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THE ADDITION THEOREM FOR 4-VELOCITIES

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1. It seems that the notion of 4-velocity (or covariant velocity) has been first introduced by H.Poincare (see, for example, Ref.[1]). By definition,

$$u^j = dx^j/d\tau, \quad j=0,1,2,3, \tag{1}$$

where τ is the invariant proper time. From the definition it follows that the u^j is a unit 4-vector, i.e.

$$u^j u_j = (u^0)^2 - (\vec{u})^2 = 1 \tag{2}$$

(We use the units in which the speed of light $c = 1$). The energy-momentum vector p^j is related to the u^j by the well-known relation

$$p^j = m u^j \tag{3}$$

(m is the rest mass of a particle).

Discussing the 4-velocity, one usually mentions only its definition and the property (2) and remarks that under Lorentz transformation the 4-vector u^j is transformed like the space-time 4-vector x^j. Although the formulae for 4-velocity transformation are sometimes presented in the case of special Lorentz transformations, in the general case one usually discusses the transformation law for conventional 3-dimensional velocity. Corresponding formulae are known as a theorem (or a law) of velocity addition.

2. The goal of this paper is to obtain the theorem of addition for the 4-velocities. In essence, such a law allows one to find the relative 4-velocity between two particles, i.e. the components of the 4-velocity of particle 1 in the rest frame of particle 2 if the 4-velocities of both particles are known in a given system of reference S. It is obvious that to find the relative 4-velocity, one needs to make a Lorentz transformation from the S-system to the S'-system, where particle 2 is at rest (i.e., its 4-velocity $u_2 = (1,0)$). It is known that under this transformation the energy-momentum of particle 1 is transformed as follows:

$$\begin{cases} E' = \gamma_2 (E_1 - \vec{\beta}_2 \vec{P}_1), \\ \vec{P}'_1 = \vec{P}_1 + \gamma_2 \vec{\beta}_2 \left(\dfrac{\gamma_2 \vec{\beta}_2 \vec{P}_1}{1+\gamma_2} - E_1 \right), \end{cases} \tag{4}$$

where γ_2 and $\vec{\beta}_2$ are the gamma-factor and the 3-velocity of particle 2 taken in the S-system. Because of

$$\begin{cases} u_1^0 = E_1/m_1 = \gamma_1, & \vec{u}_1 = \vec{P}_1/m_1 = \gamma_1 \vec{\beta}_1, \\ u_2^0 = E_2/m_2 = \gamma_2, & \vec{u}_2 = \vec{P}_2/m_2 = \gamma_2 \vec{\beta}_2 \end{cases} \tag{5}$$

(see(3)), one obtains immediately

$$\begin{aligned} u_1^{0\prime} &= u_2^0 u_1^0 - \vec{u}_2 \vec{u}_1 = u_1^j u_{2j} = (u_1 \cdot u_2) \\ \vec{u}'_1 &= \vec{u}_1 + \vec{u}_2 \left(\frac{\vec{u}_2 \vec{u}_1}{1+u_2^0} - u_1^0 \right). \end{aligned} \tag{6$'$}$$

where $(u_1 \cdot u_2)$ is the scalar product of the 4-vectors u_1 and u_2. In essence, formulae (6') are just the law of addition for the 4-velocities[3a] and the components $u_1^{0\prime}$ and $\vec{u}'_1$ are just the components of relative 4-velocities between particles 1 and 2. Therefore, we use below the notations u_{12}^0 and $\vec{u}_{12}$. Formulae (6') can be rewritten in a more symmetric form:

$$u_{12}^0 = (u_1 \cdot u_2) \tag{a}$$

$$\vec{u}_{12} = \vec{u}_1 - \vec{u}_2 \frac{u_1^0 + u_{12}^0}{u_2^0 + 1} \tag{b}$$

(6)

(One can easily verify the condition (2)).

Formula (6) is essentially another (in comparison with the conventional one) form of the general Lorentz transformation for the 4-vector A, transformed from the S-system to the S'-system, which moves relative to the S with the 4-velocity $u = (u^0, \vec{u}) = (\gamma, \gamma\vec{\beta})$:

$$\begin{cases} A^{0\prime} = (u \cdot A) \\ \vec{A}' = \vec{A} - \vec{u} \dfrac{(u \cdot A) + A^0}{1 + u^0} = \vec{A} - \vec{u} \dfrac{A^0 + A^{0\prime}}{u^0 + 1} \end{cases} \tag{7}$$

We believe that this form of the general Lorentz transformation is more suitable for practical use than the conventional one.

From formulae (6) one can see that the time-like component of the relative 4-velocity is just a scalar product of the 4-velocities u and u_2. (But, in general, $\vec{u}_{12} \neq -\vec{u}_{21}$).

We note that the $|\vec{u}_{12}|^2$ is a relativistic invariant related to the tensor

$$u_{jk} = \varepsilon_{jklm} u_1^m u_2^l, \tag{8}$$

where ε_{jklm} is Levi-Chivita's symbol, namely,

$$\begin{cases} |\vec u_{12}|^2 = -\frac{1}{2} u^{jk} u_{jk} = \vec E^2 - \vec H^2, \\ \vec E = u_1^0 \vec u_2 - u_2^0 \vec u_1, \\ \vec H = \vec u_1 \times \vec u_2 . \end{cases} \tag{9}$$

(The second invariant, related to this tensor and proportional to the product $\vec E \cdot \vec H$, is identically equal to zero.)

For relative energies it is obvious that

$$E_{12} = m_1 u^0_{12}, \qquad E_{21} = m_2 u^0_{12} = \frac{m_2}{m_1} E_{12} \tag{10}$$

and for the absolute value of relative momentum

$$|\vec P_{12}| = m_1 |\vec u_{12}|, \quad |\vec P_{21}| = m_2 |\vec u_{12}| = \frac{m_2}{m_1} |\vec P_{12}| . \tag{11}$$

3. Using the relative 4-velocity, in particular its space-like component, i.e. $\vec u_{12}$, one can considerably simplify some well-known relations.

For example, the relativistic kinetic equation (see Ref.[4]) now takes the form

$$u^j \frac{\partial f}{\partial x^j} + \frac{1}{m} F^j \frac{\partial f}{\partial u^j} = \frac{1}{2m} \int \frac{d^3 p}{p_1^0}\, d\Omega \,(f' f_1' - f f_1) |\vec u_{12}| \sigma \tag{12}$$

Here f, f' are the distribution functions, F is an external Minkowski force; σ, a cross section; and $d\Omega$, an element of the solid angle.

This form of the equation stresses its similarity to the nonrelativistic Boltzmann kinetic equation because we have used instead of the rapidity y_{12} (as in Ref.[5]) the quantity $|\vec u_{12}|$ which is a relativistic analog of the nonrelativistic velocity connected with rapidity by the known relation:

$$\begin{cases} u^0 = \mathrm{ch}\, y, \quad |\vec u| = \mathrm{sh}\, y, \\ y = \ln (u^0 + |\vec u|) \end{cases} \tag{13}$$

Another example of usefulness of the relative 4-velocity is the well-known nonrelativistic expression for a number of collisions $d\nu$ occurring in the volume element dV over the time interval dt (see, for example, Ref.[6]):

$$d\nu = \sigma\, v_{\text{отн}}\, n_1 n_2\, dV\, dt . \tag{14}$$

Here σ is the interaction cross section, n_1 and n_2 are the densities of particle beams.

The relativistic generalization of this expression looks as

$$d\nu = \sigma |\vec u_{12}|\, n_1^{(0)} n_2^{(0)}\, dV\, dt , \tag{15}$$

where $n_1^{(0)}$ and $n_2^{(0)}$ are the beam densities taken in the corresponding rest frames. This expression is more obvious than the conventional expression [6] with the so-called Møller flux.

Using Eq.(6), one can easily relate the components of the relative 4-velocity to the variable $b_{ik} = -(u_i - u_k)^2$ used in the relativistic nuclear physics studies [7]:

$$\begin{cases} u^0_{12} = 1 + \frac{1}{2} b_{12} , \\ |\vec u_{12}|^2 = b_{12} (1 + \frac{1}{4} b_{12}) . \end{cases} \tag{16}$$

(Let us note that the 4-vector ($u_1 - u_2$) is not, in general, a 4-velocity because it does not satisfy the condition (2).)

There are some other interesting properties of the 4-velocities. In particular, when both particles have equal masses, the time-like component of the relative 4-velocity also defines the components of the 4-velocities in the centre of mass frame. Indeed, in this case the c.m.s. 4-velocity is

$$u^j_c = \frac{u_1^j + u_2^j}{\sqrt{2(u_1 u_2) + 1}} = \frac{u_1^j + u_2^j}{\sqrt{2(u^0_{12} + 1)}} \tag{17}$$

and the 4-velocities of particles in the c.m.s. are, respectively,

$$\begin{cases} u^0_{1c} = u^0_{2c} = \frac{1}{2}\sqrt{2(u^0_{12}+1)} = \frac{1}{2}\sqrt{b_{12}+4} , \\ |\vec u_{1c}| = |\vec u_{2c}| = \frac{1}{2}\sqrt{2(u^0_{12}-1)} = \frac{1}{2}\sqrt{b_{12}} . \end{cases} \tag{18}$$

In the case of N particles of equal masses one has:

$$u^j_c = \frac{\sum_k^N u^j_k}{\sqrt{\sum_{n\neq m}^N u^0_{nm} + N}} = C \cdot \sum_k^N u^j_k \tag{19}$$

and

$$u^0_{kc} = C \Big(\sum_{n \neq k}^N u^0_{kn} + 1 \Big) . \tag{20}$$

This component of the 4-velocity of particle k in the c.m.s. is closely related to $b_k = -(u_k - u_c)^2$, which is widely used when jets in the 4-velocity space are analysed [8]:

$$b_k = \sqrt{2(u^0_{kc} - 1)} = \sqrt{2(\mathrm{ch}\, y_{kc} - 1)} , \tag{21}$$

where y_{kc} is the rapidity of the k-th particle in the c.m.s. of the jet.

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Теорема сложения 4-скоростей

Приводится теорема сложения 4-скоростей. Показано, что временная компонента относительной 4-скорости двух частиц /1 и 2/ определяется релятивистским инвариантом - скалярным произведением 4-скоростей этих частиц. При этом квадрат модуля пространственной части относительной 4-скорости, $|\vec{u}_{12}|^2$, представляет собой скалярное произведение /двукратную свертку/ соответствующих косых произведений исходных 4-скоростей. С помощью $|\vec{u}_{12}|$ релятивистское кинетическое уравнение представлено в явно лоренц-инвариантном виде. Обсуждается применение понятия относительной 4-скорости в физике высоких энергий.

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The Addition Theorem for 4-Velocities

The addition theorem for 4-velocities is obtained. It is shown that the time-like component of a relative 4-velocity of two particles is defined by a relativistic invariant quantity, namely, by the scalar product of their 4-velocities. The modulus squared of the spatial component of the relative 4-velocity, i.e. $|\vec{u}_{12}|^2$, is just a scalar product (double contraction) of the corresponding skew products of initial 4-velocities. Using the $|\vec{u}_{12}|$, the relativistic kinetic equation is presented in the explicitly Lorentz-invariant form. Some application of the relative 4-velocity in high energy physics is discussed.

The investigation has been performed at the Laboratory of High Energies, JINR.

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