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DYNAMICAL SYMMETRY OF A THREE-DIMENSIONAL WICK – CUTKOSKY PROBLEM

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1. Introduction

It is well known that the bound state problem of two scalar particles interacting via the scalar "photons" is exactly solved in the framework of the Bethe-Salpeter equation in the ladder approximation. This problem has exactly been solved by Wick [1], and it turned out that the solution of the zero energy equation possesses the higher dynamical O(5) symmetry. Cutkosky [1] considered the case of non-zero angular momentum and obtained the complete set of solutions.

On the other hand, the comparison of spectra of the quasipotential equation of Logunov-Tavkhelidze [2] with the Bethe-Salpeter ones shows that the first gives a good description of relativistic effects. At the same time, it does not suffer from shortcomings characteristic of the Bethe-Salpeter equation: the existence of the "relative" time, the absence of probability interpretation and so on.

Hence it is interesting to look for exactly solvable models in a three-dimensional quantum field theory and to study higher symmetry properties of solutions corresponding to quasipotential equations.

In this paper we investigate the solutions of a three-dimensional Wick-Cutkosky problem and obtain the exact solution of this problem for the Coulomb quasipotential possessing the hidden O(4)-symmetry.

...ТУТ ...ДОВАНИЙ БИБЛИОТЕКА

2. Dynamical symmetry of the Wick problem

The Bethe-Salpeter equation for two equal mass scalar particles exchanging massless and scalar "photon" in the centre of mass system (c.m.s.) with zero total energy has the form

$$(p^2+m^2)^2 \chi(p) = \frac{\lambda}{\pi^2} \int \frac{\chi(k)\, d^4k}{(p-k)^2}, \qquad (1)$$

where p is a relative 4-momentum in the c.m.s. and Wick's rotation is performed.

After mapping a four-dimensional momentum space upon the surface of a five-dimensional sphere by a stereographic projection, we derive the O(5) invariant equation for a five-dimensional spherical harmonics [3]

$$H(\alpha) = \frac{\lambda}{16\pi^2} \int \frac{d\Omega_5'\, H(\alpha')}{\sin^2 \theta/2}, \qquad (2)$$

$$H(\alpha) = \sec^6 \frac{\alpha}{2}\, \chi(\alpha).$$

Equation (2) is invariant under all the rotations of 5-sphere, $\lambda_N = N(N+1)$ and the solution may be presented as a five-dimensional spherical harmonics

$$\chi^m_{n\ell} = \int_{-1}^{1} \mathcal{G}_n(z)\, dz \frac{Y^m_\ell(\vec{p})\, C^{\ell+1}_{n-\ell-1}\left(\frac{X}{R}\right) R^{n-\ell-1}}{(m^2+p^2)^{2+n}}, \qquad (3)$$

where $R^2 = (p^2+m^2)^2 + 4p_0^2$, $X = m^2 - p^2$.

The functions $\mathcal{G}_n(z)$ of the form [1]

$$\mathcal{G}_n(z) = (1-z^2)^n\, C_k^{n+1/2}(z)$$

satisfy the differential equation of the Gegenbauer polynomials and have the following eigenvalues $\lambda = (n+k)(n+k+1)$, where k is a new quantum number which is set to zero for the right correspondence with nonrelativistic case.

3. Solution of the quasipotential equation for the Wick problem

Let us write the quasipotential equation for the bound state of two scalar particles in the c.m.s. [2]

$$\left[E^2 - 4(\vec{p}^{\,2} \mp m^2)\right] \Psi(\vec{p}) = \frac{1}{\sqrt{\vec{p}^{\,2}+m^2}} \int \Psi(\vec{k})\, V(\vec{p}-\vec{k})\, d^3k, \qquad (4)$$

where $\vec{p}$ and $\vec{k}$ are three-dimensional relative momenta and $\Psi(\vec{p})$ is the quasipotential wave function of the bound state with the following normalization condition*:

$$\int \Psi^*(\vec{p})\, \tilde{F}_0^{-1}\, \Psi(\vec{p})\, d^3p = 1. \qquad (5)$$

$\tilde{F}_0$ is the operator of the free Green function of the quasipotential equation.

There is connection between the quasipotential and the Bethe-Salpeter wave functions

$$\Psi(\vec{p}) = \int_{-\infty}^{+\infty} dp_0\, \chi(\vec{p}, p_0) \qquad (6)$$

Let us apply the quasipotential equation (4) to the Wick-Cutkosky problem. The choice of a suitable quasipotential is important.

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* $\Psi(\vec{p})$ is an eigenfunction of the coupling constant problem at fixed energy [4] .

We may derive the quasipotential for the relativistic interaction by the presciption given in [2]. It is possible to restore potential from quantum field theory by the corresponding Feynman diagrams.

Since the Wick model is a problem of searching for the zero energy bound states, if the Coulomb interaction exists, we consider the standard procedure allowing derivation of the potential from quantum field theory.

We should like to remark that an important feature of the Coulomb potential in the momentum space is the dependence of the decreasing power on the dimension of the space

$$\frac{1}{r}=(\pi\omega_{N-1})^{-1}\int\frac{d^Nq\,e^{i\vec{q}\vec{r}}}{|q|^{N-1}}\ ,\qquad \omega_N=\frac{2\pi^{N/2}}{\Gamma(N/2)}\ .$$

The kernel of the potential $V(\vec{p}-\vec{k})=|\vec{p}-\vec{k}|^{N-1}$ in a $(N+1)$ -dimensional space is the Green function of the Laplace operator

$$\Delta_u^{N+1}\left(|\vec{u}-\vec{v}|^{N-1}\right)^{-1}=-(N+1)\omega_{N+1}\delta^{N+1}(\vec{u}-\vec{v})\ .$$

It turned out that the quasipotential for spinless, equal mass particles, interacting through a massless scalar particle, evaluated from the second order of perturbation theory in the framework of quantum electrodynamics [5] has the following form in the zero energy limit:

$$\underset{E\to 0}{V(\vec{p}-\vec{k})}=\frac{1}{\sqrt{\vec{p}^2+m^2}}\cdot\frac{1}{|\vec{p}-\vec{k}|}\ . \qquad (7)$$

If we take into account (7), (4) becomes analogous to a three-dimensional Bethe-Salpeter equation of the Wick problem, and we shall show below that it describes Wick model in terms of quasipotential

$$(\vec{p}^2+m^2)^2\,\psi(\vec{p})=\frac{\lambda}{4\pi}\int\frac{\psi(\vec{k})\,d^3k}{|\vec{p}-\vec{k}|}\ . \qquad (8)$$

For the case of the zero angular momentum equation (8) is equivalent to the following differential equation:

$$\frac{1}{p}\frac{d^2}{dp^2}\left[p(p^2+m^2)^2\,\psi(p)\right]+\lambda\psi(p)=0 \qquad (9)$$

with the boundary conditions

$$p\,\psi(p)\to 0\ ,\quad p\to 0$$

$$p(p^2+m^2)\psi(p)\to \mathrm{const}\ ,\quad p\to\infty\ .$$

The ground state solution (n=1) of the s-wave problem $\psi_{n=1}(\vec{p})=C(\vec{p}^2+m^2)^{-5/2}$ coincides with the corresponding integrated solution of the Bethe-Salpeter equation (3). In the configuration space it takes the form

$$\psi_{n=1}(r)=\frac{4\pi}{3}\left(\frac{r}{m}\right)K_1(mr)\ .$$

Owing to the well-known features of the modified first kind Bessel function $K_1(z)$, this solution has correct behaviour at zero and characteristic asymptotics for the bound state wave function.

If we consider the case of non-zero angular momentum, we deal with the equation

$$\nabla_{\vec{p}}^2\left\{(\vec{p}^2+m^2)^2\,\psi(\vec{p})\right\}+\lambda\psi(\vec{p})=0\ ,$$

where the Laplace operator contains the angular variables

$$\nabla_{\vec{p}}^2=\nabla_p^2+\frac{1}{p^2}\nabla_{\theta,\varphi}^2\ .$$

If we suppose that a solution of the equation has the form $\Psi(\vec{p}) = R_{n\ell}(p)\cdot Y_{\ell m}(\theta,\varphi)$, we get the following radial equation:

$$\frac{1}{p^2}\frac{d}{dp}\left[p^2\frac{d}{dp}(p^2+m^2)^2 R_{n\ell}(p)\right]+\left[\lambda-\frac{\ell(\ell+1)}{p^2}(p^2+m^2)^2\right]R_{n\ell}(p)=0$$

which after substitution of the variables $p^2=u$, $R=u^{-1/4}(u+m^2)^{-2}\Phi(u)$ takes the form

$$\left[\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\frac{\lambda}{4u(u+m^2)^2}+\frac{1/4-\ell(\ell+1)}{4u^2}\right]\Phi_{n\ell}(u)=0 \qquad (10)$$

of differential equations of adjoined spherical functions $P_\mu^\nu(z)$ [6].

Thus, eigenfunctions and eigenvalues of the equation (8) are

$$R_{n\ell}(p)=\frac{1}{\sqrt{p}\,(p^2+m^2)^2}\,P_{n-1/2}^{-(\ell+1/2)}\left(\frac{m^2-p^2}{m^2+p^2}\right)$$

$$\lambda=4n^2-1\,,\qquad n=1,2,\ldots \qquad (11)$$

4. Dynamical symmetry of a three-dimensional Wick problem

The Bethe-Salpeter equation for two spinless particles in the case of the zero total energy possesses O(5) dynamical symmetry. This circumstance allows us to find a complete set of solutions that are a multiple of a five-dimensional spherical harmonics. The Fock stereographic projection method [3] applied with the quasipotential equation (8) gives the solution (10).

For the purpose of confirming the above-mentioned, let us introduce the stereographic coordinates of the surface of a unit 4-sphere

$$\vec{\xi}=\frac{2m\vec{p}}{m^2+\vec{p}^2}\,,\qquad \xi_4=\frac{m^2-\vec{p}^2}{m^2+\vec{p}^2}\,,\qquad \xi^2=\vec{\xi}^2+\xi_4^2=1$$

or (12)

$$\xi_1=\sin\alpha\sin\theta\sin\varphi \qquad \frac{|\vec{p}|}{m}=\operatorname{tg}\frac{\alpha}{2}$$

$$\xi_2=\sin\alpha\sin\theta\cos\varphi \qquad \frac{|\vec{p}|}{\sqrt{m^2+\vec{p}^2}}=\sin\frac{\alpha}{2}$$

$$\xi_3=\sin\alpha\cos\theta$$

$$\xi_4=\cos\alpha \qquad \frac{m}{\sqrt{m^2+\vec{p}^2}}=\cos\frac{\alpha}{2}$$

The connection between the volume element in the new coordinates and in the four-dimensional Euclidean space is given by

$$d^3k=\left(\frac{m}{2}\sec^2\frac{\alpha}{2}\right)^3 d^4\Omega$$

$$d^4\Omega=\frac{d\vec{\xi}}{\xi_4}=\sin^2\alpha\sin\theta\,d\alpha\,d\theta\,d\varphi$$

and equation (8) takes the form

$$H(\alpha)=\lambda_n\int\frac{H(\alpha')\,d^4\Omega'}{\sin\chi/2}\,,\qquad \lambda_n=\frac{\sqrt{2}}{32\pi}\lambda\,, \qquad (13)$$

where $H(\alpha)=\cos^5\frac{\alpha}{2}\,\psi(\alpha)$ and

$$\cos\chi=\cos\alpha\cos\alpha'+\sin\alpha\sin\alpha'\cos\gamma$$

$$\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi')$$

is cosine of an angle between the vectors on the surface of a 4-sphere.

Equation (13) is now written in an obviously O(4) symmetric form. If we exploit the Funk-Hecke theorem [7], we may derive the eigenfunctions

$$H(\alpha,\theta,\varphi) = P_{n\ell}(\alpha)\, Y_{\ell m}(\theta,\varphi)$$

$$P_{n\ell}(\alpha) = \frac{1}{\sqrt{\sin\alpha}}\, P^{-(\ell+1/2)}_{n-1/2}(\cos\alpha), \quad n=1,2,\ldots \tag{14}$$

and eigenvalues

$$\lambda_n^{-1} = \frac{4\pi}{n+1}\int_{-1}^{+1} \frac{dx}{\sqrt{1-x}}\, C_n^1(x)\,(1-x^2)^{1/2}$$

$$= \frac{16\,\pi\,\sqrt{2}}{4(n+1)^2-1}\,.$$

Thus, the solution of equation (8) takes the form

$$\Psi_{n\ell}(\vec{p}) = \frac{1}{(m^2+p^2)^{5/4}}\,\frac{\sqrt{m^2+p^2}}{\sqrt{2p}}\, P^{-(\ell+1/2)}_{n-1/2}\left(\frac{m^2-p^2}{m^2+p^2}\right) Y_{\ell m}(\theta,\varphi) \tag{15}$$

which is in complete accordance with the result derived in the previous section.

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Appendix

We consider here the connection of the general solution of three-dimensional Wick problem with the corresponding solution of the Bethe-Salpeter equation.

For the 1s-wave ($n=1$, $\ell=0$, $\lambda=3$) we have

$$R_{10}(p) = \frac{1}{\sqrt{p}\,(p^2+m^2)^2}\, P^{-1/2}_{1/2}\left(\frac{m^2-p^2}{m^2+p^2}\right) = \frac{2}{\sqrt{\pi}}\,\frac{1}{(m^2+p^2)^{5/2}}$$

Analogously, using $P_\nu^{-\nu}(\cos\alpha) = \frac{2^{-\nu}}{\Gamma(\nu+1)}(\sin\alpha)^{\nu/2}$ we get for the 2p, 2s, 3d-waves

$$R_{21} = \frac{4}{3\pi^{1/2}}\,\frac{|\vec{p}|}{(m^2+\vec{p}^{\,2})^{7/2}} \qquad R_{20} = \frac{2}{\sqrt{\pi}}\,\frac{m^2-\vec{p}^{\,2}}{(m^2+\vec{p}^{\,2})^{7/2}}$$

$$R_{32} = \frac{8}{15\sqrt{\pi}}\,\frac{|\vec{p}|^2}{(m^2+\vec{p}^{\,2})^{9/2}}$$

and in general ($\ell=n-1$)

$$R_{n,n-1}(p) = \frac{2^n}{\sqrt{\pi}(2n-1)!!}\,\frac{|\vec{p}|^{n-1}}{(m^2+\vec{p}^{\,2})^{n+3/2}}\,. \tag{A.1}$$

On the other hand, eigenfunctions of the Bethe-Salpeter equation and functions integrated over energy have the form

$$\chi_{10}(p) = \frac{4}{3}\,\frac{1}{(p^2+m^2)^3}\,, \qquad \Psi_{10}(\vec{p}) = \frac{\pi}{2}\,\frac{1}{(m^2+\vec{p}^{\,2})^{5/2}}\,.$$

If we take into account that $C_1^1(z) = 2z$ we receive for the 2p, 2s, 3d-waves

$$\chi_{21}(p) = \frac{16}{15}\,\frac{|p|}{(m^2+p^2)^4} \qquad \Psi_{21}(\vec{p}) = \frac{\pi}{3}\,\frac{|\vec{p}|}{(m^2+\vec{p}^{\,2})^{7/2}}$$

$$\chi_{20}(p) = \frac{32}{15}\,\frac{m^2-p^2}{(m^2+p^2)^4} \qquad \Psi_{20}(\vec{p}) = 16\pi^2\,\frac{m^2-\vec{p}^{\,2}}{(m^2+\vec{p}^{\,2})^{7/2}}$$

$$\chi_{32}(p) = \frac{32}{35}\,\frac{|p|^2}{(m^2+p^2)^5} \qquad \Psi_{32}(\vec{p}) = \frac{\pi}{4}\,\frac{|\vec{p}|^2}{(m^2+\vec{p}^{\,2})^{9/2}}$$

and in general $n=\ell+1$

$$\chi_{n,\,n-1}(P) = N\,\frac{|P|^{n-1}}{(m^2+p^2)^{n+2}} \tag{A.2}$$

$$\psi_{n,\,n-1}(\vec{P}) = \frac{\pi}{n+1}\,\frac{|\vec{P}|^{n-1}}{(m^2+\vec{P}^2)^{n+3/2}}\,. \tag{A.3}$$

The comparison of expressions (A.1) and (A.3) completes the search for connection between them.

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Динамическая симметрия трехмерной задачи Вика – Куткоского

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Предлагается исследование решения трехмерной задачи Вика – Куткоского и получено точное решение этой задачи для кулоновского квазипотенциала, обладающее скрытой O(4)-симметрией. Рассмотрена связь общего решения трехмерной задачи Вика и соответствующего решения уравнения Бете – Солпитера.

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Dynamical Symmetry of a Three-Dimensional Wick – Cutkosky Problem

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The solutions of a three-dimensional Wick – Cutkosky problem are investigated and the exact solution of this problem for the Coulomb quasipotential possessing the hidden O(4)-symmetry is obtained. The connection of the general solution of three-dimensional Wick problem with the corresponding solution of the Bethe – Salpeter equation is considered.

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