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# ON TWO-PARTICLE RELATIVISTIC RUIJSENAARS-SCHNEIDER SYSTEMS IN AN EXTERNAL FIELD

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Ruijsenaars and Schneider[1-3] have shown that "relativistic" generalizations of the known classical integrable systems of particles on a straight line are also integrable, trajectories of the particles being directly connected with the characteristics of multisoliton solutions to the sine-Gordon equation. Relativistic invariance (more precisely, invariance under the Poincare transformations of a (1+1)-dimensional space) implies the existence of three generators, the Hamiltonian H, momentum P, and boost generator B (dependent on the particle rapidities $\theta_j$ and canonically conjugate positions $q_j$), with the Poisson brackets $\{H, P\} = 0$, $\{H, B\} = P$, $\{P, B\} = H/c^2$ [1]. The Hamiltonian chosen in [1]

$$H_0 = c^2 \sum_{j=1}^{N} \mathrm{ch}\,\theta_j \prod_{k \neq j} f(q_j - q_k) \tag{1}$$

leads automatically to integrable Hamiltonian systems if the above formulated condition of invariance is fulfilled: it is valid only when a certain functional relation holds for f with the general solution $f^2(\xi) = 1 + \frac{g^2}{c^2}\wp(\xi)$. There exists an isospectral Lax matrix L whose explicit form is given in [1]. However, the Lax equation can no longer be written for systems (1) in an explicit form owing to a very complicated structure of M, though the latter can in principle be found.

Schneider[2] has also proposed to consider systems with an external field switched-on and the Hamiltonian

$$H = H_0 + \sum_{j=1}^{N} W(q_j). \tag{2}$$

These systems, when $c \to \infty$, transform into the known nonrelativistic systems. He has found that the integrability takes place provided that

$$f^2(\xi) = 1 + \frac{g^2}{c^2}(\mathrm{sh}\,a\xi)^{-2}, \quad W(\xi) = A\exp(\pm 2a\xi). \tag{3}$$

For a more simple Hamiltonian $\tilde{H}_0$ differing from (1) by the change $\mathrm{ch}\theta_j \to \exp(\theta_j)$, the integrability has been proved for the case $W(\xi) = A\exp(2\xi) + B\exp(-2\xi)$. In that case, because of $H_0$ being simple, it turned out to be possible to determine the $\{L, M\}$ pair in an explicit form and to integrate, as for (3), the equations of motion. In the limit $c \to \infty$, eq.(3) represents the nonrelativistic Adler system[4].

In my papers[6] and[5], I have shown more general than (3) potentials of the external field $W(\xi)$ for which the nonrelativistic systems of particles are integrable:

$$W(\xi) = A\,\mathrm{ch}(4\xi) + B\,\mathrm{ch}(2\xi \pm \gamma). \tag{4}$$

Here $A$, $B$ and $\gamma$ are arbitrary constants. This note is aimed at answering the question (in part set in[3]): will the systems (2) with the potential $W(\xi)$ (4) be integrable and if not, what sort of generalization of the Hamiltonian (2) is necessary for systems of the type (2) being reduced to the systems indicated in[5] and[6] as $c \to \infty$? It is also instructive to set a more general problem of constructing systems of the type (2) reducing in the limit $c \to \infty$ to the known integrable systems with potentials constructed out of the systems of roots of classical Lie algebras[7] and[8].

I cannot solve this problem for an arbitrary number of degrees of freedom, determining an appropriate L,M-pair. However, at N = 2 it can be solved in a more simple manner. Consider a Hamiltonian of the general form

$$H_0 = c^2 \sum_{j=1}^{2} S_i\,\mathrm{ch}\,\theta_j + v, \tag{5}$$

where $S_i$ and $v$ are certain functions of positions $q_1$, $q_2$ canonically conjugate to the rapidities $\theta_j$, and establish the conditions of existence of an extra constant of motion I with the structure

$$I = \mathrm{ch}\theta_1\,\mathrm{ch}\theta_2\,\lambda + \mathrm{sh}\theta_1\,\mathrm{sh}\theta_2\,\mu + \rho_1\,\mathrm{ch}\theta_1 + \rho_2\,\mathrm{ch}\theta_2 + r. \tag{6}$$

The choice of I (6) is made by analogy with the nonrelativistic canstant of motion I have presented in[9]. Calculating the Poisson brackets of (5) and (6) and equating to zero the coefficients of linear-independent combinations of the rapidities, we get the following system of 10 equations for $S$, $v$, $\lambda$, $\mu$, $\rho_1$, $\rho_2$ and $r$



$$S_1\frac{\partial\lambda}{\partial q_1} - \lambda\frac{\partial S_1}{\partial q_1} - \mu\frac{\partial S_1}{\partial q_2} = 0, \quad S_2\frac{\partial\lambda}{\partial q_2} - \lambda\frac{\partial S_2}{\partial q_2} - \mu\frac{\partial S_2}{\partial q_1} = 0, \tag{7}$$

$$S_1\frac{\partial\mu}{\partial q_1} - \lambda\frac{\partial S_1}{\partial q_2} - \mu\frac{\partial S_1}{\partial q_1} = 0, \quad S_2\frac{\partial\mu}{\partial q_2} - \lambda\frac{\partial S_2}{\partial q_1} - \mu\frac{\partial S_2}{\partial q_2} = 0,$$

$$S_1\frac{\partial\rho_1}{\partial q_1} - \rho_1\frac{\partial S_1}{\partial q_1} = 0, \quad S_2\frac{\partial\rho_2}{\partial q_2} - \rho_2\frac{\partial S_2}{\partial q_2} = 0, \tag{8}$$

$$S_1\frac{\partial r}{\partial q_1} - S_2\frac{\partial\mu}{\partial q_2} - \rho_1\frac{\partial v}{\partial q_1} = 0, \quad S_2\frac{\partial r}{\partial q_2} - S_1\frac{\partial\mu}{\partial q_1} - \rho_2\frac{\partial v}{\partial q_2} = 0, \tag{9}$$

$$S_1\frac{\partial\rho_2}{\partial q_1} - \rho_1\frac{\partial S_2}{\partial q_1} - \mu\frac{\partial v}{\partial q_2} - \lambda\frac{\partial v}{\partial q_1} = 0,$$

$$S_2\frac{\partial\rho_1}{\partial q_2} - \rho_2\frac{\partial S_1}{\partial q_2} - \mu\frac{\partial v}{\partial q_1} - \lambda\frac{\partial v}{\partial q_2} = 0. \tag{10}$$

It is convenient to look for solutions to (7)-(10) in succession. From (7) it follows that the dependence of $S_{1,2}$ on $q_1$, $q_2$ and $q_1 \pm q_2$ can be factorized. The general solution to (7) contains four arbitrary functions

$$S_1 = f(q_1 - q_2)\,g(q_1 + q_2)\,C_1(q_1),$$

$$S_2 = f(q_1 - q_2)\,g(q_1 + q_2)\,C_2(q_2), \tag{11}$$

$$\lambda + \mu = -C_1(q_1)\,C_2(q_2)\,g^2(q_1 + q_2),$$

$$\lambda - \mu = -C_1(q_1)\,C_2(q_2)\,f^2(q_1 - q_2), \tag{12}$$

where $f$, $g$, $C_1$ and $C_2$ are functions of one argument to be determined from the other equations of the system. From (8) we find $\rho_1$ and $\rho_2$:

$$\rho_1 = S_1 d_1(q_2), \quad \rho_2 = S_2 d_2(q_1). \tag{13}$$

From (9) we get the derivatives $\frac{\partial r}{\partial q_1}$ and $\frac{\partial r}{\partial q_2}$, the equality of the second derivatives requires that $v$, $f$, $g$, $C_1$, $C_2$, $d_1$ and $d_2$ obey the the following equation

$$\frac{\partial^2 v}{\partial q_1 \partial q_2}(d_1 - d_2) + \frac{\partial v}{\partial q_1} d_1' - \frac{\partial v}{\partial q_2} d_2' + \frac{\partial}{\partial q_2}\Big[\frac{C_2 C_2'}{2}(f^2 - g^2) - C_2^2 (ff' + gg')\Big]$$

$$- \frac{\partial}{\partial q_1}\Big[\frac{C_1 C_1'}{2}(f^2 - g^2) + C_1^2 (ff' - gg')\Big] = 0 \tag{14}$$

where the prime means differentiation of a function with respect to its argument. And finally, upon substituting (11), (12) and (13) into (10) we obtain

$$\frac{\partial v}{\partial q_1} + \frac{\partial v}{\partial q_2} = -2ff'(d_2 - d_1) - f^2(d_2' + d_1')$$

$$\frac{\partial v}{\partial q_1} - \frac{\partial v}{\partial q_2} = -2gg'(d_2 - d_1) - g^2(d_2' - d_1'). \tag{15}$$

For convenience, let us introduce the notation

$$f = \sqrt{1 + \tilde f}, \quad g = \sqrt{1 + \tilde g}, \quad C_1 = \sqrt{1 + \tilde C_1}, \quad C_2 = \sqrt{1 + \tilde C_2} \tag{16}$$

and consider some of possible solutions to the overdetermined system (14), (15). Equations (15) become identities if $v = d_1 = d_2 = 0$, and (14) transforms into the functional equation

$$2(\tilde f'' - \tilde g'')(\tilde C_2 - \tilde C_1) + (\tilde f - \tilde g)(\tilde C_2'' - \tilde C_1'') - 3\tilde f'(\tilde C_1' + \tilde C_2') + 3\tilde g'(\tilde C_1' - \tilde C_2') = 0, \tag{17}$$

which exactly coincides with the functional equation for nonrelativistic systems I have derived in[9]. All the partial solutions presented in that paper lead thus to integrable systems of the type (5), for instance,

$$\tilde f(\xi) = \tilde g(\xi) = \frac{\lambda^2}{c^2}\wp(a\xi), \quad \tilde C_1(\xi) = \tilde C_2(\xi) = c^{-2}\Big[\lambda_1 \wp(a\xi) + \lambda_2 \wp\Big(a\xi + \frac{\omega_1}{2}\Big) +$$

$$+ \lambda_3 \wp\Big(a\xi + \frac{\omega_2}{2}\Big) + \lambda_4 \wp\Big(a\xi + \frac{\omega_1 + \omega_2}{2}\Big)\Big] \tag{18}$$

($\omega_1$ and $\omega_2$ are periods of the Weierstrass function $\wp(\xi)$ ) or

$$\tilde g(\xi) = 0, \quad \tilde f(\xi) = \frac{\lambda^2}{c^2}(\mathrm{sh}\, a\xi)^{-2},$$

$$\tilde C_1(\xi) = \tilde C_2(\xi) = c^{-2}[A\,\mathrm{ch}\,4a\xi + B\,\mathrm{ch}(2a\xi + \gamma)]. \tag{19}$$



It is just the solution (19) that, according to (16), (11), (5), gives, when $c \to \infty$, nonrelativistic integrable systems of two particles in the external field (4). The solution (18) provides an answer to the question put at the beginning of the note concerning the generalization of nonrelativistic potentials constructed from systems of the roots of classical Lie algebras[7] and [8].

Proceed now to the case $v$, $d_1$, $d_2 \neq 0$. It is seen that under the condition

$$v = -d_1 - d_2$$

equation (14) is still equivalent to (17), whereas (15) transform into the functional equations

$$\tilde f'(d_2 - d_1) + \tilde f(d_1' + d_2') = 0$$

$$\tilde g'(d_2 - d_1) + \tilde g(d_2' - d_1') = 0 \tag{20}$$

which I have for the first time solved in paper[5]:

$$\tilde f(\xi) = \frac{\lambda^2}{c^2}(\mathrm{sh}\, a\xi)^{-2}, \quad \tilde g(\xi) = \frac{\lambda_1^2}{c^2}(\mathrm{sh}\, a\xi)^{-2}, \quad d_1(\xi) = d_2(\xi) = \tilde A\,\mathrm{ch}(2a\xi) \tag{21}$$

$$\tilde f(\xi) = \lambda^2 c^{-2}(\mathrm{sh}\, a\xi)^{-2}, \quad \tilde g(\xi) = 0, \quad d_1(\xi) = d_2(\xi) = \tilde A\,\mathrm{ch}(2a\xi + \gamma) + \tilde B. \tag{22}$$

Formulae (22) and (19) determine the most general potential of an "external field" which may be added to the Hamiltonian without breaking integrability. It is seen that it is a particular case of the nonrelativistic potential (4) but does not coincide with it. As a result, we find that the "relativistic" generalization of the systems with potential (4) is defined by the Hamiltonian of the form (2),

$$H = c^2 \sum_{j=1}^{2} \mathrm{ch}\,\theta_j\,[1 + \lambda^2 c^{-2}\mathrm{sh}^{-2} a(q_1 - q_2)]^{1/2}[1 + c^{-2}(A\,\mathrm{ch}\,4aq\ +$$

$$+ B\,\mathrm{ch}(2aq_j + \gamma))]^{1/2} + \sum_{j=1}^{2} \tilde A\,\mathrm{ch}\,2a(q_j + \gamma_1). \tag{23}$$

It is interesting that this Hamiltonian depends on a larger amount of parameters than its nonrelativistic analog: in (23) B and $\tilde A$, $\gamma_1$ and $\gamma$ are independent of each other. In the nonrelativistic limit $q_j \to x_j c$, $\theta_j = \mathrm{arch}(p_j c^{-1})$, $a = \tilde{a} c^{-1}$, $c \to \infty$ the dependence of H on the set $\{B, \tilde{A}, \gamma, \gamma_1\}$ reduces to the dependence on two effective arbitrary parameters entering into the second summand of (4).

By analogy with the systems considered in [5] and [6] it is natural to expect that integrability in the cases (18) and (23) will take place also for an arbitrary number of degrees of freedom $N > 2$. The proof of this statement requires either explicit construction of the (L, M) pair, or another, similar to the one made in [1], construction of extra constants of motion. Along the first line, the best we may achieve is to determine the structure of the L-matrix. The second approach is also very cumbersome owing to the dependence of (18) and (23) on a great amount of parameters, and a similar program will be of interest only for physical or mathematical application of the systems with potentials (18) and (23).

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