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GREEN FUNCTION OF A THREE-DIMENSIONAL WICK PROBLEM

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1. Introduction

It has long been known that the degeneracy of the bound states in the nonrelativistic Coulomb problem can be described by a four-dimensional orthogonal symmetry group. The same approach was used by Schwinger (1964) to obtain an explicit construction for the Green function of this problem.

Recently (Matveev, Slepchenko and Vardiashvili, 1988), an exact solution of a three-dimensional coulombic Wick-Cutkosky problem (Wick, 1954, Cutkosky, 1954) has been obtained which possesses the hidden O(4)-symmetry. Here we shall give the derivation of the corresponding Green function and consider its connection with the asymptotic behaviour of the scattering amplitude.

2. The Green function

Let us consider the elastic scattering of the two scalars with the mass m in the ladder approximation. The exchanged particles are assumed scalar and massless. The corresponding Bethe-Salpeter equation for the Green function is

$$(m^2-v)(m^2-\omega)f(v,\omega,t)=-\frac{1}{(p-p')^2}+\frac{\lambda}{\pi^2}\int\frac{f(v'',\omega'',t)}{-(p-p'')^2}\,d^4p'' \quad (1)$$

We consider the case of the t-channel elastic forward scattering $p_\mu=(\vec{p},E)=0$

$$(m^2-p^2)^2 f(p,t)=-\frac{1}{(p-p')^2}+\frac{\lambda}{\pi^2}\int\frac{f(p'',t)}{-(p-p'')^2}\,d^4p'' \quad (2)$$

The corresponding three-dimensional quasipotential equation (Logunov, Tavkhelidze, 1963) for the off-shell scattering amplitude can be written in the form

$$T(\vec{p},\vec{p}')=-\tilde{V}(\vec{p},\vec{p}';E)-\frac{i}{2\pi}\int d\vec{q}\,\tilde{V}(\vec{p},\vec{q};E)\tilde{F}(\vec{q})T(\vec{p}',\vec{q}), \quad (3)$$

where

$$\tilde{F}(\vec{q},E)=\frac{2\pi i}{W(\vec{q})}\left[E^2-4W^2(\vec{q})\right]^{-1}, \quad W(\vec{q})=\sqrt{\vec{q}^{\,2}+m^2}$$

and coulombian quasipotential in the zero-energy E=0 case has the form (Matveev, Slepchenko and Vardiashvili, 1988)

$$\tilde{V}(\vec{p},\vec{p}';0)=\lambda\pi^{-1}W^{-1}(\vec{p})\,|\vec{p}-\vec{p}'|^{-1}. \quad (4)$$

Thus

$$(\vec{p}^{\,2}+m^2)^2\, g(\vec{p},\vec{p}')=\frac{\lambda}{\pi|\vec{p}-\vec{p}'|}+\frac{\lambda}{4\pi}\int\frac{d\vec{q}\; g(\vec{q},\vec{p}')}{|\vec{p}-\vec{q}|}, \quad (5)$$

here

$$g(\vec{q},\vec{p})=T(\vec{q},\vec{p})(\vec{q}^{\,2}+m^2)^{-3/2}.$$

This equation is equivalent to the following differential equation:

$$\nabla^2_{\vec{p}}\{(\vec{p}^{\,2}+m^2)g(\vec{p},\vec{p}')\}+\lambda g(\vec{p},\vec{p}')=-4\lambda\delta(\vec{p}-\vec{p}'). \quad (6)$$

We use now the explicit form of the exact solution for the corresponding homogeneous equation (Matveev, Slepchenko and Vardiashvili, 1988)

$$\psi_{nlm}=u(\vec{p})Z_{nlm}(\vec{p}), \quad Z_{nlm}=P_{nl}(p)Y_{lm}(\theta,\varphi), \quad (7)$$

$$u(\vec{p})=(\vec{p}^{\,2}+m^2)^{-5/2},$$

$$P_{n\ell} = \sqrt{\frac{n\,\Gamma(n+\ell+1)}{\Gamma(n-\ell)}}\;\frac{\sqrt{m^2+\vec{p}^{\,2}}}{\sqrt{2p}}\;P_{n-1/2}^{-(\ell+1/2)}\left(\frac{m^2-\vec{p}^{\,2}}{m^2+\vec{p}^{\,2}}\right).$$

Then, for the Green function

$$G(\vec{p},\vec{p}\,';\lambda) = \sum_{n\ell m} \frac{Z_{n\ell m}(\vec{p})\, Z^{*}_{n\ell m}(\vec{p}\,')}{(Z,KZ)_{n\ell m}(\lambda-\lambda_n)}$$

after applying the addition theorem

$$\frac{\sqrt{2\pi}}{n}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Z^{*}_{n\ell m}(\beta',\theta',\varphi')\, Z_{n\ell m}(\beta'',\theta'',\varphi'') = Z_{n00}(\gamma)$$

$$Z_{n00}(\gamma) = \frac{1}{\sqrt{2\pi^2}}\;\frac{\sin n\beta}{\sin\beta}$$

$$\cos\gamma = \cos\beta'\cos\beta'' + \sin\beta'\sin\beta''\cos\Theta$$

$$\cos\Theta = \cos\theta'\cos\theta'' + \sin\theta'\sin\theta''\cos(\varphi'-\varphi'')$$

we obtain

$$G(\vec{p},\vec{p}\,';\lambda) = u(p)\,u(p')\sum_{n=1}^{\infty}\frac{n}{2\pi^2}\;\frac{1}{\lambda-\lambda_n}\;\frac{\sin n\psi}{\sin\psi} \qquad (8)$$

$$\cos\psi = 1 - \frac{2(\vec{p}-\vec{p}\,')^2}{(m^2+\vec{p}^{\,2})(m^2+\vec{p}\,'^2)}\,.$$

Thus, instead of the usual expansion in the angular momentum, our Green function is expanded in the principal quantum number n . This fact gives a possibility of investigating the scattering amplitude in the complex n - plane and asymptotic behaviour in the 3-momentum transfer and to get information about Regge poles in the Wick problem.

3. Asymptotic behaviour of the scattering amplitude

Rewrite now the Green function (8) in the form of the off-shell scattering amplitude

$$G(\vec{p},\vec{p}\,';\lambda) = g_0(\vec{p})\,A(\cos\psi,\lambda)\,g_0(\vec{p}\,')$$

$$A = \frac{1}{a(p)\,a(p')}\;\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{n}{\lambda+1-4n^2}\;\frac{\sin n\psi}{\sin\psi} \qquad (9)$$

$$g_0(\vec{p}) = a(p)^{-3/2} = (\vec{p}^{\,2}+m^2)^{-3/2}\,.$$

Transition in (9) on the mass shell must be done carefully as the scattering amplitude is divergent there. This divergence is of the infrared nature and is induced by the distortion of incident and reflected waves in the long-range Coulomb potential of the Wick problem.

In quantum electrodynamics, infrared divergences are connected with the emission of soft virtual photons and are cancelled when the real photons and soft ones are taken into account. This situation takes place in any order of perturbation theory and in our case of scalar photons as well. All this exists in the Bethe-Salpeter formulation of the Coulomb problem (Okubo and Feldman,1960, Nakanishi,1964a). There are some difficulties in the last case as the Bethe-Salpeter equation does not correspond to the conventional perturbation theory. Thus, we must take into account real soft "photons" nonperturbatively.

Let us consider the possibility of the low energy cut off of virtual photons. For this purpose we must derive the convenient integral representation. By using the equation

$$\sum_{n=1}^{\infty}\frac{n}{\lambda+1-4n^2}\;\frac{\sin n\psi}{\sin\psi} = \frac{(-)}{8}\;\frac{1}{2i\sin\psi}\sum_{n=1}^{\infty}\left(\frac{x_+^n}{n+\nu}+\frac{x_+^n}{n-\nu}-\frac{x_-^n}{n-\nu}-\frac{x_-^n}{n+\nu}\right), \qquad (10)$$

where $x_{\pm} = e^{\pm i\psi}$, $\nu = \frac{\sqrt{\lambda+1}}{2}$ and the integral

$$\sum_{N=1}^{\infty}\frac{x^N}{N-a}=\int_0^1 y^{-a-1}\,\frac{xy}{1-xy}\,dy$$

we may write the off-mass-shell amplitude

$$A(\psi;\lambda)=-\frac{1}{2\pi^2}\cdot\frac{1}{a(p)a(p')}\cdot\frac{1}{8}\int_0^1\frac{y^{\nu}+y^{-\nu}}{1-2y\cos\psi+y^2}\,dy\,. \qquad (11)$$

Expression (11) is actually the Mellin transformation with transform $(1+y^2-2y\cos\psi)^{-1}$, or as the $\cos\psi$ has the form (8) we may write

$$A(\psi;\lambda)=\frac{(-)}{16\pi^2}\int_0^1\frac{y^{\nu}+y^{-\nu}}{(1-y)^2a(p)a(p')+4ty}\,dy \qquad (12)$$

$$t=(p-p')^2.$$

Let us consider the amplitude (12) for the large momentum transfers of t (when p^2 and p'^2 are fixed). Changing $4ty = z$ we may in this limit extend the area of integration to infinity in (12)

$$\frac{1}{4t}\int_0^{\infty}\frac{\left(\frac{z}{4t}\right)^{\nu}+\left(\frac{z}{4t}\right)^{-\nu}}{a(p)a(p')+z}=\left(\frac{1}{4t}\right)^{-\nu+1}\int_0^{\infty}\frac{z^{-\nu}\,dz}{a(p)a(p')+z}+O\left(\frac{1}{t^{\nu}}\right)$$

and if we use the integral

$$\int_0^{\infty}\frac{y^{\alpha}dy}{y-x-i\varepsilon}=2\pi i\,\frac{x^{\alpha}}{1-e^{2\pi i\alpha}}=-\frac{\pi e^{-i\pi\alpha}x^{\alpha}}{\sin\pi\alpha}$$

we shall have

$$A(t;\lambda)_{t\to\infty}=\frac{1}{16\pi^2}\,\frac{\pi e^{i\pi\frac{\sqrt{\lambda+1}}{2}}}{\sin\pi\frac{\sqrt{\lambda+1}}{2}}\left(\frac{1}{4t}\right)^{-\nu+1}\left|\frac{1}{a(p)a(p')}\right|^{\nu}+O(t^{-\nu})\,. \qquad (13)$$

So we have shown that all infinities connected with the transition on the mass shell may be avoided by the low energy cutoff in the integral

$$\int_0^{\infty}\frac{z^{-\nu}\,dz}{a(p)a(p')+z}\longrightarrow\int_{\delta}^{\infty}\frac{z^{-\nu}dz}{z+a(p)a(p')}\,.$$

4. Regge poles of the Wick-Cutkosky problem

In the framework of the nonrelativistic potential theory Regge has established the exact connection between the asymptotic behaviour of the scattering amplitude in the crossing channel and the solutions of the bound state Schrödinger equation (Regge 1959, 1960). Further, in (Nakanishi, 1964b) it turned out that the normal solutions of the Bethe-Salpeter equation are connected with the Green function asymptotics in the crossing channel. We shall consider the standard way of investigating the Regge asymptotics of the scattering amplitude using the Sommerfeld-Watson transformation. Instead of the traditional expansion of the amplitude in the Legendre polynomials we shall use more a convenient for our purpose expansion in the Tchebysheff polynomials in principal quantum number n .

Let us write the scattering amplitude in the form

$$A(t,\lambda)=\frac{1}{a(p)a(p')}\,\frac{1}{2\pi^2}\,\tilde{A}(t,\lambda), \qquad (14)$$

$$\tilde{A}(t,\lambda)=\sum_{n=1}^{\infty}n f_n(\lambda)\,U_n(\cos\psi)\quad,\qquad U_n(\cos\psi)=\frac{\sin n\psi}{\sin\psi}\,.$$

The analytic continuation of $f_n(\lambda)$ in the area of complex n allows us to write the amplitude as the following integral:

$$\tilde{A}(t,\lambda)=\frac{1}{2i}\oint_C\frac{n\,dn}{\sin\pi n}\,f(n,\lambda)\,U_n[\cos(\pi-\psi)]\,. \qquad (15)$$

where the contour C rounds the poles of the function $\sin^{-1}\pi n$. For getting the asymptotics of (15) in the direct (t) channel, as $t \to -\infty$, we must perform the analytic continuation from cross-channel $|\cos\psi| < 1$ on the area $|\cos\psi| > 1$.

As $\sin^{-1}\pi n \sim e^{-\pi|\eta|}$, $U_n[\cos(\pi-\psi)] \sim e^{|\pi-\psi||\eta|}$, $n = \xi + i\eta$, the integral exists in the nonphysical area $0 < \psi < \pi$. Now, the integral on the circle for a big radius tends to zero and the integral on the imaginary axis is convergent. We may consider analogously the convergence of expansion of the amplitude, which is continued in the physical area $|\cos\psi| > 1$, taking into account the relations

$$\psi = \pi + i\tau \, , \quad U_n[\cos(\pi-\psi)] \sim \frac{e^{\xi\tau} e^{i\tau\eta} - e^{-\xi\tau} e^{-i\tau\eta}}{e^{\tau} - e^{-\tau}} .$$

The behaviour for $\tau \to \infty$ depends on ξ and is minimal when $\xi = 0$. As $\mathrm{Re}\,\psi = \pi > 0$ there will be no problem for convergence and we shall recieve

$$\tilde{A}(t,\lambda) = \frac{i}{2}\int_{-i\infty}^{+i\infty} \frac{n\,dn}{\sin\pi n} f(n,\lambda) U_n[\cos(\pi-\psi)] + \frac{1}{2}\sum_i \frac{\mathrm{Res}\, f(\lambda)}{\sin\pi\alpha_i} e^{i\pi\alpha_i} U_{\alpha_i}[\cos(\pi-\psi)], \quad (16)$$

when $|\cos\psi| > 1$. As $U_n(z) \underset{z\to\infty}{\sim} 2^n z^{n-1}$, $\mathrm{Re}\, n > 0$ integral on the imaginary axis as $t \to -\infty$ is of an order of z^{-1} and finally we get the set of poles

$$\tilde{A}(t,\lambda) \sim \frac{i}{16\pi^2} \sum_i \frac{e^{i\pi\alpha_i}}{\sin\pi\alpha_i} \beta_i(p)\, \beta_i(p') (-t)^{\alpha_i - 1}$$

$$\beta_i(p) = \left[\frac{2}{a(p)}\right]^{\alpha_i} , \quad \alpha_i = \frac{\sqrt{\lambda+1}}{2} \qquad (17)$$

for the values λ = 3, 15, 35,..., i.e. for the constants of the corresponding bound state solutions of the homogeneous equation.

The correct asymptotic states for the long-range Coulomb potential are the waves in the form of the packet. According to (Finkelstein and Levy, 1967)

$$\left[\frac{1}{a(p)\,a(p')}\right]^{b} \xrightarrow[a(p),\,a(p') \to 0]{} \left[\frac{1}{\Gamma(1-\beta)}\right]^{2} (-)^{1+b}$$

and

$$\sin^{-1}\pi\alpha = \Gamma(1-\alpha_i)\Gamma(1+\alpha_i)$$

we may finally write the renormalized asymptotic of the scattering amplitude as $t \to -\infty$

$$\tilde{A}(t,\lambda) = \frac{i}{(4\pi)^2} \frac{1}{t} \sum_i \frac{\Gamma(1+\alpha_i)}{\Gamma(1-\alpha_i)} e^{i\pi\alpha_i} (t)^{\alpha_i} + O(1/t^{\alpha_i}) \qquad (18)$$

which has the set of daughter poles in the ℓ plane $\ell = \alpha_i - 1$, $\alpha_i - 2, \ldots$, when $\alpha_i = n$, n = 1, 2, 3, ...

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Дан вывод точной функции Грина трехмерной задачи Вика - Куткоского, обладающий скрытой O(4) симметрией, а также рассмотрена связь решения задачи на связанные состояния с асимптотическим поведением соответствующей амплитуды рассеяния.

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