right)\right], \qquad (16)$$

where $Y = (p_+ y_- - i0) e^{\mathbb{C}}$.

To verify relation (12), we substitute obtained one-loop values of the subgraphs into (12) and reproduce expression (9).

The function $\phi(x)$ thus determined has the spectral property:

$$\phi(x) = 0 \ , \ \text{as} \ x > 1. \qquad (17)$$

To prove it, we rewrite equation (15) as follows:

$$S = \sum_N \langle 0| \ E_{-p}(0,\infty) |N\rangle \ \langle N| \left( E_{-p}(0,\infty)\right)^+ |0\rangle \ e^{-iN_+ y_-} \ ,$$

where $N_\mu$ is the momentum of a state $|N\rangle$. Substituting this relation into (12) we find that $\phi(x)$ differs from zero at $N_+ = p_+(1-x) \geq 0$. As a consequence of the spectral property (17) and (12), S being a function of $y_-$ has poles only in the upper half-plane of the complex $y_-$ :

$$S = S\left[(y_- p_+ - i0)^2 \frac{\mu^2}{m^2}, \ \frac{4(pn)^2}{m^2 n^2}\right] .$$

Using the explicit form of relation (12) we will demonstrate in the next section that $\phi(x)$ satisfies the evolution equation.

## 4. The evolution equation

We differentiate both the sides of (12) with respect to $\mu$:

$$D\,\phi(x) = \frac{p_+}{2\pi} \int dy_- e^{iy_- p_+ (1-x)} \ (HJ)^2 \ S \ D \log(S) + D \log(JH)^2 \ \phi(x) + O((1-x)^0)$$

and perform the identical transformation of the last relation:

$$D\,\phi(x) = \int_{-\infty}^{+\infty} dz \ \phi(z) \ P\left[x-z+1, \ \frac{m^2}{\mu^2}, \ \frac{(pn)^2}{n^2\mu^2}\right] + O((1-x)^0) \ , \qquad (18)$$

where the following notation is used:



$$P\left[z, \ \frac{m^2}{\mu^2}, \ \frac{(pn)^2}{n^2\mu^2}\right] = \delta(1-z) \ D \log(JH)^2 + \frac{p_+}{2\pi} \int dy_- e^{iy_- p_+ (1-z)} \ D \log S\left[(y_- p_+ - i0)^2 \frac{\mu^2}{m^2}, \ \frac{4(pn)^2}{m^2 n^2}\right]. \qquad (19)$$

In the integral in the second term of (19) the contour can be closed without enclosing a pole unless $z \leq 1$. Hence

$$P\left[z, \ \frac{m^2}{\mu^2}, \ \frac{(pn)^2}{n^2\mu^2}\right] = 0 \ , \ \text{as} \ z > 1 .$$

Taking into account this property and spectral property (17) we derive from (18) the following equation :

$$D\,\phi(x) = \int_x^1 \frac{dz}{z} \ \phi(z) \ P\left[\frac{x}{z}, \ \frac{m^2}{\mu^2}, \ \frac{(pn)^2}{n^2\mu^2}\right] + O((1-x)^0) \qquad (20)$$

similar to the APL equation (3).

Let us now examine the properties of the kernel P(...). We notice that contribution (15) of the soft subgraph to $\phi(x)$ entering into the expression (19) may be rewritten as a contour functional /15,16/:

$$S = \langle 0| \ P\exp(ig \int_C dz_\mu \ A^a_\mu(z) t_a) |0\rangle \ ,$$

where the contour C is pictured by a double line in fig.2. Both the rays in fig.2 are directed in the Minkowski space along the vector $p_\mu$. 0 is the endpoint of one of them and the beginning of the other is placed at point $y_\mu = (0, \ y_-, \ 0_T)$.

It is well known that a contour functional possesses ultraviolet divergences /17/ which are renormalized multiplicatively in the case of the contour C (fig.2) involved in the expression for S. As a result, the dependence of S on the mass unit $\mu$ is described by the following equation /15/:

$$D\,S = -2\,\Gamma_{end}(g) \ S , \qquad (21)$$

where $\Gamma_{end}(g)$ is the end anomalous dimension of the contour functionals. $\Gamma_{end}(g)$ is gauge dependent and in the axial gauge it equals /18/:

$$\Gamma_{end}(g) = \Gamma_{cusp}(\gamma, g) , \qquad (22)$$

where $\Gamma_{cusp}(\gamma, g)$ is the cusp anomalous dimension of the contour functionals and $\gamma$ is the angle in the Minkowski space formed by the vectors $n_\mu$ and $p_\mu$ :

$$\cosh^2\gamma = (pn)^2/n^2m^2 \gg 1, \quad \gamma = \frac{1}{2}\log\frac{4(pn)^2}{n^2m^2} = \frac{1}{2}\log\frac{p_+n_-}{p_-n_+} .$$

The properties of the cusp anomalous dimension are well known /17,19/. It is essential for us that in the limit of large $\gamma$ $\Gamma_{cusp}$ has the following asymptotics /19/:

$$\Gamma_{cusp}(\gamma,g) = \frac{1}{2}\log\frac{4(pn)^2}{n^2m^2}\,\Gamma_{cusp}(g) + O\left[\log^0\frac{(pn)^2}{n^2m^2}\right], \qquad (23)$$

where to the two-loop order in the $\overline{MS}$-scheme we have:

$$\Gamma_{cusp}(g) = \frac{\alpha_s}{\pi}C_F + \left(\frac{\alpha_s}{\pi}\right)^2 C_F\left[C_A\left(\frac{67}{36} - \frac{\pi^2}{12}\right) - \frac{5}{9}T_F\right] + O(\alpha_s^3).$$

After substituting eqs.(21),(22) into (19) we find:

$$P\left[z, \frac{m^2}{\mu^2}, \frac{(pn)^2}{n^2\mu^2}\right] = \delta(1-z)\left[D\log(JH)^2 - 2\Gamma_{cusp}(\gamma,g)\right]$$

which in its turn leads to the following equation for $\phi(x)$:

$$D\phi(x) = \phi(x)\left[D\log(JH)^2 - 2\Gamma_{cusp}(\gamma,g)\right].$$

The comparison of this relation with equation (8) yields:

$$D\log(JH)^2 - 2\Gamma_{cusp}(\gamma,g) = 2\gamma_q(g).$$

Since the r.h.s. of this equality does not depend on variables $\mu^2/m^2$ and $(pn)^2/n^2\mu^2$, we conclude that /16/:

$$\frac{d}{d\log m^2}D\log(J) + \frac{1}{2}\Gamma_{cusp}(g) = -\frac{d}{d\log n^2}D\log(H) + \frac{1}{2}\Gamma_{cusp}(g) = 0, \qquad (24)$$

where the asymptotics (23) has been used. We are convinced that the one-loop values of H and J and (eqs.(13) and (14)) are in agreement with (24).

## 5. The evolution kernels

So, we have in detail studied the properties of the function $\phi(x)$ defined in (6) as $x \to 1$ in the axial gauge with $n^2 \neq 0$. Now we turn to equation (20) and take the limit $n^2 \to 0$. In this limit



the function $\phi(x)$ and kernel P(...) coincide according to (7) with the quark distribution in a quark $f_{q/q}(x)$ and evolution kernel $P_{qq}(z)$, respectively. From equation (20) we get:

$$D f_{q/q}(x) = \int_x^1 \frac{dz}{z} f_{q/q}(z)\,P_{qq}\left(\frac{x}{z}\right) + O((1-x)^0). \qquad (25)$$

However, this limit is not quite obvious. The function $\phi(x)$ and contributions of the soft and hard subgraphs to it (eqs.(9),(13) and (16)) contain terms $\log(n^2)$ so that the limit $n^2 \to 0$ does not exist. If we put $n^2 = 0$ from the beginning, these dangerous terms are replaced by the ultraviolet poles in the dimensional regularization parameter and after subtraction they are revealed as an additional dependence of $\phi(x)$, S and H on $\mu$. For example, at $n^2=0$ expression (9) is replaced by (10), and the one-loop calculation of the soft subgraph gives, instead of (16):

$$S = 1 - \frac{\alpha_s}{4\pi}C_F\left[\log^2\left(\frac{\mu^2}{m^2}Y^2\right) - 2\log\left(\frac{\mu^2}{m^2}Y^2\right)\right], \quad Y = (p_+y_- - i0)e^{C}. \qquad (26)$$

It may be noticed that expression (26) does not obey equation (21) since the cusp anomalous dimension (23) entering into this equation becomes divergent at $n^2 = 0$. It means that in the light-like axial gauge both the multiplicative renormalizability of the contour functional and as a consequence equation (21) are violated. Nevertheless, using relations (21)-(23) we derive

$$\frac{d}{d\log m^2}D\log S = \Gamma_{cusp}(g). \qquad (27)$$

This relation is fulfilled at arbitrary values of $n^2$ and it is not changed at $n^2 = 0$. Thus, in the light-like axial gauge the contribution of the soft subgraph S satisfies equation (27) although its renormalization properties are drastically changed as compared with the case $n^2 \neq 0$.

Using the explicit expression (19) for the evolution kernel we get at $n^2 = 0$:

$$P_{qq}(z) = P\left(z, \frac{m^2}{\mu^2}, \infty\right) = \delta(1-z)\,D\log(JH)^2$$

$$+ \frac{p_+}{2\pi}\int dy_-\,e^{iy_-p_+(1-z)}\,D\log S\left[(y_-p_+ - i0)^2\frac{\mu^2}{m^2}, \infty\right]. \qquad (28)$$

The evolution kernel $P_{qq}(z)$ does not depend on $m^2/\mu^2$ since after differentiating both the sides of the last equality we have:

$$\frac{d}{d\log m^2} P_{qq}(z) = 2\delta(1-z)\frac{d}{d\log m^2} D\log(J)$$
$$+\frac{P_+}{2\pi}\int dy_- e^{iy_-p_+(1-z)}\frac{d}{d\log m^2} D\log S\left[(y_-p_+-i0)^2\frac{\mu^2}{m^2},\infty\right]$$
$$= -\Gamma_{cusp}(g)\delta(1-z) + \Gamma_{cusp}(g)\delta(1-z) = 0,$$

where the relations (24) and (27) valid at $n^2 = 0$ are substituted.

The evolution kernel $P_{qq}(z)$ as $z \to 1$ satisfies, as it follows from (28), the following equation:

$$(1-z)P_{qq}(z) = \frac{i}{2\pi}\int dy_- e^{iy_-p_+(1-z)}\frac{d}{dy_-} D\log S\left[(y_-p_+-i0)^2\frac{\mu^2}{m^2},\infty\right]$$
$$= -\frac{i}{\pi}\int\frac{dy_-}{y_--i0} e^{iy_-p_+(1-z)}\frac{d}{d\log m^2} D\log S\left[(y_-p_+-i0)^2\frac{\mu^2}{m^2},\infty\right]$$
$$= -\frac{i}{\pi}\int\frac{dy_-}{y_--i0} e^{iy_-p_+(1-z)}\Gamma_{cusp}(g)$$
$$= 2\,\Gamma_{cusp}(g)\theta(1-z).$$

Solving this equation we determine the asymptotic behaviour of $P_{qq}(z)$ as $z \to 1$:

$$P_{qq}(z) = 2\,\Gamma_{cusp}(g)\frac{1}{(1-z)_+} + C_q(g)\,\delta(1-z) + O((1-z)^0). \qquad (29a)$$

The one-loop value of $C_q = \frac{3}{2}\frac{\alpha_s}{\pi}C_F$ is obtained after substituting (10) into (25).

As a consequence of (29a), the anomalous dimension $\gamma_{qq}(N)$ has the following asymptotics at large N:

$$\gamma_{qq}(N) = -2\,\Gamma_{cusp}(g)\log(N e^{\mathbb{C}}) + C_q(g) + o(N^0). \qquad (29b)$$

Comparing equations (3) and (25) we conclude that the quark distribution function $f_{q/q}(x)$ when $x \to 1$ does not mix under renormalization with gluon and antiquark distributions in a quark, and therefore:

$$P_{qg}(z) = O((1-z)^0), \qquad \gamma_{qg}(N) = o(N^0). \qquad (29c)$$

Moreover, the spectral properties (a) of the parton distributions



stated in section 2 imply that

$$P_{\bar q\bar q}(z) = P_{qq}(z). \qquad (29d)$$

All the relations (29) are in agreement with the two-loop calculations (5). To determine the remaining evolution kernels and anomalous dimensions, we have to consider the properties of the gluon distribution in a gluon.

6. The gluon distribution as $x \to 1$

Our study of the gluon distribution in a gluon $f_{g/g}(x)$ for $x \to 1$ is analogous in many respects to the previous analysis of $f_{q/q}(x)$. We again introduce a new function in the axial gauge $(nA^a(x)) = 0$:

$$\Phi(x) = \frac{1}{2\pi x p_+}\int dy_- e^{-ixp_+y_-}$$
$$\langle p|F^b_{+\nu}(0,y_-,0_T)P\exp(ig\int_0^{y_-}ds A^a_+(0,s,0_T)\sigma_a)F^b_{\nu-}(0)|p\rangle, \qquad (30)$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ and $|p\rangle$ is a spin-averaged state of a gluon with a momentum $p_\mu$. In the special case of the light-like axial gauge $A^a_+(x) = 0$ the function $\Phi(x)$ coincides with the gluon distribution in a gluon (2). Since the quark function $\phi(x)$, as it follows from (9) and (10), has the logarithmic dependence on the quark mass m, it is natural to expect for the gluon function $\Phi(x)$ to possess the divergences caused by the zero gluon mass. To regularize the divergences, we introduce "by hand" a fictitious gluon mass $\lambda$ and work in the frame where $p_- = \lambda^2/2p_+ \ll p_+$.

The Feynman diagrams contributing to $\Phi(x)$ as $x \to 1$ differ from the diagrams, fig.1(a),(b), only by the replacement of the external quark lines by the gluon ones /6,14/. They as before contain three types of subgraphs: hard, collinear and soft ones. The definitions of the subgraphs distinguish from (11) only by substitution of the gluon mass $\lambda$ for the quark mass m. Specific features of the function $\Phi(x)$ are revealed when one factorizes the contribution of the soft subgraph into the diagram fig.1(b). In case of the quark distribution all the effects caused by the attachments of soft gluons to the quark collinear subgraphs are accumulated by the path-ordered exponentials $E_{-p}(y,\infty)$ defined in

(15) /14,15/. In the diagrams for the $\Phi(x)$ soft gluons are attached to the gluon collinear subgraphs. Nevertheless, the factorization of the soft subgraph is valid as before /14,15/ but with the factors $E_{-p}(y,\infty)$ being replaced by the factors $\mathcal{E}_{-p}(y,\infty)$ in expression (15). They equal:

$$\mathcal{E}_{-p}(y,\infty) = P\exp(-ig\int_0^\infty ds\, e^{-\varepsilon s}p^\mu A_\mu^a(y-ps)\,\sigma_a),\ \varepsilon\to 0$$

and differ from $E_{-p}(y,\infty)$ only by substitution of the generators $\sigma_a$ of the adjoint representation for the generators $t_a$ of the fundamental representation of SU(3) group. Hence the resulting expression for $\Phi(x)$ as $x\to 1$ with factorized subgraphs has the same form as expression (12). The contribution of the soft subgraph S to $\Phi(x)$ differs from (15) only by the replacement of the representation of the color matrices. That is why all the properties (21) and (27) of the soft subgraph are preserved for $\Phi(x)$ with $\Gamma_{cusp}$ being replaced by $\gamma_{cusp}$ where $\gamma_{cusp}$ is the cusp anomalous dimension of contour functionals in the adjoint representation of SU(3) group. There is simple relation between $\Gamma_{cusp}$ and $\gamma_{cusp}$. To the lowest orders of PT the expression for $\Gamma_{cusp}$ has the following structure /19/:

$$\Gamma_{cusp}(g) = \sum_{n>k} \alpha_s^n\, C_F\, C_A^{n-1-k}\, T_F^k\, a_{nk},$$

where $a_{nk}$ are numerical coefficients. The above-mentioned replacement of the representation may be reduced to the transformation of Casimir operators: $C_F \to C_A$ in the last equation, that is to the lowest orders of PT:

$$\gamma_{cusp}(g) = \frac{C_A}{C_F}\,\Gamma_{cusp}(g)\,. \qquad (31)$$

The contributions to $\Phi(x)$ of the collinear and hard subgraphs possess property (24), with $\Gamma_{cusp}$ being changed by $\gamma_{cusp}$ but their particular values differing from the analogous ones in the quark distribution case.

When $n^2\neq 0$ the function $\Phi(x)$ satisfies the equation similar to (8):

$$D\,\Phi(x) = 2\,\gamma_g(g)\,\Phi(x),$$

where $\gamma_g$ is the anomalous dimension of a gluon field in the axial gauge.



Taking the limit $n^2\to 0$ and taking into account the equality $\Phi(x)=f_{g/g}(x)$ we find with all intermediate steps reproducing equations (25) and (29) that the gluon distribution in a gluon satisfies the evolution equation:

$$D\, f_{g/g}(x) = \int_x^1 \frac{dy}{y}\, f_{g/g}(y)\, P_{gg}\left[\frac{x}{y}\right] + O((1-x)^0).$$

The involved evolution kernels have the following asymptotics as $z\to 1$:

$$P_{gg}(z) = 2\,\gamma_{cusp}(g)\frac{1}{(1-z)_+} + C_g(g)\delta(1-z) + O((1-z)^0)$$

$$P_{gq}(z) = O((1-z)^0) \qquad (32a)$$

and the behaviour of the anomalous dimensions at large N is

$$\gamma_{gg}(N) = -2\,\gamma_{cusp}(g)\,\log(N e^{\mathbf{C}}) + C_g(g) + o(N^0)$$

$$\gamma_{gq}(N) = o(N^0)\,. \qquad (32b)$$

The derived relations (32) are in agreement with the two-loop calculation (5). The constant $C_g$ in equation (32) depends on the contributions of the collinear and hard subgraphs and it is not identical with the analogous constant in (29).

Comparing relations (29) and (32) with the two-loop calculation (5) we obtain: $C_a = -2B_a$, $a = q, g$. In the first terms of the perturbative expansion of $C_q$ and $C_g$ we easily recognize the one-loop values of the anomalous dimensions $\gamma_q$ and $\gamma_g$ of quark and gluon fields in the nonlight-like axial gauge. (The anomalous dimension of a gluon field in the axial gauge is proportional to the beta function of QCD: $\gamma_g = \beta(g)/g$). The coincidence is not accidental and we prove now that there is a deep connection between $C_a$, $\gamma_q$ and $\gamma_g$.

Let us represent equations (21) and (22) and relation derived at the end of section 4 as follows:

$$2\,\Gamma_{cusp}(\gamma,g) = -D\log S = D\log(JH)^2 - 2\gamma_q(g),$$

where S, J and H denote the contributions of the corresponding subgraphs to the quark function $\phi(x)$ as $n^2\neq 0$. An analogous relation may be written for the gluon function $\Phi(x)$ with the same notations:

$$2\,\gamma_{cusp}(\gamma, g) = -\,D \log S = D \log(JH)^2 - 2\gamma_g(g).$$

Both the equations are valid when $n^2 \neq 0$ since in the limit $n^2 \to 0$ the cusp anomalous dimension defined in (23) possesses divergences. The l.h.s. of these equations differ from each other only by replacement of the representation of the gauge group and to the lowest orders of PT they are related by the equation similar to (31)/15/: $C_A \Gamma_{cusp}(\gamma, g) = C_F \gamma_{cusp}(\gamma, g)$. Its use allows us to derive:

$$0 = C_A\, D \log S - C_F\, D \log S = C_A\left[D \log(JH)^2 - 2\gamma_g\right] - C_F\left[D \log(JH)^2 - 2\gamma_q\right].$$

Since the l.h.s. of the equality does not depend on $n^2$, we get from (28) that in the limit $n^2 \to 0$ to the lowest orders of PT:

$$C_A\left[P_{qq}(z) - 2\gamma_q(g)\delta(1-z)\right] = C_F\left[P_{gg}(z) - 2\gamma_g(g)\delta(1-z)\right].$$

Substituting (29) and (32) into this relation we find:

$$C_A\left[C_q(g) - 2\gamma_q(g)\right] = C_F\left[C_g(g) - 2\gamma_g(g)\right]. \qquad (33)$$

To the one-loop order, as it follows from (5), both the sides of (33) equal zero. We cannot verify (33) to the two-loop order since the two-loop value of $\gamma_q$ is unknown. Nevertheless, the validity of this relation may be checked: after substituting $\gamma_g^{(2)}(g) = \left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{17}{24}C_A^2 - \frac{5}{12}C_A T_F - \frac{1}{4}C_F T_F\right]$ into the r.h.s. of (33) the forbidden color weight $C_F^2\, T_F$ appears but its numerical coefficient turns out equal to zero.

## 7. Conclusion

Summarizing the investigation of the parton distributions we state that for $z \to 1$ and $N \gg 1$:

$$P_{ab}(z) = \delta_{ab}\left[2\,\Gamma^{(a)}_{cusp}(g)\,\frac{1}{(1-z)_+} + C_a(g)\delta(1-z)\right] + O((1-z)^0)$$

$$\gamma_{ab}(N) = -\delta_{ab}\left[2\,\Gamma^{(a)}_{cusp}(g)\,\log(N e^{\mathbb{C}}) - C_a(g)\right] + o(N^0).$$



where $\Gamma^{(a)}_{cusp}(g)$ is the cusp anomalous dimension of the contour functionals in the fundamental ($a$="quark") and adjoint ($a$="gluon") representations of SU(3) group, respectively, and to the lowest orders $C_a$ are related to the anomalous dimension of a quark field in the axial gauge and the beta function of QCD by a simple equation (33).

Let us substitute the obtained asymptotics of the evolution kernels into the APL equation and determine the evolution of the parton distributions $f_{a/A}(x)$ in a hadron $A$ when $x \to 1$. It is supposed that the behavour of the parton distributions as $x \to 1$ is:

$$f_{a/A}(x) = (1-x)^{N^{(a)}(\mu)},$$

where the constant $N^{(a)}(\mu)$ cannot be calculated within perturbative QCD. Solving the APL equation with this ansatz we derive the evolution law:

$$D\, N^{(a)}(\mu) = 2\,\Gamma^{(a)}_{cusp}.$$

Since $\Gamma^{(a)}_{cusp}$ is a positive definite function /16/, $N^{(a)}(\mu)$ is an increasing function of the renormalization parameter $\mu$. Thus, when $x \to 1$, the parton distributions tend to zero faster with increasing $\mu$.

In conclusion we notice that the above consideration may be easily generalized to the investigation of the asymptotics of the kernels governing the evolution of the decay (or fragmentation) functions.


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Асимптотика ядер эволюции Алтарелли - Паризи - Липатова для партонных распределений

Исследуется асимптотика ядер эволюции $P_{ab}(z)$, $a = q, \bar{q}, g$ функций распределения партонов в адроне при $z \to 1$. Доказано, что во всех порядках пертурбативной КХД ядра эволюции имеют однопетлевую асимптотику, определяемую угловой аномальной размерностью контурных функционалов в фундаментальном и присоединенном представлениях калибровочной группы. Получено уравнение, связывающее асимптотику ядер эволюции в низших порядках теории возмущения.

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Korchemsky G.P. E2-88-717

Asymptotics of the Altarelli - Parisi - Lipatov Evolution Kernels of Parton Distributions

The asymptotics of the evolution kernels $P_{ab}(z)$ of parton distributions is investigated as $z \to 1$. It is proved that to all orders of perturbative QCD the evolution kernels have one-loop asymptotics determined by the cusp anomalous dimensions of the contour functionals in the fundamental and abjoint representations of gauge group. A simple equation is found connecting the asymptotics of the evolution kernels.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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