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DECAY OF A PSEUDOSCALAR MESON INTO A LEPTON PAIR IN A THREE-DIMENSIONAL VERSION OF THE BOUND STATE THEORY

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I. INTRODUCTION

The decays of pseudoscalar mesons into leptons $P \to \ell\bar{\ell}$ are processes of the fourth order in electromagnetic interaction and are directly related to the radiative $P \to \gamma\gamma$ and conversion $P \to \gamma\ell\bar{\ell}$-decays. The first experimental data on the decay $\eta \to \mu^+\mu^-$ were published in 1969[1]. Theoretical studies of $P \to \ell\bar{\ell}$ were started in the late fifties (see e.g. review[2]). The probability of decay $P \to \ell\bar{\ell}$ normalized to the probability of decay $P \to \gamma\gamma$ (in what follows that ratio will be denoted by $Br(P \to \ell\bar{\ell}/\gamma\gamma)$) is proportional to the sum of squares of real (or dispersive) $Re\, R$ and imaginary (or absorptive) $Im\, R$ parts of the dimensionless amplitude of the process under consideration. The use of $|Im\, R|^2$ alone leads to the lowest ("unitary") limit, $Br^{unit}(P \to \ell\bar{\ell}/\gamma\gamma)$. Basic difficulties in calculations of the complete ratio $Br(P \to \ell\bar{\ell}/\gamma\gamma)$ are connected with the quantity $Re\, R$.

The quantity $Re\, R$ for a point vertex has a logarithmic divergence, therefore calculations are accompanied by a cut-off defined by the form factor of transition "meson-virtual photons". Without going into details of the history of theoretical estimates of $Re\, R$ (for review, see[2]), we only mention some of the works. For instance, in[3] and[4] the $Re\, R$ for the decay $\pi^0 \to e^+e^-$ was calculated with the use of the triangle quark diagrams (the model of quark triangle loops) and a once-subtracted dispersion relation. In this way the expression for $Re\, R$, in addition to the term with an intermediate $\gamma\gamma$-state, included the term corresponding to the "soft-meson" approximation ($m_\pi^2 \to 0$). The authors[3] have concluded that the ReR depends only on the hadron parameter of cut-off $\Lambda \gg \sqrt{m_\pi^2}$ (this dependence is logarithmic) that originates from the "soft" part of $Re\, R$ and does not depend on the detailed structure of the relevant form factor. In their previous paper[5] the authors[3] have computed the $Re\, R$ by using a dispersion relation with no subtraction and including contributions of intermediate states: $\gamma\gamma$, quark-antiquark, quark-antiquark - γ. An approximate equality $Re\, R \cong Im\, R$ in the limit $m_\pi \to 0$ was obtained in[6], with the account taken of influence of the transition form factor dependent on the mass of a constituent nonstrange quark. Using the quark-triangle mechanism and taking account of the transition form factor the authors of ref.[7] have found, for the quark masses $m \sim$ 100-150 MeV, the ratio $Br(\pi^0 \to e^+e^-/\gamma\gamma)$ being 2-3 times as small as the experimental one.

All the mentioned authors described the pseudoscalar-quark-antiquark vertex ($P q\bar{q}$) in a quark triangle with the γ_5 -matrices, i.e. they did not consider the structure of the $Pq\bar{q}$ - vertex, the nature of relative motion of quarks inside the meson, its influence on the yield of decay products. In the potential theory of bound states, the levels of quarkonia are predicted with the use of particular types of potentials, such as a QCD potential, $\sim 1/r$, and the one following from the lattice gauge theory, $\sim \lambda r$. For vector mesons, it is conventional to employ a nonrelativistic formula[8] connecting the wave function at the coordinate origin with the lepton-decay width. The formula is to be added with relativistic gluon corrections[9]. These corrections could diminish the magnitude of decay widths.

In this paper, a new scheme is proposed for inclusion in the calculations of $Br(P \to \ell\bar{\ell}/\gamma\gamma)$ of an intermediate transition two-photon form factor dependent on a three-dimensional relativistic wave function of a two-quark system (quarkonium). The results of calculations show first the explicit dependence of $Br(P \to \ell\bar{\ell}/\gamma\gamma)$ on the given model, i.e. on the quarkonium wave function, mass of constituent quarks, and the cut-off hadron parameter.

The paper is organized as follows: in Sect. 2 a three-dimensional relativistic equation and the wave function of a quarkonium are considered; in Sect. 3 a general scheme is given for construction of the amplitude of decay $P \to \ell\bar{\ell}$; in Sect. 4 the two-photon transition form factor is thoroughly analysed; in Sect. 5 dispersion calculations with the relativistic-oscillator wave function are performed; and in Sect. 6 conclusions are made.

2. WAVE FUNCTION. THREE-DIMENSIONAL RELATIVISTIC EQUATION

A bound state of a quark and antiquark can be described by the two-particle Bethe-Salpeter amplitude $\chi(p_1,p_2)$, where p_1 and p_2 are quark and antiquark momenta. Let $P = p_1 + p_2$ be the momentum of the bound state; and $2p = p_1 - p_2$, the relative momentum of the quarks. In the c.m.s., $P = (m_P, \vec{0})$, $p_1 = (m_P/2 + p^0, \vec{p})$ and

$p_2 = (m_P/2 - p^0, -\vec{p})$, where m_P is the mass of the bound state. The amplitude $\chi(p_1,p_2) \to \chi(p)$ fulfils a Bethe-Salpeter equation

$$\chi(p) = S_q^{\prime(1)}(p_1)\, S_q^{\prime(2)}(p_2) \int \frac{d^4p'}{(2\pi)^4}\, K(p-p')\,\chi(p'), \qquad (1)$$

where $S_q^{\prime(i)}(p_i)$ (i = 1,2) denotes the full propagator of the quark (antiquark) and the kernel $K(p-p')$ denotes the most general two-particle irreducible interaction. In what follows the amplitude $\chi(p)$ will as usual be approximated by changing the full quark propagators $S_q^{\prime(i)}(p_i)$ to the free propagators $S_{qF}^{(i)}(p_i)$.

The interaction kernel $K(p)$ is approximated by the formula

$$K(p) = K_{QCD}(p) + K_{conf.}(p), \qquad (2)$$

where $K_{QCD}(p)$ is a quantum-chromodynamics term corresponding to the one-gluon exchange and $K_{conf.}(p)$ is the term responsible for quark confinement. To write eq.(1) for the zeroth order approximation, the kernel $K(p)$ should be separated in the instantaneous approximation $K^I(\vec{p})$. Then equation (1) assumes the form $(\chi(p) \to \chi^I(p))$:

$$\chi^I(p) = S_{qF}^{(1)}(p_1)\, S_{qF}^{(2)}(p_2) \int \frac{d^4p'}{(2\pi)^4}\, K^I(p-p')\,\chi^I(p'). \qquad (3)$$

In what follows we will employ the one-time approach to quantum field theory[10-13] with the three-dimensional amplitude (the wave function) $\Psi(\vec{p})$ describing the relative motion of particles. Specifically, for quarks with momenta p_1 and p_2 and resp. polarization indices σ_1 and σ_2 the wave function (w.f.) $\Psi^{\sigma_1\sigma_2}(\vec{p})$ is determined through the Bethe-Salpeter w.f. as follows[12,13]:

$$\Psi^{\sigma_1\sigma_2}(\vec{p}) = \bar{u}_\alpha^{\sigma_1}(\Lambda_{\lambda_P}^{-1}p_1)\, v_\beta^{\sigma_2}(\Lambda_{\lambda_P}^{-1}p_2) \int d^4x \cdot \exp[i(p_1-p_2)\cdot x] \cdot \qquad (4)$$

$$\cdot\, \delta(\lambda_P \cdot x) \langle 0 | T(\psi_\alpha(x/2)\bar{\psi}_\beta(-x/2)) | m_P, \vec{P}, s, \sigma \rangle ,$$

where $\psi(x/2)$ and $\bar{\psi}(-x/2)$ are field operators of a quark and an antiquark, $x = x_1 - x_2$, the vector $|m_P, \vec{P}, s, \sigma\rangle$ specifies a composite system with mass m_P , spin s , its projection σ , and moving with momentum $\vec{P}$, $\lambda_P^\mu = P^\mu/\sqrt{P^2} = (p_1+p_2)^\mu/\sqrt{P^2}$, $u^{\sigma_1}(\Lambda_{\lambda_P}^{-1}p_1)$ and $v^{\sigma_2}(\Lambda_{\lambda_P}^{-1}p_2)$ are bispinors of a free quark and antiquark with polarizations σ_1 and σ_2 and momenta $\Lambda_{\lambda_P}^{-1}p_1$ and $\Lambda_{\lambda_P}^{-1}p_2$ respectively. For a more detailed account of the formalism of matrices of the pure Lorentz transformation as applied to a composite system with the total momentum P , see refs.[14] and[15].

The wave function $\Psi_{\sigma_1\sigma_2}(\vec{p})$ obeys the following three-dimensional equation[16]:

$$2\varepsilon(\vec{p})[m_P - 2\varepsilon(\vec{p})]\,\Psi_{\sigma_1\sigma_2}(\vec{p}) =$$

$$= \frac{1}{(2\pi)^3}\sum_{\sigma_1'\sigma_2'}\int\frac{d^3\vec{p}\,'}{2\varepsilon(\vec{p}\,')}\,V_{\sigma_1,\sigma_2;\sigma_1',\sigma_2'}(\vec{p},\vec{p}\,')\,\Psi_{\sigma_1'\sigma_2'}(\vec{p}\,'), \tag{5}$$

where $\varepsilon(\vec{p}) = (\vec{p}^{\,2} + m_q^2)^{1/2}$, m_q is the quark mass, $V_{\sigma_1,\sigma_2;\sigma_1',\sigma_2'}(\vec{p},\vec{p}\,')$ is a quasipotential. Using the approximation

$$\varepsilon(\vec{p}) \simeq m_q + \vec{p}^{\,2}/2m_q, \tag{6}$$

$$E_B = 2m_q - m_P, \tag{7}$$

$$m_R = m_q/2, \tag{8}$$

we obtain from (5) the nonrelativistic Schrödinger equation, $\Psi(\vec{p}) \to \Psi^{N.R.}(\vec{p})$ (for the weak coupling $V^{N.R.}(\vec{p}-\vec{p}\,') \simeq$ $\simeq V(\vec{p},\vec{p}\,')/m_R^2$):

$$(E_B + \vec{p}^{\,2}/2m_R)\,\Psi^{N.R.}_{\sigma_1\sigma_2}(\vec{p}) =$$

$$= -\sum_{\sigma_1'\sigma_2'}\int\frac{d^3\vec{p}\,'}{(2\pi)^3}\,V_{\sigma_1,\sigma_2;\sigma_1',\sigma_2'}(\vec{p}-\vec{p}\,')\cdot\Psi^{N.R.}_{\sigma_1'\sigma_2'}(\vec{p}\,'). \tag{9}$$

Further we shall calculate realistic physical amplitudes using the spinor representation of the relativistic w.f. (in the rest frame of the bound state with momentum $P=(m_P,\vec{0})$)[15] in the form:

$$\Psi_{\sigma_1\sigma_2}(\vec{p}) = \bar{u}_{\sigma_1}(p)\,\gamma_5\,v_{\sigma_2}(\tilde{p})\cdot\frac{\phi(\vec{p})}{2\varepsilon(\vec{p})}, \tag{10}$$

where $\phi(\vec{p})$ is a reduced w.f. of a quarkonium, $\tilde{p} = (p^0, -\vec{p})$. The three-dimensional relativistic equation for $\phi(\vec{p})$ with the quasipotential

$$V_{\sigma_1,\sigma_2;\sigma_1',\sigma_2'}(\vec{p},\vec{p}\,') =$$

$$= u_{\sigma_1}(p)\,\gamma_\mu\,u_{\sigma_1'}(p')\,\bar{v}_{\sigma_2'}(\tilde{p}')\,\gamma^\mu\,v_{\sigma_2}(\tilde{p})\cdot V_s(Q^2), \tag{11}$$

$V_s(Q^2) = -g^2/Q^2$, $Q = p - p'$, $\tilde{p}' = (p'^0, -\vec{p}\,')$ (g is the quark-gluon interaction constant) is of the form

$$2\varepsilon(\vec{p})[m_P - 2\varepsilon(\vec{p})]\,\phi(\vec{p}) =$$

$$= \frac{2}{(2\pi)^3}\int\frac{d^3\vec{p}\,'}{2\varepsilon(\vec{p}\,')}\cdot\left\{2\varepsilon(\vec{p})\varepsilon(\vec{p}\,') - m_q^2\right\}\cdot V_s(\vec{p}-\vec{p}\,')\cdot\phi(\vec{p}\,'). \tag{12}$$

Since the normalization condition for the w.f. $\Psi_{\sigma_1\sigma_2}(\vec{p})$ has the form[17]

$$\int\frac{d^3\vec{p}}{(2\pi)^3}\cdot\frac{\Psi^+_{\sigma_1\sigma_2}(\vec{p})}{\sqrt{2m_P}}\cdot\frac{\Psi^{\sigma_1\sigma_2}(\vec{p})}{\sqrt{2m_P}} = 1, \tag{13}$$

we obtain, using the representation (10), the following normalization condition for the reduced w.f. $\phi(\vec{p})$:

$$\int\frac{d^3\vec{p}}{(2\pi)^3}\cdot\left|\frac{\phi(\vec{p})}{\sqrt{m_P}}\right|^2 = 1. \tag{14}$$

Taking advantage of the relation

$$\phi(\vec{p}) \Longrightarrow 2\varepsilon(\vec{p})\cdot m_P^{-1/2}\cdot\tilde{\Psi}^{N.R.}(\vec{p}) \tag{15}$$

between the reduced w.f. $\phi(\vec{p})$ and nonrelativistic w.f. $\tilde{\Psi}^{N.R.}(\vec{p})$ and taking account of the relations (6)-(8), we derive from (12) the nonrelativistic equation for $\tilde{\Psi}^{N.R.}(\vec{p})$:

$$(E_B + \vec{p}^{\,2}/2m_R)\,\tilde{\Psi}^{N.R.}(\vec{p}) =$$

$$= -\int\frac{d^3\vec{p}\,'}{(2\pi)^3}\,V_s^{N.R.}(\vec{p}-\vec{p}\,')\cdot\tilde{\Psi}^{N.R.}(\vec{p}\,') \tag{16}$$

with the normalization condition

$$\int\frac{d^3\vec{p}}{(2\pi)^3}\cdot\left|\tilde{\Psi}^{N.R.}(\vec{p})\right|^2 = 1. \tag{17}$$

3. THE AMPLITUDE OF DECAY $P \to \ell\bar{\ell}$

Consider the decay $P \to \ell\bar{\ell}$ drawn in a two-loop form in Fig. 1. In the one-time approach to quantum field theory the $P \to \ell\bar{\ell}$ - decay amplitude to lowest order in QED and QCD is given by

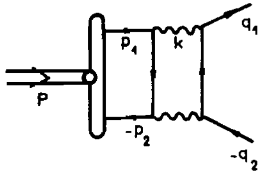

Fig.1

$$A(P\to\ell\bar\ell)=\frac{1}{(2\pi)^3}\int\frac{d^3\vec p}{2\varepsilon(\vec p)}\cdot\frac{d^4k}{(2\pi)^4}Tr\left\{\Psi^{\sigma_1\sigma_2}(\vec p)\cdot T^{*}_{\sigma_1\sigma_2}(p,P;k,q_1)\right\}+ \tag{18}$$
$$+\ \text{crossed},$$

where the amplitude $T_{\sigma_1,\sigma_2}(p,P,k,q_1)$ is of the form

$$T_{\sigma_1\sigma_2}(p,P;k,q_1)=\frac{\bar u_{\sigma_1}(p)(-ie\gamma^{\mu})(\hat p-\hat k+m_q)(-ie\gamma^{\nu})v_{\sigma_2}(\bar p)}{(p-k)^2-m_q^2+i\varepsilon}\cdot L_{\mu\nu} \tag{19}$$

with

$$L_{\mu\nu}=\frac{\bar u(q_1)(-ie\gamma_\mu)(\hat q_1-\hat k+m_e)(-ie\gamma_\nu)v(-q_2)}{[k^2+i\varepsilon][(P-k)^2+i\varepsilon][(q_1-k)^2-m_e^2+i\varepsilon]}. \tag{20}$$

Due to CP-invariance, the amplitude (18) in the Feynman representation may be written as follows:

$$A(P\to\ell\bar\ell)=i\,\bar u(q_1)\gamma_5\,W(t)\,v(-q_2), \tag{21}$$

where

$$t=(q_1+q_2)^2=P^2.$$

Inserting $\Psi_{\sigma_1\sigma_2}(\vec p)$ (10) in the spinor representation into (18) and comparing the r.h.s. of eqs. (18) and (21), we obtain the following expression for the decay form factor $W(t)$:

$$\bar u(q_1)\gamma_5 W(t)v(-q_2)=$$
$$=4e^2m_q^2\int\frac{d^4k}{(2\pi)^4}\cdot\varepsilon^{\alpha\mu\beta\nu}P_\alpha k_\beta L_{\mu\nu}\cdot \tag{22}$$
$$\cdot\frac{1}{(2\pi)^3}\int\frac{d^3\vec p}{2\varepsilon(\vec p)}\cdot\frac{\phi(\vec p)}{2\varepsilon(\vec p)}\cdot\frac{1}{(p-k)^2-m_q^2+i\varepsilon}.$$

Considering that in the decay $P\to\ell\bar\ell$ the only possible final state is the ${}^1S_0$ -state$(CP|(\ell\bar\ell)_{{}^1S_0}\rangle=-|\ell\bar\ell\rangle_{{}^1S_0})$ we should insert into the r.h.s. of (22) the in-projection operator $P^{(0)}_{in}(q_1,q_2)$ /18/

$$P^{(0)}_{in}(q_1,q_2)=$$
$$=\frac{-1}{2(2t)^{1/2}}\left[2m_e(\hat q_1+\hat q_2)\gamma_5+\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}(q_2^{\rho}q_1^{\sigma}-q_1^{\rho}q_2^{\sigma})\sigma^{\mu\nu}+t\gamma_5\right] \tag{23}$$

that transforms the tensor $L_{\mu\nu}$ into a singlet lepton state $(\ell\bar\ell)_{{}^1S_0}$. Note that

$$Tr\left\{P^{(0)}_{in}(q_1,q_2)\gamma^{\sigma}(\hat q_1-\hat k+m_e)\gamma^{\rho}\right\}=\frac{-4i\,m_e}{(2t)^{1/2}}\varepsilon^{\rho\nu\sigma\mu}P_\mu k_\nu. \tag{24}$$

Introducing into the l.h.s. of eq.(22) the out-projection operator $P^{(0)}_{out}(q_1,q_2)$ /18/

$$P^{(0)}_{out}(q_1,q_2)=$$
$$=\frac{1}{2(2t)^{1/2}}\left[-2m_e(\hat q_1+\hat q_2)\gamma_5+\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}(q_1^{\rho}q_2^{\sigma}-q_2^{\rho}q_1^{\sigma})\sigma^{\mu\nu}+t\gamma_5\right] \tag{25}$$

we obtain the following expression for the form factor $W(t)$

$$W(t)=2ie^2m_e\cdot m_P^{-2}\cdot$$
$$\cdot\int\frac{d^4k}{(2\pi)^4}\cdot\frac{2[P^2\cdot k^2-(P\cdot k)^2]\cdot F_P(k^2,(P-k)^2)}{[k^2+i\varepsilon][(P-k)^2+i\varepsilon][(q_1-k)^2-m_e^2+i\varepsilon]}, \tag{26}$$

where the two-photon transition form factor $F_P(k^2,(P-k)^2)$ is determined by the reduced w.f. $\phi(\vec p)$. In the end, we will be interested in the branching ratio of the decay $P\to\ell\bar\ell$:

$$Br(P\to\ell\bar\ell/\gamma\gamma)=\frac{8}{m_P^2}(1-4m_e^2/m_P^2)^{1/2}\cdot\left|\tilde W(m_P^2)\right|^2, \tag{27}$$

where m_e is the lepton mass, $\tilde W(m_P^2)=W(t)/F_P(0,0)$ is the relative form factor of the decay (26) with $t=m_P^2$, $F_P(0,0)$ is the two-photon form factor for on-shell photons.

4. TWO-PHOTON TRANSITION FORM FACTOR

Drell in his earlier calculations/19/ has found that an integral like (26) is divergent if F_P = const at any values of the arguments. So, we shall define the form factor $F_P(k^2,(P-k)^2)$ as follows/20/:

$$F_P(k^2,(P-k)^2)=\frac{\alpha\cdot m_P\cdot m_q}{\pi\cdot\lambda^{1/2}}\cdot Z_q\int_0^{\infty}\frac{dp\cdot p}{p^2+m_q^2}\cdot\phi(p)\ln\left|\frac{m_P\cdot k^2-(m_P^2+\varkappa)\varepsilon(p)+\sqrt{\lambda}\cdot p}{m_P\cdot k^2-(m_P^2+\varkappa)\varepsilon(p)-\sqrt{\lambda}\cdot p}\right| \tag{28}$$
$$+\ \text{crossed},$$

where $p\equiv|\vec p|$, Z_q is the charge-colour factor of quarks,

$$\lambda\equiv m_P^4+S_1^2+S_2^2-2m_P^2\cdot S_1-2m_P^2\cdot S_2-2S_1\cdot S_2;$$
$$S_1\equiv k^2,\quad S_2\equiv(P-k)^2,\quad \varkappa\equiv S_1-S_2,\quad \varepsilon(p)\equiv(p^2+m_q^2)^{1/2}.$$

In what follows it will be convenient to consider the quantity $\tilde{F}_P$ defined by the form factor (28) normalized to the $P \to \gamma\gamma$ decay constant $F_P(0,0)$:

$$\tilde{F}_P = F_P(k^2, (P-k)^2) / F_P(0,0), \tag{29}$$

where

$$F_P(0,0) = \frac{-4\cdot\alpha\cdot m_q}{\pi\cdot m_P}\cdot z_q \cdot \int_0^\infty \frac{dp\cdot p}{p^2+m_q^2}\cdot \phi(p)\cdot \ln\left|\frac{\varepsilon(p)+p}{m_q}\right|. \tag{30}$$

The form factor $F_P(0,0)$ can be expressed in terms of the quarkonium w.f. at zero relative separation (the so-called static limit)

$$F_P^{STAT}(0,0) = -\frac{8\cdot\pi\cdot\alpha}{m_P\cdot m_q^2}\cdot z_q\cdot \tilde{\Psi}(0).$$

The asymptotics of the transition form factor $\tilde{F}_P$ (29) when $k^2 \to \infty$ has the form

$$\tilde{F}_P \xrightarrow[k^2\to\infty]{} \frac{\alpha\cdot m_P\cdot m_q}{\pi\cdot k^2}\cdot N(m_P; m_q), \tag{31}$$

where

$$N(m_P; m_q) = F_P^{-1}(0,0)\cdot \int_0^\infty \frac{dp\cdot p}{p^2+m_q^2}\cdot \phi(p)\cdot \ln\left|\frac{m_q^2 - m_P\cdot(\varepsilon(p)+p)}{m_q^2 - m_P\cdot(\varepsilon(p)-p)}\right|. \tag{32}$$

Thus, at large k^2 the transition form factor behaves like k^{-2} (see[21]) and the integral (26) is convergent.

In the weak-coupling limit ($m_P \simeq 2m_q$, m_q , are masses of c-, b-, t-quarks, and m_P are masses of $c\bar{c}$-, $b\bar{b}$-, $t\bar{t}$-quark bound states) we have

$$\tilde{F}_P \xrightarrow[k^2\to\infty]{} \frac{\alpha\cdot m_P^2}{4\cdot\pi\cdot k^2}\cdot N_1(m_P; m_q), \tag{33}$$

where

$$N_1(m_P; m_q) = F_P^{-1}(0,0)\cdot \int_0^\infty \frac{dr}{r+(m_P/2)^2}\cdot \varphi(r)\cdot \ln\left|\frac{m_P - 4\sqrt{r+(m_P/2)^2}+\sqrt{r}}{m_P - 4\sqrt{r+(m_P/2)^2}-\sqrt{r}}\right| \tag{34}$$

and $\varphi(r)$ is the w.f. of a weakly bound quark system in the momentum representation.

In recent years the two-photon transition form factor has been calculated within the QCD approach[22] in the relativistic theory of a bound state. There the function $F_P(s_1, s_2)$ was expressed in terms of a three-dimensional integral of the w.f. in the momentum space which at least for heavy quark states may be interpreted as a solution to the Schrödinger equation with one of the phenomenological "true" potentials. Note that $F_P(s_1, s_2)$, according to formula (7) from[23], and the same form factor at $s_1 = s_2 = 0$ are proportional to $m_P^{-5/2}\cdot m_q$, which is incorrect as compared to earlier results obtained by the same author and others[9] for $F_P(0,0)$ and differing from the latter from[23] by a factor $m_P\cdot m_q^{-1}$ (the wave functions in both refs. are meant the same and normalized in the same manner). Making use of relation (15), we immediately obtain from formula (30) the expression for $F_P(0,0)$[9] up to a factor e^2 . For further calculations we need the value of the w.f. that may be found as a solution to eq.(5) with the phenomenological kernel of quark interaction (quasipotential). Note that the quark-interaction mechanism inside a hadron admitting quark confinement, i.e. at long distances, cannot be yet explained within QCD. Therefore an interaction of that sort is to be approximated with phenomenological linear, logarithmic (or a sume of them), or oscillator types of potentials which for the most part determined the shape of the mass spectrum of light mesons.

If we set ourselves the task of gaining any information on the transition form factor (28), without knowing the explicit form of the w.f., we should investigate the static limit of the form factor $\tilde{F}_P^{STAT}$ (the w.f. in the integrand in (28) is taken at $|\vec{p}| = 0$)

$$\tilde{F}_P^{STAT}(s_1, s_2) = \frac{m_P^3}{2\cdot m_q\cdot\lambda}\cdot(2m_q\cdot m_P - s_1 - s_2), \tag{35}$$

where

$$\lambda \equiv m_P^4 + s_1^2 + s_2^2 - 2m_P^2\cdot s_1 - 2m_P^2\cdot s_2 - 2s_1\cdot s_2,$$

$$s_1 \equiv k^2, \quad s_2 \equiv (P-k)^2.$$

For decays of heavy quarkonia, in the weak-coupling limit we get

$$\tilde{F}_P^{STAT}(x_1, x_2) \cong \left(1 - x_1 - x_2 - \frac{4\cdot x_1\cdot x_2}{1-x_1-x_2}\right)^{-1}, \tag{36}$$

where $x_i = s_i/m_P^2$, $i = 1, 2$.

It is instructive to compare formula (36) with the phenomenological form factor with correct analytic properties obtained in[24]:

$$\tilde{F}(s_1,s_2) = \Lambda^2/(\Lambda^2 - s_1 - s_2), \tag{37}$$

where the parameter Λ is determined by the ρ -meson mass ($\Lambda^2 \simeq m_\rho^2$) from the analysis of the data on $P \to \gamma \ell \bar{\ell}$ - decay. For decays of heavy particles at small x_i the form factor (36) takes the form

$$\tilde{F}_P^{STAT}(x_1, x_2) \cong 1 + x_1 + x_2 . \tag{38}$$

From comparison of (38) with (37) we see that $m_P^2 \cong \Lambda^2$, i.e. the mass parameter Λ gets large, which is inconsistent with the assumption[24] that $\Lambda^2 \simeq m_\rho^2$ throughout the whole range of variation of x_i.

It is known[20] that the "inclusion" of the transition form factor in the static limit leads, e.g. in decays $P \to \gamma \ell \bar{\ell}$, to overestimated values of the decay widths, whereas the application of nonrelativistic Coulomb and oscillator wave functions leads to a 2--3-times decrease of the same quantities as compared to the experimental data. We will here compute the transition two-photon form factors exactly. So, some difficulties in theoretical analysis of the light mesons as bound states of a constituent quark and antiquark lead to that when working beyond the static approximation it is necessary to employ model relativistic wave functions.

5. THE FORM FACTOR OF $P \to \ell \bar{\ell}$ - DECAY. DISPERSION CALCULATIONS.

In[3] the quantity $Re\,W(0)$ has been computed in the quark triangular-loop model and in the vector dominance model. Both the models are specified by their particular "model" constant terms. Specifically, in the quark triangular-loop model[3] based on analogous though incorrect computations of the baryon loop[25] it has been found that[3]:

$$Re\,\tilde{W}(0) = -\frac{3\alpha \cdot m_e}{2\cdot\pi}\cdot\left[\ln(m_q/m_e) + 17/12 + \cdots\right], \tag{39}$$

where the second term in the r.h.s. is a "model" constant neglected in subsequent calculations. However, at $m_q = 0.300$ GeV taken in[3] the second term in (39) amounts to about 26% of the first one.

Scadron and Visinescu[6] from analogous computations within the soft momentum approximation in the quark triangular-loop model have made the conclusion that the width of $\pi^0 \to e^+e^-$, $\eta \to \mu^+\mu^-$, $K_L \to \mu^+\mu^-$ decays do not depend on the quark masses. Direct computation of $W(t)$

by formula (26) with (28) is practically a complicated problem. Therefore, we make use of the once-subtracted dispersion relation

$$Re\,\tilde{W}(t) = Re\,W(t_0) + \frac{t - t_0}{\pi}\int_0^\infty \frac{dt'\, Im\,\tilde{W}(t')}{(t'-t)(t'-t_0)}, \tag{40}$$

where the first term in the r.h.s. is a contribution with the quarkonium-mass squared approaching zero. If $Im\,\tilde{W}(t)$ is calculated only with the two-photon intermediate state (that mainly contributes, at least, for light mesons), we get the model independent quantity ($t > 4m_e^2$)

$$Im\,\tilde{W}(t) = -\frac{\alpha\cdot m_e}{4}\cdot\frac{1}{\beta_t}\cdot\ln\left(\frac{1+\beta_t}{1-\beta_t}\right), \tag{41}$$

where $\beta_t = (1 - 4m_e^2/t)^{1/2}$ and m_e is the lepton mass. The imaginary part of the form factor at $t = m_P^2$ does not depend on the mass of a constituent quark, m_q.

With (41) the second term in the r.h.s. of (40) can be expressed as follows:

$$Re\,\tilde{W}(\beta) = \frac{\alpha\cdot m_e}{2\cdot\pi\cdot\beta}\cdot\left[\frac{1}{4}\cdot\ln^2\left(\frac{1-\beta}{1+\beta}\right) + \frac{\pi^2}{12} - \mathcal{L}i_2\left(\frac{\beta-1}{\beta+1}\right)\right], \tag{42}$$

where $\mathcal{L}i_2(z)$ is a dilogarithmic function. Expanding (42) in powers of m_e/m_P we obtain:

$$Re\,\tilde{W}(\beta) \cong \frac{\alpha\cdot m_e}{2\cdot\pi}\cdot\left[\ln^2(m_e/m_P) + \pi^2/12\right]. \tag{43}$$

The first term in the r.h.s. of (40) is model- dependent and is a function of

$$Re\,\tilde{W}(m_P^2 \cong 0) \cong -\frac{2\alpha\cdot m_e}{\pi}\cdot\ln\left(\frac{\Lambda}{m_e}\right)\cdot\left[1 + \frac{\Delta(m_q)}{2\cdot m_q^2\cdot\tau(m_q)}\right], \tag{44}$$

where

$$\Delta(m_q) \equiv \int_0^\infty \frac{d|\vec{p}|\cdot\vec{p}^{\,2}}{(\vec{p}^{\,2} + m_q^2)^{1/2}}\cdot\phi(|\vec{p}|), \tag{45}$$

$$\tau(m_q) \equiv \int_0^\infty \frac{d|\vec{p}|\cdot|\vec{p}|}{\vec{p}^{\,2} + m_q^2}\cdot\ln\left|\frac{\mathcal{E}(\vec{p}) + |\vec{p}|}{m_q}\right|\cdot\phi(|\vec{p}|), \tag{46}$$

the hadron cut-off parameter Λ ($\Lambda^2 > m_P^2$) may be taken to be $\Lambda \simeq 2m_q$, on the basis of Q^2 -duality formulated in the analysis of total cross sections of e^+e^- - annihilation. The Q^2 -duality implies equivalence of the calculations made within the phenomenological vector-dominance model and the dynamic model of quark triangular loops (with constituent quark masses) with the quark-hadron vertex (wave) function in the bound-state theory. Expression (43) for $Re\,\tilde{W}(\beta)$ is an exact result as in the case of light mesons there do not exist, besides $\gamma\gamma$ -, other intermediate states that would contribute to the amplitude as a function of the constant α . Generally speaking, there is possible one more contribution to the amplitude dependent on details of the transitional form factor and wave function. This contribution caused by other intermediate states, e.g. by $q\bar{q}$ - and $q\bar{q}\gamma$ - states, may appear only for $t > 4m_q^2$ In our consideration contributions like that are suppressed owing to $m_P^2 < 4m_q^2$.

In the static limit expression (44) becomes model-independent and looks as follows:

$$Re\,\tilde{W}^{STAT}(m_P^2=0) \cong -\frac{3\cdot\alpha\cdot m_e}{\pi}\cdot \ln(\Lambda/m_e). \qquad (47)$$

For comparison with the earlier results/3-7/, we explicitly write the expression for $Re\,\tilde{W}(m_P^2)$:

$$Re\,\tilde{W}(m_P^2)=\frac{\alpha\cdot m_e}{2\cdot\pi}\left\{\ln^2\left(\frac{m_e}{m_P}\right)+\frac{\pi^2}{12}-4\cdot\ln\left(\frac{\Lambda}{m_e}\right)\cdot\left[1+\delta(m_q)\right]\right\}, \qquad (48)$$

where

$$\delta(m_q) = \Delta(m_q)/[2\cdot m_q^2\cdot\tau(m_q)]. \qquad (49)$$

Formula (48) differs from the results of refs./3-7/. First, the $Re\,\tilde{W}(m_P^2)$ in (48) depends on the wave function (specified by the model) of a bound quark state; second, it is a logarithmic function of the cut-off parameter Λ ; third, the squared mass of a constituent quark, m_q^2 , governs the behaviour of the decay form factor $\tilde{W}(m_P^2)$.

For quantitative estimates of $Br(P\to\ell\bar{\ell}/\gamma\gamma)$ by formula (27), e.g. for the decay $\pi^0\to e^+e^-$, we make use of the model relativistic wave function

$$\phi(\vec{p}^{\,2}) = C_N\cdot exp\{-\vec{p}^{\,2}/2\nu^2\}, \qquad (50)$$

obeying eq. (12) with the quark-confinement potential (as a generalized function):

$$V(\vec{p}-\vec{p}\,')=-(2\pi)^3\cdot\frac{m_{u,d}^2\cdot\nu^4\cdot\mathcal{E}(\vec{p})\cdot\mathcal{E}(\vec{p}\,')\cdot\nabla_{\vec{\pi}}^2\,\delta^{(3)}(\vec{p}-\vec{p}\,')}{4\cdot[\mathcal{E}(\vec{p})+m_{u,d}]\cdot[2\cdot\mathcal{E}(\vec{p})\cdot\mathcal{E}(\vec{p}\,')-m_{u,d}^2]}, \qquad (51)$$

where $\nabla_{\vec{\pi}}^2 \equiv d^2/d\vec{\pi}_P^{\,2}$ is the Laplacian in the three-dimensional Euclidean space of half-momenta $\pi_P = (\pi_P^0, \vec{\pi}_P)$

$$\vec{\pi}_P^{\,2} = \vec{p}^{\,2}\cdot\frac{m_{u,d}}{2\,[\mathcal{E}(\vec{p})+m_{u,d}]}, \qquad (52)$$

C_N is a constant determined by the normalization condition (14), ν is a scale parameter of the potential of an order of the meson mass, $m_{u,d}$ is the $u-(d)$-quark mass. In the nonrelativistic limit the potential (51) becomes/26/

$$V^{N.R.}(\vec{p}-\vec{p}\,')=-(2\pi)^3\cdot\frac{m_{u,d}\cdot\nu^4}{2}\cdot\nabla_{\vec{p}}^2\,\delta^{(3)}(\vec{p}-\vec{p}\,'), \qquad (53)$$

where $\nabla_{\vec{p}}^2 \equiv d^2/d\vec{p}^{\,2}$ is the Laplacian in the three-dimensional Euclidean space of momenta $p=(p^0,\vec{p})$. The dependence of $Re\,\tilde{W}(t)/Im\,\tilde{W}(m_\pi^2)$ on the squared meson mass t is shown in Fig. 2 at $m_{u,d}$ = 0.300 GeV; 0.330 GeV; 0.400 GeV. The parameter ν =0.450 GeV has been determined from the calculation of the mass difference between the π -meson excitations. Throughout the whole range $(2\cdot m_e)^2 \leq t \leq m_\pi^2$,the quantity $Re\,\tilde{W}(t)<0$.The curves in Fig. 2 differ from the results obtained in/5/ where the function $Re\,W^{\Phi\iota}(t)$ changes in sign at $t \cong 0.2\,m_\pi^2$ and at $t=m_\pi^2$ $Re\,\tilde{W}^{\Phi\iota}(m_\pi^2)/Im\,\tilde{W}^{\Phi\iota}(m_\pi^2)=-0.45$. The negative sign of the function $Re\,\tilde{W}^{\Phi\iota}(t)/Im\,\tilde{W}^{\Phi\iota}(m_\pi^2)$ at $t \gtrsim 0.2\,m_\pi^2$ /5/ is due to a large contribution to $Re\,\tilde{W}^{\Phi\iota}(t)$ from the intermediate $q\bar{q}$ -state (almost twice of the contribution of the intermediate $\gamma\gamma$ -state) that is opposite in sign to $Im\,\tilde{W}^{\Phi\iota}(m_\pi^2)$. That contribution was a result of the approximation of the coupling of π -meson with constituent quarks ($m_{u,d}$ = 0.300 ÷ 0.400 GeV) only by the γ_5 - matrix. This is not quite correct as in the quark triangular-loop model the influence of intermediate (besides $\gamma\gamma$-) states is possible only for $m_P^2 > \Lambda^2$ (or, owing to the Q^2 -duality, for $m_P^2 \gtrsim (2m_q)^2$), which is clearly in poor agreement with the constituent u - and d - quarks in the case of π -meson. In Fig. 3 we present the results of calculations of $\Delta Br(\pi^0\to e^+e^-/\gamma\gamma) = Br^{THEOR} - Br^{exp}$ for the values of constituent quark masses $m_{u,d}$ = (0.300÷0.400) GeV and $Br^{exp}(\pi^0\to e^+e^-/\gamma\gamma)=$ = (1.82 ± 0.61) 10^{-7}/27/. Note that the variation of $m_{u,d}$ in the interval from 0.300 GeV to 0.400 GeV slightly influences the magnitude of ΔBr . Total deviation $|\Delta Br|$ amounts to ~ $0.3\cdot10^{-7}$, which

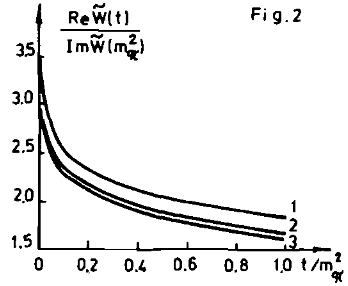

Fig.2

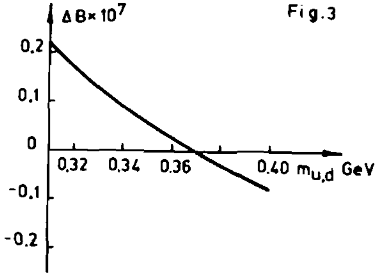

Fig.3

is half the error in determination of $Br^{exp}(\pi^o \to e^+e^-/\gamma\gamma)$ /27/. At $m_{u,d}$ = 0.370 GeV we get $Br^{THEOR}(\pi^o \to e^+e^-/\gamma\gamma) = 1.82\cdot 10^{-7}$. The analysis of (48) has revealed a weak model (m_q-) dependence of $Re\,\tilde{W}(t)$ in the region of large t ($t \gtrsim 0.6\, m_p^2$).

6. CONCLUSIONS

1. In the one-time approach to quantum field theory the expression is found for $Re\,\tilde{W}(m_p^2)$ (48) that explicitly depends on the model relativistic quarkonium wave function $\phi(\vec{p}^{\,2})$ and hadron cut-off parameter Λ .

2. The hadron cut-off parameter $m_p \le \Lambda \le 2m_q$ is characteristic of the given model and is significant for calculation of $Re\,\tilde{W}(m_p^2{=}0)$ (44).

3. The constituent-quark mass squared m_q^2 in (48) defines the scale of the decay form factor $\tilde{W}(m_p^2)$.

4. The ratio $Br^{THEOR}(\pi^o \to e^+e^-/\gamma\gamma)$ has been calculated with the wave function $\phi(\vec{p}^{\,2})$ (50) that makes an extra positive contribution to $Re\,\tilde{W}(m_\pi^2)$ (see Fig. 3). The results for ΔBr are in good agreement with the experimental data /27/ $Br^{exp}(\pi^o \to e^+e^-/\gamma\gamma) = (1.82 \pm 0.61)\cdot 10^{-7}$.

5. The ratio $Br^{THEOR}(\pi^o \to e^+e^-/\gamma\gamma)$ turns out to be slightly sensitive to the change of the parameter $m_{u,d}$ (for a fixed ν) entering into the wave function $\phi(\vec{p}^{\,2})$ within the considered model with constituent quarks.

Our further purpose is to find numerical estimates for $Br(\eta \to e^+e^-/\gamma\gamma)$, $Br(\eta \to \mu^+\mu^-/\gamma\gamma)$ with the use of wave functions of the $u\bar{u}(d\bar{d})$-and $s\bar{s}$ -quarkonia and to study the model and Λ - dependence of differences of the real parts of the amplitudes of decays $\pi^o \to e^+e^-$, $\eta \to e^+e^-$, $\eta \to \mu^+\mu^-$, $K_L^o \to e^+e^-$, $K_L^o \to \mu^+\mu^-$.

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REFERENCES

1. Hyams B.D. et al. - Phys.Lett., 1969, 29B, 128.
2. Landsberg L.G. - Phys. Rep ., 1985, 128, 301.
3. Bergström L. et al. - Phys. Lett., 1983, 126B, 117.
4. Ametller L. et al. - Phys. Rev., 1984, D30, 251.
5. Ametller L. et al. - Nucl. Phys., 1983, 228B, 301.
6. Scadron M.D., Visinescu M. - Phys. Rev., 1984, D29, 911.
7. Pich A., Bernaben J. - Z. Phys., 1984, 22C, 197.
8. Bergström L. et al.,-Z.Phys., 1980, 4C, 215.
9. Bergström L. et al. - Phys. Lett., 1979, 82B, 419.
10. Logunov A.A., Tavkhelidze A.N. - Nuovo Cimento, 1963, 29, 380.
11. Kadyshevsky V.G. - Sov. J. JETPh., 1964, 46, 654; 872.
12. Logunov A.A. et al.-Sov. J. Theor. Math. Ph., 1971, 6, 157.
13. Faustov R.N. - Ann. of Phys., 1973, 78, 176.
14. Shirokov Yu.M.-Sov. J. JETPh., 1958, 35, 1005.
15. Savrin V.I., Skachkov N.B., in:Proc. of the V Int. Sem. on High Energy Physics and Quantum Field Theory, Protvino, 1982, v. II, 229.
16. Kadyshevsky V.G., Mateev M.D. - Nuovo Cim., 1968, 55A, 275.
17. Faustov R.N., Helashvily A.A. - Sov. J. Nucl. Phys., 1969, 10, 1085.
18. Martin B.R. et al. - Phys. Rev., 1970, D2, 179; (E: Phys. Rev., 1971, D4, 272).
19. Drell S.D. - Nuovo Cimento, 1959, II, 693.
20. Kozlov G.A. et al. - Z. Phys. 1983, 21, 63.
21. Cornwall J.M. - Phys. Rev. Lett., 1966, 16, 1174; Suura H. et al. - Nuovo Cimento, 1972, 4, 505; Walsh T.F., Zerwas P. - Nucl.Phys., 1972, 41B, 551.
22. Bergström L. - Royal Inst. Tech. Stockholm Preprint TRITA-TRY-82-5 and Thesis, 1981.
23. Bergström L. - Z. Phys., 1982, 14C, 129.
24. Berman S.H., Geffen D.A. - Nuovo Cimento, 1960, 18, 1193.
25. Pratap M., Smith J. - Phys. Rev., 1972, D5, 2020.
26. Кулешов С.П., Сидоров А.В., Скачков Н.Б. - Сообщения ОИЯИ Р2-82-7, Дубна, 1982.
27. Mischke R.E. et al. - Phys. Rev. Lett., 1982, 48, 1153.

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