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QUANTUM MECHANICS IN RIEMANNIAN SPACE-TIME.

General Covariant Schrödinger Equation with Relativistic Corrections

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1. Introduction

Generalization of (nonrelativistic) quantum mechanics of a particle to the general Riemannian space-time $V_{1,3}$ with the metric tensor $g_{\alpha\beta}(x)$ seems to be interesting from different points of view. Firstly, this theory is of applied interest for investigations of quantum effects due to an external gravitational field and noninertiality of motion, particularly, in astrophysics or gravitational wave experiments. Methodical aspect of the problem is of interest too because formulation of quantum mechanics on a wider geometrical basis can contribute to a deeper insight to the theory and to finding out its connections with theories of a higher level. However, such a generalization could hardly be done immediately on the nonrelativistic level. It seems more reliable to extract the nonrelativistic content of a general relativistic (i.e. generally covariant and based on $V_{1,3}$) structure which may conditionally be called the general relativistic quantum mechanics (GRQM). The nonrelativistic quantum mechanics (NRQM) with relativistic corrections will be obtained here as an asymptotics of the GRQM for a small parameter proportional to c^{-2} , e.g. the inverse Compton frequency $\omega_c^{-1} = \hbar/2mc^2$, c and m being the velocity of light and the mass of the particle, respectively. In fact, the asymptotical expansions will prove to be powers of the ratios of the energies of slow motions in the system to the rest energy of the particle. Of course, one may speak about the slowness of motion only implying a frame of reference (FR) which is necessarily noninertial in the framework of General Relativity, see Sec. 2. So, the term "nonrelativistic" is used here in the sense of slowness of motions in an appropriate FR.

The most important use of the NRQM thus constructed may turn out to be in quantum theory of the field φ in $V_{1,3}$ (with further "application" to particle creation processes in the early Universe): the one-particle wave functions of the NRQM point out the Fock representation of the canonical commutator relations of quantum field, see the discussion of this problem in /1/, pp. 184-187. So the notion of a particle will depend on the choice of FR, which corresponds to the point of view widely adopted now /2,3/.

The problem of general construction of the NRQM in $V_{1,3}$ has received little attention. The most complete results and references may be found in the book by A.C.Gorbatzevich /4/ where the NRQM in $V_{1,3}$ for spin 1/2 particle is formulated as a modification of the standard scheme of quantum mechanics with a priori Hilbert structure of the state space. Nevertheless, it has proved necessary to appear to the general relativistic Dirac equation for obtaining a Hamiltonian. Except for this point, one might call Gorbatzevich's approach an inductive one while ours is completely deductive in this sense. Comparison of results of these two approaches is of independent interest. Moreover, consideration of the bosonic case is important for the mentioned applications to quantum processes in the early Universe.

It seems to be reasonable to follow the physical level of rigour in the paper which is aimed at working out an approach to the problem of construction of the NRQM in V ; so it is suggested that the functions under consideration have the properties which are necessary for either assertion.

2. Transition to the nonrelativistic asymptotics

As the original GRQM we shall consider the formal structure consisting of

1) <u>the Klein-Gordon-Fock equation</u> for the complex scalar field in $V_{1,3}$ with a metric tensor $g_{\alpha\beta}(x)$, $x \in V_{1,3}$ and in an external electromagnetic field $A_\alpha(x)$ *)

$$g^{\alpha\beta}\tilde{\nabla}_\alpha \partial_\beta \varphi + \left(\frac{mc}{\hbar}\right)^2 \varphi = 0 , \quad \alpha, \beta, \gamma, \ldots = 0, 1, 2, 3 \tag{2.1}$$

$\tilde{\nabla}_\alpha$ being the covariant derivative, $\tilde{\nabla}_\alpha = \nabla_\alpha - \frac{ie}{\hbar c} A_\alpha$, $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$;

2) <u>the indefinite bilinear form</u>

$$(\varphi_1, \varphi_2)_\Sigma \equiv i \int_\Sigma d\sigma^\alpha(x) \left(\bar{\varphi}_1 \tilde{\partial}_\alpha \varphi_2 - \overline{\tilde{\partial}_\alpha \varphi_1}\, \varphi_2\right), \tag{2.2}$$

*) We ought to consider the conformal-invariant (for $m = 0$) equation

$$g^{\alpha\beta}\tilde{\nabla}_\alpha \tilde{\partial}_\beta \varphi + \frac{R}{6}\varphi + \left(\frac{mc}{\hbar}\right)^2 \varphi = 0, \tag{2.3}$$

where R is the scalar curvature. This would be especially appropriate because eq.(2.3) was originally introduced in a physical context in paper /5/ for reasons relevant to the present paper.

where $\bar{\varphi}$ is the complex conjugate of φ and the integration is taken over a space-like hypersurface $\Sigma = \{x \in V_{1,3} / \Sigma(x) = const\}$; if φ_1 and φ_2 are solutions of eq.(2.1), then $(\varphi_1, \varphi_2)_\Sigma$ does not depend on Σ ;

3) <u>operators of observables</u>, generally nonconserving; they, or more strictly their matrix elements in the sense of the "scalar product" (2.2), will be introduced from the principles of general covariance, hermiticity and correspondence to the particular cases where expressions for them are known as differential operators; this problem will be discussed in a separate paper.

In the Minkowski space-time, transition to the nonrelativistic asymptotics proceeds from extraction of the "fast" phase from the relativistic wave function:

$$\varphi = const \cdot \exp(-i\omega_c t)\, \psi(x).$$

The condition $|\partial(\arg \psi)/\partial t| \ll \omega_c$ points out the class of FR in which the motion of the quantum system described by the wave function is nonrelativistic.

In the General Relativity any particular coordinates are scalar functions of the general coordinates. More accurately, four real scalar functions with the nonzero Jacobian $f^{(\alpha)}(x)$, $\det \| \partial_\beta f^{(\alpha)} \| \neq 0$, locally determine a holonomic system of coordinates. Therefore, in the general coordinates $\{x\}$, $x \in V_{1,3}$, extraction of the "fast" phase will be done in the general covariant form

$$\varphi = \sqrt{2\frac{mc}{\hbar}}\, e^{-i\frac{mc}{\hbar} S(x)}\, \phi(x) \tag{2.4}$$

$S(x)$ being a real function to be defined. If one considers eq.(2.1) as an equation with the small parameter $(\hbar/mc)^2$ of the higher derivatives, then substitution of eq.(2.4) leads to the quasiclassic asymptotic estimation of the wave function as $\hbar \to 0$. However, this would not be a physically sapid asymptotics for $c \to \infty$ because there appear additional powers of c^{-1} after the transition to the real, i.e., macroscopically measurable, coordinate of time t . For this reason, the transition to the asymptotics for c^{-2} or ω_c^{-1} through eq. (2.4) does not lead to asymptotic expansions for the wave function but provides asymptotic equations for it thus leading to a new theory. This theory is valid under the condition of smallness of phase variations of $\phi(x)$ with respect to $(mc/\hbar) S(x)$, i.e.

$$|\partial_\alpha S\, \partial^\alpha \arg \phi| \ll \frac{mc}{\hbar} |\partial_\alpha S\, \partial^\alpha S| .$$

Substitution of (2.4) into eq.(2.1) leads to the equation

$$(\nabla_\alpha S' \nabla^\alpha S' - 1)\phi + 2\frac{i\hbar}{mc}\left(\nabla_\alpha S' \tilde{\nabla}^\alpha + \frac{1}{2}\Box S'\right)\phi - \left(\frac{\hbar}{mc}\right)^2 \tilde{\Box}\phi = 0, \tag{2.5}$$

$$\Box \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta .$$

Now if the function $S'(x)$ is the complete integral the equation

$$\nabla_\alpha S' \nabla^\alpha S' = 1 \tag{2.6}$$

which is evidently the Hamilton-Jacobi equation for a free particle in $V_{1,3}$, then the vector field $s^\alpha \equiv \nabla^\alpha S'$ will be the tangential field to the geodesic congruence orthogonal to the hypersurfaces $S' = \{x \in V_{1,3} \mid S(x) = const\}$. This congruence or equivalently, the vector field s^α is an example of what we shall call the frame of reference (FR), following e.g. /6/, namely this is a normal Gaussian FR. From the physical point of view it may be conceived as a flow of free falling bodies each of which might be the classical part of the "quantum object + measuring device" system.

Now we introduce the field of 4-velocities $\tau^\alpha \equiv c s^\alpha$, $\tau_\alpha \tau^\alpha = c^2$, and the tensor field

$$h^\alpha{}_\beta \equiv c^{-2}\tau^\alpha\tau_\beta - \delta^\alpha{}_\beta \tag{2.7}$$

which is the projection onto the infinitesimal 3- area orthogonal to τ^α. Then, the operator of the covariant derivative may be represented as

$$\nabla_\alpha = c^{-2}\tau_\alpha \tau^\beta \nabla_\beta - D_\alpha, \quad D_\alpha \equiv h_\alpha{}^\beta \nabla_\beta . \tag{2.8}$$

For the geodesical conguence we evidently have

$$\tau^\alpha \nabla_\alpha \cdot \tau_\beta = 0 \tag{2.9}$$

and consequently

$$\tau^\alpha \nabla_\alpha h^\beta{}_\gamma = 0, \tag{2.10}$$

From eqs.(2.5), (2.6) by virtue of (2.7)-(2.10) one obtains the following equation for

$$i\hbar \tilde{T}\phi = \mathcal{H}\phi, \tag{2.11}$$

where

$$\tilde{T} \equiv \tau^\alpha \nabla_\alpha + \frac{1}{2}\nabla_\alpha \tau^\alpha, \tag{2.12}$$

$$\mathcal{H} \equiv H_0 - \frac{\hbar^2}{2mc^2}\left(\frac{1}{2}\tau^\alpha\nabla_\alpha\nabla_\beta\tau^\beta + \frac{1}{4}\nabla_\alpha\tau^\alpha\nabla_\beta\tau^\beta - \tilde{T}^2\right), \tag{2.13}$$

$$H_0 \equiv -\frac{\hbar}{2m}\tilde{\nabla}_\alpha \cdot \tilde{D}^\alpha = -\frac{\hbar}{2m}\tilde{\Delta}_{S'} \tag{2.14}$$

$\tilde{\Delta}_{S'}$ is the generalised Laplacean (with the gauge covariant derivatives $\tilde{\nabla}_\alpha \equiv \nabla_\alpha - \frac{ie}{\hbar c}A_\alpha$) on the hypersurface $S'(x) = const$ and the point between differential operators denotes their operator product, e.g. $\nabla_\alpha \cdot D_\beta \phi \equiv \nabla_\alpha (D_\beta \phi)$. Note also that

$$\sigma^2_{(\tau)} \equiv -\frac{1}{2}\tau^\alpha\nabla_\alpha\nabla_\beta\tau^\beta - \frac{1}{4}\nabla_\alpha\tau^\alpha\nabla_\beta\tau^\beta = \sigma^2 - \frac{1}{2}R_{\alpha\beta}\tau^\alpha\tau^\beta,$$

σ^2 and $R_{\alpha\beta}$ being the shear of congruence and the Ricci tensor correspondingly, see /6/.

Combination in the operator $\tilde{T}$ of the term $\nabla_\alpha\tau^\alpha/2$ with the derivative $\tau^\alpha\nabla_\alpha$ along the congruence seems still rather arbitrary but this is a very important point and it will be justified by reasons of Sec. 4. Shortly, the quantity $\nabla_\alpha\tau^\alpha$ characterises the variation of the space volume along the vector field τ^α, see e.g. /6/, and it is to be taken into account for the right definition of the wave function of the one-particle state.

Now we pass from the exact equation (2.11) to approximate equations that can be obtained by the iteration scheme used in /7/ for the transition to the Schrödinger-Pauli equation from the Dirac equation with the external electromagnetic field (without gravitation). The approximate equation of the zeroth order

$$i\hbar\tilde{T}\phi = H_0\phi \tag{2.15}$$

is used for substitution of $\tilde{T}$ in the right-hand side of eq.(2.11). The resulting equation of the first approximation is used in the same way for obtaining the second approximation, etc. For the N-th approximation one has

$$i\hbar\tilde{T}\phi = H_N\phi, \tag{2.16}$$

$$H_N = \sum_{n=0}^{N} \omega_c^{-n} h_n, \tag{2.17}$$

$$h_0 = H_0, \quad h_1 = \hbar\sigma^2_{(\tau)} - i[\tilde{T}, H_0] - \hbar^{-1}H_0^2, \tag{2.18}$$

$$h_n = -i[\tilde{T}, h_{n-1}] - \sum_{k=0}^{n-1}\hbar^{-1}h_k \cdot h_{n-k-1}, \quad n \geq 2. \tag{2.19}$$

The important property of H_N is that it includes no derivative along τ^α (i.e. derivative in time). In fact, due to eqs.(2.7), (2.10), (2.12) one has

$$[\widetilde{\mathcal{T}}, \widetilde{D}_\alpha]\phi = -\left(\nabla_\alpha \tau_\beta \widetilde{D}^\beta + \frac{ie}{\hbar c}\tau^\beta \mathcal{F}_{\alpha\beta} + \frac{1}{2} D_\alpha \nabla_\beta \tau^\beta\right)\phi . \tag{2.20}$$

where $\mathcal{F}_{\alpha\beta} \equiv \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$. Consequently, the commutator of $\widetilde{\mathcal{T}}$ with an operator which includes only D_α, i.e. the derivatives along the hypersurface S (space derivatives) gives again an operator including only these derivatives. This assertion is invalid in the case of the nongeodesic vector field τ^α because in each iteration there may appear the derivatives in the direction of τ^α in the right-hand side of eq.(2.16). However this is not a serious obstacle because time derivatives may again be excluded by means of eq.(2.16) of the preceding order of approximation.

Let ϕ_N be a sufficiently smooth solution of eq.(2.16). Then evidently

$$(i\hbar\mathcal{T} - \mathcal{H})\phi_N = O(\omega_c^{-N-1}). \tag{2.21}$$

This relation does not mean strictly that ϕ_N provides an asymptotic estimation of a solution of eq.(2.11). However, the physical level of rigour allows one to keep off this question and to consider equation (2.16) and the asymptotic structure based on it as an independent theory, namely, the quantum mechanics with relativistic corrections up to the order N.

Denote by Φ the space of all appropriate functions ϕ with a "slow" phase and by Φ_{oN} the space of such solutions of eq.(2.16) that $\Phi_{oN} \subset \Phi$. Linear differential operators on Φ can be divided into two classes. The first class consists of the operators (including the operator of multiplication by a scalar function) that contain no derivative in the direction of τ^α and thus act along the hypersurface S. We call then S-operators and denote them by the usual italic letters e.g. H_N, h_n. The second class contains the operators with derivatives in the direction of τ^α. These operators will be called t-operators and we denote them by the cursive letters, e.g. $\mathcal{T}$, $\mathcal{H}$.

3. The scalar product and hermiticity of the Hamiltonian

Consider now, as has been planned at the beginning of Sec. 2, the indefinite bilinear form (2.2) and obtain an asymptotic scalar

product induced by it in Φ_{oN}. Choosing $S = \{x \in V_{1,3} \,|\, S(x) = const\}$ as Σ and substituting expression (2.4) into (2.2), one gets the following bilinear form:

$$\{\phi_1,\phi_2\}_S \equiv \int_S d\sigma(x)\{\bar\phi_1\phi_2 + i\omega_c^{-1}(\bar\phi_1\mathcal{T}\phi_2 - \mathcal{T}\bar\phi_1\phi_2)\} = (\psi_1,\psi_2)_S . \tag{3.1}$$

where $d\sigma(x) \equiv c^{-1}\tau_\alpha d\sigma^\alpha(x)$ is the elementary invariant 3-volume of S. The form (3.1) is evidently positive definite under the condition

$$-2\int_S d\sigma\, \tau^\alpha \nabla_\alpha \arg\phi\, |\phi|^2 < \omega_c \int_S d\sigma |\phi|^2 . \tag{3.2}$$

Under our suppositions on the space Φ the condition is valid.

If $\phi_1, \phi_2 \in \Phi_{oN}$, one can write instead of eq.(3.1)

$$\{\phi_1,\phi_2\}_S = \int_S d\sigma\, \bar\phi_1 \left(1 + \frac{H_N^\dagger + H_N}{2mc^2}\right)\phi_2 + O(\omega_c^{-N-1}). \tag{3.3}$$

We denote by $Z^\dagger$ the Hermitian conjugation of the S-operator Z with respect to the standard scalar product

$$(\psi_1,\psi_2) = \int_S d\sigma\, \bar\psi_1 \psi_2 , \tag{3.4}$$

i.e.,

$$(\psi_1, Z^\dagger \psi_2) = (Z\psi_1, \psi_2). \tag{3.5}$$

In other words, the S-operator $Z^\dagger$ is the covariantly transposed and complex conjugated S-operator Z.

Further, we introduce the functions $\psi = V^{-1}\phi$, V being an S-operator defined by the equation

$$V^\dagger V = \left(1 + \frac{H_N^\dagger + H_N}{2mc^2}\right)^{-1} . \tag{3.6}$$

Its formal particular solution is

$$V = \left(1 + \frac{H_N^\dagger + H_N}{2mc^2}\right)^{-1/2} \tag{3.7}$$

It is determined up to the left multiplication by a unitary S-operator.

If $\varphi_1, \varphi_2 \in \Phi_{oN}$, then it follows from eq.(3.3) that

$$\{\Phi_1, \Phi_2\}_S = (\Psi_1, \Psi_2) + O(\omega_c^{-N-1}). \tag{3.7}$$

Thus, asymptotically the basic bilinear form (2.2) induces through (3.3) generalization (3.4) of the standard scalar product to the case of curved space-like hypersurface for Ψ's if the latter satisfy the equation

$$i\hbar \bar{T} \Psi = \hat{H}_N \Psi, \tag{3.8}$$

where

$$\hat{H}_N = V^{-1} \cdot \left(H_N \cdot V - [i\hbar \bar{T}, V]\right), \tag{3.9}$$

following from eq.(2.18). We call it the Schrödinger equation with relativistic corrections up to the N-th order.

We will prove now that the operator $\hat{H}_N$, which will be called the Hamiltonian, is asymptotically Hermitian in the space Ψ_{oN} of the solutions of eq.(3.8), i.e.,

$$\delta_N \equiv \left(\Psi_1, (\hat{H}_N - \hat{H}_N^{\dagger})\Psi_2\right) = O(\omega_c^{-N-1}) \tag{3.10}$$

for $\Psi_1, \Psi_2 \in \Psi_{oN}$. To prove this we substitute eq.(3.9) into eq. (3.10) and use the hermiticity of V and the relation

$$[i\hbar \bar{T}, Z]^{\dagger} = -[i\hbar \bar{T}, Z^{\dagger}], \tag{3.11}$$

which is valid for any scalar S-operator Z but is not so evident as it might seem at a first sight. The matter is that $\bar{T}$ is t-operator, not s-operator, and therefore, the Hermitian conjugation with respect to the scalar product (3.4) is not defined for it. As a result, we have

$$\delta_N = \int_S d\sigma\, \bar{\Phi}_1 \left(V^{-2} \cdot H_N - H_N^{\dagger} \cdot V + [i\hbar \bar{T}, V^{-2}]\right)\Phi_2 =$$

$$= \int_{S'} d\sigma\, \bar{\Phi}_1 \left(H_N + \frac{H_N^2 + [i\hbar \bar{T}, H_N]}{2mc^2} - H_N^{\dagger} - \frac{H_N^{\dagger 2} + [i\hbar \bar{T}, H_N]^{\dagger}}{2mc^2}\right)\Phi_2 .$$

Using eq.(2.16) we obtain

$$\delta_N = \int_S d\sigma\, \bar{\Phi}_1 \left(H_N + \frac{(i\hbar \bar{T})^2}{2mc^2}\right)\Phi_2 - \int_S d\sigma\, \overline{\left(H_N + \frac{(i\hbar \bar{T})^2}{2mc^2}\right)\Phi_1}\, \Phi_2$$

and hence, in view of eq.(2.21),

$$\delta_N = \int_S d\sigma\, \bar{\Phi}_1 \left(\mathcal{H} + \frac{(i\hbar \bar{T})^2}{2mc^2} + O(\omega_c^{-N-1})\right)\Phi_2 -$$

$$- \int_{S'} d\sigma\, \overline{\left(\mathcal{H} + \frac{(i\hbar \bar{T})^2}{2mc^2} + O(\omega_c^{-N-1})\right)\Phi_1}\, \Phi_2$$

Further, as a consequence of eq.(2.13)

$$\delta_N = \int_S d\sigma\, \bar{\Phi}_1 \left(H_0 + \sigma^2_{(\tau)} + O(\omega_c^{-N-1})\right)\Phi_2 -$$

$$- \int_S d\sigma\, \overline{\left(H_0 + \sigma^2_{(\tau)} + O(\omega_c^{-N-1})\right)\Phi_1}\, \Phi_2$$

Hence, in view of obvious hermiticity of H_0, we come to (3.10), i.e., asymptotical hermiticity of $\hat{H}_N$.

In conclusion of the section, we demonstrate the expression for $\hat{H}_2$, i.e., for the Hamiltonian with relativistic corrections of the second order

$$\hat{H}_2 = H_0 - \frac{1}{2mc^2}\left(H_0^2 - \frac{\hbar^2}{2}\sigma^2_{(\tau)}\right) + \tag{3.12}$$

$$+ \left(\frac{1}{2mc^2}\right)^2 \left(2H_0^3 + \left[i\hbar \bar{T} - \frac{3}{2}H_0, [i\hbar \bar{T}, H_0]\right] - \hbar^2\left(H_0 \sigma^2_{(\tau)} + \sigma^2_{(\tau)} H_0\right)\right).$$

The commutator $[i\hbar \bar{T}, H_0]$ is calculated on the basis of eq. (2.20) and reads

$$[i\hbar \bar{T}, H_0] = \frac{i\hbar^3}{m}\left(\nabla^{\alpha}\tau^{\beta}\tilde{D}_{\alpha}\tilde{D}_{\beta} - \nabla_{\gamma}\nabla_{\alpha}\tau^{\gamma}D^{\alpha} - \frac{ie}{\hbar c}\tau^{\gamma}\mathcal{F}_{\gamma\alpha}D^{\alpha} + \right.$$

$$\left. + \frac{ie}{2\hbar c}\tau^{\gamma}\nabla^{\beta}\mathcal{F}_{\gamma\beta} - \frac{1}{4}\nabla_{\alpha}\nabla^{\alpha}\nabla_{\gamma}\tau^{\gamma}\right).$$

One clearly sees from eq.(3.12) that we in fact have an asymptotics in the ratios of kinetic energies of motions in the system under consideration to the rest mass of the scalar particle which is described by the field φ.

4. Conservation of the norm of the wave function

The hermiticity of $\hat{H}_N$ just established provides conservation of the norm of the wave function $\Psi \in \Psi_{oN}$ in the direction of τ^{α}

(of course, asymptotically up to $O(\omega_c^{-N-1})$). To show this we introduce a normal system of coordinates in $V_{1,3}$ which is associated with the given FR, i.e., we take for x^0 the canonical parameter ct on the time-like geodesic in the direction of τ^α, each geodesic being in turn numerated by three numbers ξ^i, $i,j,k=1,2,3$. The ξ's form curvilinear coordinates on each hypersurface S. The metric form of $V_{1,3}$ has the following form in these coordinates $\{\tau, \xi^i\}$:

$$ds^2 = c^2dt^2 - \omega_{ij}(t,\xi)d\xi^i d\xi^j. \tag{4.1}$$

Consider now

$$i\hbar \frac{d}{dt}\|\psi\|^2 \equiv i\hbar \frac{d}{dt}\int_S d\sigma \bar{\psi}\psi .$$

Taking into account that $d\sigma = d\xi^1 d\xi^2 d\xi^3 \sqrt{\omega(t,\xi)}$, $\omega \equiv det\|\omega_{ij}\|$ one obtains

$$i\hbar \frac{d}{dt}\|\psi\|^2 = i\hbar \int d\sigma \left\{ \frac{1}{\sqrt{\omega}}\frac{\partial\sqrt{\omega}}{\partial t}\bar{\psi}\psi + \frac{\partial\bar{\psi}}{\partial t}\psi + \bar{\psi}\frac{\partial\psi}{\partial t}\right\}.$$

Remember now that

$$\nabla_\alpha \tau^\alpha = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}\,\tau^\alpha\right) = \frac{1}{\sqrt{\omega}}\frac{\partial\sqrt{\omega}}{\partial t} \tag{4.2}$$

Therefore,

$$i\hbar \frac{d}{dt}\|\psi\|^2 = i\hbar \int_{t=const} d\sigma \left\{ \bar{\mathcal{T}}\bar{\psi}\psi + \bar{\psi}\mathcal{T}\psi \right\} =$$

$$= \int_{t=const} d\sigma\, \bar{\psi}\left(H_N - H_N^\dagger\right)\psi = O(\omega_c^{-N-1})$$

Q.E.D.

The conservation of the norm of the wave function and the principle of correspondence serve as a basis for the standard Born interpretation of $|\psi(x)|^2 d\sigma(x)$ as the probability for finding the particle in the element of volume $d\sigma(x)$ of the hypersurface S.

In this connection let us return to the question of defining t-operator $\mathcal{T}$ in eqs. (2.11), (2.16), (3.8) as an analogue of time derivative in the usual Schrödinger equation. Of course, it does not matter from the mathematical point of view where will the term $i\hbar\nabla_\alpha\tau^\alpha/2$ be set and what will be called the Hamiltonian H_N or $H_N - i\hbar\nabla_\alpha\tau^\alpha/2$. However, in the latter case the Hamiltonian will be non-Hermitean. According to the standard physical interpretation, this is an indication of instability of the one-particle state. This instability, however, would be connected not with the probability of a real decay of the particle but with the variation of the probability of finding the particle in a small space region because of the dependence of the metric on time.

It is worth noting that one might work from the very beginning in the coordinates t, ξ^i and obtain very familiar formulae:

$$i\hbar\frac{\partial\psi_\omega}{\partial t} = \hat{H}_{\omega N}\psi_\omega ,$$

where $\psi_\omega \equiv \omega^{1/4}\psi$, $\hat{H}_{\omega N} \equiv \omega^{1/4}\hat{H}_N \cdot \omega^{-1/4}$,

and

$$(\psi_{\omega 1}, \psi_{\omega 2}) \equiv \int d\xi^1 \int d\xi^2 \int_{t=const} d\xi^3\, \bar{\psi}_{\omega 1}\psi_{\omega 2} = (\psi_1, \psi_2).$$

However, the approach presented here has the virtue of general covariance which is important in itself, clarifies the role of FR's and is necessary for supposed construction of the NRQM for nongeodesic frames of reference. Indeed, the choice of the Hamilton-Jacobi equation for $S(x)$ in the form of eq.(2.6) seems to be natural in our problem, but not be necessary. We might consider the Hamiltonian dynamics of reference bodies interacting with some external fields and compensate the change in the Hamilton-Jacobi equation by inclusion of the corresponding terms into the equation for $S(x)$. Particularly, the electromagnetic field $A_\alpha(x)$ which participates already in equation (2.5) might be this external field. This would mean that the (classical) reference bodies are charged and interact with the field as well as the quantum particle does. In general, for each concrete problem there will be the most adequate FR. With respect to which the quantum system will be"mostly nonrelativistic" for the "longest time interval".

Leaving details of these questions for a special investigation we note nevertheless the principal idea that for extraction of a nonrelativistic component of quantum dynamics of a particle, it is apparently necessary that the FR were defined dynamically as a Hamiltonian system but not kinematically.

Appendix

To prove eq.(3.11) we consider a differential S-operator $\hat{Z}$ in $\Phi(\Sigma)$ using the coordinates $\{\tau, \xi^i\}$ introduced in Sec. 4 :

$$\hat{Z} = Z_0(\tau,\xi) + \sum_{\kappa=1}^{n} Z^{i_1\ldots i_\kappa}(\tau,\xi)\partial_{i_1}\cdots\partial_{i_\kappa} , \quad \partial_i = \frac{\partial}{\partial\xi^i} .$$

According to (3.5) and (3.4), we obtain as a result of integrations by parts that

$$\hat{Z}^{\dagger} = \bar{Z}_0 + \frac{1}{\sqrt{\omega}} \sum_{k=1}^{n} \partial_{i_1} \dots \partial_{i_k} \left(\sqrt{\omega}\, \bar{Z}^{i_1 \dots i_k}\right). \tag{A.1}$$

Evidently, it will be sufficient to consider the operator $\mathcal{T}$.
Using (2.12) and (4.2) we have

$$[\mathcal{T}, \hat{Z}] = \frac{\partial \hat{Z}}{\partial \tau} + \frac{1}{2}\left(\frac{\partial \ln\sqrt{\omega}}{\partial \tau} \cdot \hat{Z} - \hat{Z} \cdot \frac{\partial \ln\sqrt{\omega}}{\partial \tau}\right), \tag{A.2}$$

and consequently

$$[\mathcal{T}, \hat{Z}] = \left(\frac{\partial \hat{Z}}{\partial \tau}\right)^{\dagger} - \frac{1}{2}\left[\frac{\partial \ln\sqrt{\omega}}{\partial \tau}, \hat{Z}^{\dagger}\right]. \tag{A.3}$$

According to (A.1)

$$\left(\frac{\partial \hat{Z}}{\partial \tau}\right)^{\dagger} = \frac{\partial \bar{Z}_0}{\partial \tau} + \frac{1}{\sqrt{\omega}} \sum_{k=1}^{n} \partial_{i_1} \dots \partial_{i_k} \left(\sqrt{\omega}\, \frac{\partial \bar{Z}^{i_1 \dots i_k}}{\partial \tau}\right).$$

Differentiation of (A.1) by τ shows that

$$\left(\frac{\partial \hat{Z}}{\partial \tau}\right)^{\dagger} = \frac{\partial \hat{Z}}{\partial \tau} - \left[\hat{Z}^{\dagger}, \frac{\partial \ln\sqrt{\omega}}{\partial \tau}\right]. \tag{A.4}$$

Substituting into eq.(A.2) the operator $\hat{Z}^{\dagger}$ instead of $\hat{Z}$ and comparing the result with (A.3) we obtain in view of (A.4) that

$$[\mathcal{T}, \hat{Z}]^{\dagger} = [\mathcal{T}, Z^{\dagger}],$$

Q.E.D.

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Квантовая механика в римановом пространстве-времени. Общековариантное уравнение Шредингера с релятивистскими поправками

Исследуется асимптотика по c^{-2} /c - скорость света/ теории комплексного скалярного поля в общем римановом пространстве-времени, взаимодействующего с внешним электромагнитным полем. В свободнопадающей /нормальной гауссовой/ системе отсчета получен общековариантный аналог уравнения Шредингера для скалярной частицы во внешних гравитационных и электромагнитном полях с релятивистскими поправками произвольного порядка. Показано, что учет геометрического изменения во времени элемента пространственного объема приводит к гамильтониану, который /асимптотически/ эрмитов относительно стандартного скалярного произведения, что служит основанием для борновского истолкования соответствующих волновых функций.

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Tagirov E.A. E2-88-678

Quantum Mechanics in Riemannian Space-Time. General Covariant Schrödinger Equation with Relativistic Corrections

The asymptotics for c^{-2} (c being the velocity of light) of the theory of a complex scalar field in the Riemannian space-time and external electromagnetic field is considered. The general-covariant Schrödinger equation with relativistic corrections for a scalar particle in external gravitational and electromagnetic fields is obtained for the case of normal Gaussian systems of reference. Account of the geometric variation of the spatial volume element along the geodesics of the system of reference leads to a Hamiltonian which is (asymptotically) self-abjoint with respect to the standard scalar product. This fact is considered as a ground for the Born interpretation of the wave functions.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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