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E2-88-155

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SPONTANEOUS COMPACTIFICATION TO HOMOGENEOUS SPACES

Submitted to "Zeitschrift für Physik C. Particles and Fields"

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1988

I. Introduction

Lately we have witnessed a growing interest both in the interpretation of multidimensional models from the point of view of four-dimensional space-time (dimensional reduction) /1-6/, and in the development of methods of finding classical solutions of multidimensional Einstein-Yang-Mills (EYM) equations, corresponding to the factorized structure of space-time: $E = M^4 \times K$, where M^4 is the Minkowski space-time and K is a compact "internal" space with a size of the Plank's length. In the theory of spontaneous compactification (SC) such solutions are treated as vacua of the multidimensional theory /7-15/.

Recently an interesting link between dimensional reduction and SC, was discovered /8-10,13/. It consists in the fact that, on one hand, solutions of EYM equations can be interpreted in four dimensions and, on the other hand, multidimensional background gauge fields leading to SC are easily obtained from certain solutions of the reduced theory (the extrema of the scalar fields potential) /8-14/.

So far the majority of papers on these subjects (especially in SC theory) utilized symmetric coset spaces S/R as internal spaces K. The reason is that the Einstein part of SC equations for symmetric coset spaces is trivially satisfied.

However it is of interest to investigate the general case of coset spaces /13-17/. In the present paper we extend the results obtained by other authors, with the use of a "normal" metric on S/R /13,15/, by taking the most general form of an

S-invariant metric which gives us a possibility of obtaining a considerably wider set of compactifying solutions. The normal metric is the metric obtained by one parameter rescaling of a vielbein (see /13,18/) or equivalently is the metric proportional - at the "origin" of S/R (i.e. the coset $[e]$=R) - to the restriction of the Killing form in $\mathcal{S}$:=Lie(S) to the second term in the reductive decomposition $\mathcal{S}=\mathcal{R}\oplus\mathcal{M}$, $\mathcal{R}$ =Lie(R)/19/.

This paper is organized as follows. In sect.2 we introduce notations and discuss the link between coset space dimensional reduction (CSDR) scheme and SC theory. In sect.3 we show that Einstein equation in the SC theory is equivalent to a set of nonlinear algebraic equations for the parameters $\{M\}$ which define the Riemannian geometry of S/R. If one restricts oneself to the normal metrics on S/R these equations are equivalent to those found in /13,15/. In the last section we apply the developed theory to searching and analysing different solutions leading to spontaneous compactification of extra dimensions to six-dimensional non-symmetric space SO(5)/SU(2)$\times$ U(1). In addition to the solution with the normal metric and the gauge group G=R=SU(2)$\times$U(1) found in /15/, we find new solutions for a model discussed in /16/. It turns out that in this case there exist no solutions with the normal metric. In this model, as was shown in /16/, the spontaneous symmetry breaking in the reduced theory occurs only for a subset of invariant metrics on SO(5)/SU(2)$\times$ U(1) (In /17/ another model was found with the same properties). The solutions of the SC equations correspond to this subset, i.e. dynamics imposes a spontaneous symmetry breaking.

In the following, the space symmetry group and the gauge group G are supposed to be compact simple Lie groups.

2. CSDR Scheme and the SC Equations

We start from the standard EYM action for a coupled system of gravity and gauge fields in 4+d - dimensional space-time

$$S = S_{gauge} + S_{grav}, \qquad (1a)$$

where

$$S_{gauge} = \frac{1}{8e^2}\int_E d^{4+d}z\,\sqrt{-g}\,\langle F_{MN}, F^{MN}\rangle \qquad (1b)$$

$$S_{grav} = \int_E d^{4+d}z\,\sqrt{-g}\left\{\frac{1}{16\pi k}R - \Lambda\right\}. \qquad (1c)$$

here M=0,...,3+d, k and e are 4+d - dimensional gravitational and gauge constants, Λ is the cosmological constant and $\langle\, ,\rangle$ denotes the canonical invariant scalar product in the Lie algebra $\mathcal{G}$=Lie(G) (for the sake of definiteness both $\langle\, ,\rangle$ and the canonical scalar product on $\mathcal{S}$ -(,), are normalized so that the squares of the longest roots, with respect to them, equal 2).

As is usual in the SC theory, we look for solutions of multidimensional EYM equations corresponding to the following factorization of the space-time E:E=$M^4\times$ S/R with a metric of the type g= $\eta\oplus\gamma$ (where η =diag(-1,1,1,1) is the Minkowski metric and γ is a metric on S/R).

If we now make the ansatz A_μ =0 and $A_\alpha = A_\alpha(y)$ for the gauge fields (here $z^M=(x^\mu,y^\alpha)$,μ=0,1,2,3 and α =1,...,d) we get from the EYM equations the SC equations

$$R_{\alpha\beta} = -\frac{4\pi k}{e^2}\langle F_{\alpha\gamma}, F_\beta{}^\gamma\rangle. \qquad (2a)$$

$$\nabla_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}] = 0. \qquad (2b)$$

For the solutions of (2) to be solutions of EYM equations we have to choose Λ in (1C) to be equal to

$$\Lambda = -(1/8e^2)\langle F_{\alpha\beta}\ ,\ F_{\alpha'\beta'}\rangle \gamma^{\alpha\alpha'}\gamma^{\beta\beta'}.$$

Let us restrict ourselves to S - symmetric solutions $\gamma_{\alpha\beta}$ and A_α of SC equations. As is usual in CSDR scheme the S - symmetry of the theory means that physical observables are invariant under the action of S, i.e., γ remains invariant, whereas A_α undergoes a gauge transformation /1,2/.

The S-symmetry turns out to be very important in solving SC equations, because, in the class of S-symmetric fields, this very complicated set of nonlinear differential equations is equivalent to a set of nonlinear algebraic equations. This is because S-symmetry enables us to solve SC equations at one point $y_0 \in$ S/R and then to extend the solutions to the whole S/R.

The aim of SC theory is to find solutions of EYM equations which could be interpreted as vacuum expectation values of the bosonic fields g and A_M

$$\langle g_{MN}\rangle_0 = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \gamma_{\alpha\beta} \end{pmatrix},\ \langle A_M\rangle_0 = (0, A_\alpha).$$

Therefore S-symmetry of the fields is naturally required by the space homogeneity of vacuum solutions.

Besides finding solutions of (2) various physical properties of these solutions, for example, stability under classical fluctuations /8,14,20/, are analysed in literature. Below we will propose, in agreement with /8,13/, a physical classification for SC solutions.

It appears that the CSDR scheme is very useful in finding and interpreting S-symmetric solutions of (2b) /8–13/. Let us summarize for later convenience the main aspects of this scheme when applied to the dimensional reduction of pure gauge theories. It is well known /1,2,21/ that every S-symmetric gauge field $A_M(x,y)$ on $M^4 \times S/R$ defines a homomorphism τ : R $\rightarrow$ G. Moreover, this field is in a one-to-one correspondence with a pair of objects on M^4: $\{\tilde{A}_\mu(x), \phi(x)\}$ where $\tilde{A}_\mu(x)$ is a gauge field with the gauge group H, H= $\{g \in G,\ g\tau(r)g^{-1} = \tau(r)$ for all $r \in R\}$ (i.e. the centralizer of τ(R) in G) and $\phi(x)$ is, for all x e M^4, a linear mapping $\phi(x): \mathcal{M} \rightarrow \mathcal{G}$ pertinent to the scalar fields on M^4 and satisfying the constraints

$$Ad\,\tau(r) \circ \phi(x) = \phi(x) \circ Ad\,r\ , \quad for\ all\ r \in R, \qquad (3)$$

which are a reminder of the S-symmetry of A_M. Conversely A_M can be reconstructed from $\{\tilde{A}_\mu(x), \phi(x)\}$ as follows. Let $\sigma(y) \in S$ be representatives in cosets $\sigma(y)$R (i.e. a local section in the principal fibre bundle S $\rightarrow$ S/R). The Cartan 1-form on S/R is defined as the pull back of the canonical left invariant form on the group S

$$e_y = \sigma^{-1}(y)\, d\sigma(y), \qquad (4a)$$

$$e = \tilde{T}^a T_a + \tilde{T}^m T_m := \omega + \theta\ , \qquad (4b)$$

where $\{T_A\} = \{T_a, T_m\}$, $T_a \in \mathcal{R}$, $T_m \in \mathcal{M}$ is an orthonormal basis for $\mathcal{S}$: $(T_A, T_B) = -\delta_{AB}$. Then

$$Tr\, ad\,T_A\, ad\,T_B = -C_2^S(Ad)\,\delta_{AB}\,, \qquad (5)$$

$C_2^S(Ad)$ being the eigenvalue of the quadratic Casimir of S in the adjoint representation, and the structure constants C^C_{AB} are totally antisymmetric and real in this basis.

The S-symmetric gauge field corresponding to $\{\tilde{A}_\mu(x), \phi(x)\}$ has the form

$$A_{x,y} = \tilde{A}_\mu(x)\, dx^\mu + \tau(\omega_y) + \phi(x)(\theta_y). \qquad (6)$$

Here we designate the homomorphism of the Lie algebra $\mathcal{R}$ to $\mathcal{G}$ induced by τ: R $\longrightarrow$ G with the same letter. Using the properties of $\tilde{A}_\mu, \tau, \phi$ and the definition (4), it is easy to prove that the field $A_M(x,y)$ defined by (6) is really S-symmetric. With the help of the Maurer-Cartan equations for e_y /14,21/ we evaluate the field strength of $A_{x,y}$

$$F_{\mu\nu}(x,y) = \tilde{F}_{\mu\nu}(x) = \partial_\mu \tilde{A}_\nu(x) - \partial_\nu \tilde{A}_\mu(x) + [\tilde{A}_\mu(x), \tilde{A}_\nu(x)], \qquad (7a)$$

$$F_{\mu\alpha}(x,y) = \tilde{T}^m{}_\alpha(y) F_{\mu m}(x) = \tilde{T}^m{}_\alpha(y)\, \mathcal{D}_\mu \phi_m(x), \qquad (7b)$$

$$F_{\alpha\beta}(x,y) = \tilde{T}^m{}_\alpha(y)\tilde{T}^n{}_\beta(y) F_{mn}(x) = \qquad (7c)$$

$$= \tilde{T}^m{}_\alpha(y)\tilde{T}^n{}_\beta(y)\left\{[\phi_m(x), \phi_n(x)] - C^a_{mn}\tau_a - C^k_{mn}\phi_k(x)\right\},$$

where $\mathcal{D}_\mu \phi_m(x) = \partial_\mu \phi_m(x) + [\tilde{A}_\mu(x), \phi_m(x)]$, $\tau_a = \tau(T_a)$ and $\phi_m(x) = (\phi(x))(T_m)$.

For S-symmetric A_M and S-invariant metric g ($\gamma_{mn}=$ const) the Lagrangian in (1b) does not depend on y and therefore integrating over it we get a four-dimensional action of a minimally coupled Yang-Mills-Higgs system for the pair $\{\tilde{A}_\mu, \phi\}$

$$S_{gauge} = \frac{Vol(S/R)}{8e^2}\int_{M^4} d^4x \left\{\langle \tilde{F}_{\mu\nu}, \tilde{F}^{\mu\nu}\rangle + 2\langle \mathcal{D}_\mu\phi_m, \mathcal{D}^\mu\phi_n\rangle \gamma^{mn} - V(\phi)\right\}. \qquad (8)$$

where Vol(S/R) is the volume of S/R, and V (ϕ) is a quartic self-interaction potential

$$V(\phi) = -\langle F_{mn}, F_{k\ell}\rangle \gamma^{mk}\gamma^{n\ell}, \qquad (9)$$

defined with the help of (7C).

If we now put the ansatz used in (2), $\tilde{A}_\mu = 0$ and $\partial_\mu \phi = 0$ into (8), we obtain the effective action $S_{eff}=V(\phi)$ with constraints (3) for S-symmetric solutions of (2b) (in agreement with the fact that extrema in the class of S-symmetric fields are extrema in the class of all fields /22/). This important method of obtaining solutions of (2b) was developed in /8-14/, and it gives us an algorithm to find solutions of SC equations.

Step 1. Find all linearly independent solutions of (3) ϕ^L (by utilizing Schur's lemma) and introduce the unconstrained scalar fields $\{f\}$: $\phi = \phi^L f_L$. Express the general S-invariant metric γ on S/R as function of a set of parameters $\{M\}$ with dimension of mass (similar to the components ϕ_m , the S-symmetry does not allow the components γ_{mn} to be independent (see next section)). The SC equations transform into equations for these two sets $\{f\}$ and $\{M\}$.

Step 2. Find an extremum of V($\{f\}$, $\{M\}$), taking $\{M\}$ as constants

$$\{f^{ext}\} = \{f^{ext}\}(\{M\}).$$

Substitute this value and $\tilde{A}_\mu=0$ in (6) and obtain a solution of (2b) for the metric $\gamma(\{M\})$.

Different methods of finding extrema of V have been studied in literature /5,13/. We will assume that these extrema are known to us (in the simpler cases they can be found trivially (see sect.4)). For a more detailed analysis of step 2. see /8-14/.

Step 3. Put $\gamma(\{M\})$ and $\{f^{ext}\}(\{M\})$ into (2a). The number of equations coincides with the number of unknown parameters $\{M\}$ *(see next section)*.

Therefore if $\gamma(\{M^{(0)}\})$ is a solution of equation (2a), **the solution of** SC equations is given by

$$A_y = \tau(\omega_y) + f_i^{ext}(\{M^{(0)}\})\, \phi^i(\theta_y),$$

$$\gamma = \gamma(\{M^{(0)}\}) .$$

Notice that $\{f\}=0 \Leftrightarrow \phi=0$ is almost always an extremum of the potential. More precisely, it was shown in /14/ that for $\phi=0$ not to be an extremum of V, it is necessary that there exist representations in ad$\mathcal{R}$ and ad$\mathcal{R}(\mathcal{M})$ equivalent to each other, where ad$\mathcal{R}(\mathcal{M})$ denotes the representation of $\mathcal{R}$ in $\mathcal{M}$ induced by the commutation relations of $\mathcal{S}$. For symmetric spaces S/R $\phi=0$ is a local maximum and thus the potential leads to spontaneous symmetry breaking of the reduced theory /6,14/. If we choose $\phi^{ext}=0$ the steps 1. and 2. are carried out trivially.

If we have two solutions of (2), the one corresponding to the absolute minimum of V (ϕ) (Higgs vacuum) and the other corresponding to another extremum, then we can expect that the second is not stable with respect to the first one /8,14/. In /8/ the solutions corresponding to the absolute minimum were called stable compactifying EYM solutions. We shall denote them by SCS while the solutions which are not SCS will be denoted by NSCS.

Recall that, for a given model (i.e. for a given triple $[S/R,G,\tau]$), SCS may not exist. An example is the case when τ can be extended to a homomorphism of S to G. Besides not having SCS, models with such τ do not lead to chiral fermions after the dimensional reduction procedure /2,23/.

If the isotropy group R is simple, it is not easy to find physically interesting models with homomorphism τ which cannot be extended. In /6/ with the use of non-regular embeddings $\tau(R)\subset G$ models $[S/R=S^n=SO(n+1)/SO(n), \tau, SU(nj+m)]$ with nonvanishing v.e.v. of the potential and thus with SCS /12/ were found.

In the case of nonsimple R there exist an easy way of preventing τ from being extended to a homomorphism of S. Let λ_i be the ratios of the indices of the embeddings $\tau(\mathcal{R}_i)\subset\mathcal{G}$ and $\mathcal{R}_i\subset\mathcal{S}$ (i.e. $\lambda_i = \langle\tau(r_i),\tau(r_i)\rangle/(r_i, r_i)$ for all $r_i : r_i \neq 0$ and $r_i\in\mathcal{R}_i$=Lie (R_i), where $\mathcal{R}_i$ are ideals in $\mathcal{R}$). For τ to be extended to a homomorphism of S it is necessary that all λ_i are equal to each other. It was shown in /3/, for S/R= $\mathbb{C}P^m$ =SU(m+1)/SU(m) × U(1) and S/R=$G_{2,m+2}(\mathbb{R})$= = SO(m+2)/SO(m) × SO(2) and for a wide class of classical simple Lie groups G, that there are interesting Higgs models with $\lambda_1\neq\lambda_2$ and therefore with SCS. Another possibility is to utilize a non injective homomorphism τ/17/. Formally this situation corresponds to the vanishing of some indices λ_i. Explicit formulae for multidimensional SCS in local coordinates were derived in /10/.

Notice that an SCS may not lead to spontaneous symmetry breaking in the reduced theory. Indeed it is known that for non-symmetric spaces $\{f\}=0$ can be the absolute minimum of V /14,16,17/ and thus for such SCS no symmetry breaking occurs. Since these two situations are very different physically, we will distinguish the SCS leading to spontaneous symmetry breaking (SBSCS) from the others (NSBSCS). Thereby we come to the following classification of solutions of SC equations

$$CS \begin{cases} NSCS \\ SCS \begin{cases} NSBSCS \\ SBSCS \end{cases} \end{cases} .$$

3. The Einstein **Equation**

The S-symmetry of γ_{mn} and A_m implies that the Ricci tensor $R_{mn}(\gamma)$, the tensor $Q_{mn}(\gamma, A) := -\langle F_{m\ell}, F_n{}^{\ell} \rangle$, as well as γ_{mn} are S-invariant symmetric tensors of the second rank. It is a common knowledge in differential geometry /19,21/ that the set of S-invariant symmetric tensors of the second rank on S/R is in a one-to-one correspondence with the set of AdR-invariant bilinear symmetric forms on $\mathcal{M}$. This correspondence is established in the following way. The fundamental vector fields generated by T_A are defined as: $K_A(y) = \frac{d}{dt} \exp(T_A t) y|_{t=0}$. These vector fields are Killing vectors for any S-invariant metric γ. Besides we have $[K_A, K_B] = -C^D_{AB} K_D$. If $\tilde{\gamma}$ is an S-invariant tensor, then the AdR-invariant form $\tilde{B}$ corresponding to it is given by

$$\tilde{B}(T_m, T_n) = \tilde{\gamma}_o(K_m, K_n) . \qquad (10)$$

In (10) the point $y = 0$ denotes the coset $[e]$ of S/R.

The decomposition of Killing vectors $K_A(y)$ with respect to the basis $\{\tilde{T}_m(y)\}$ (dual to the basis of 1-forms $\{\tilde{T}^m(y)\}$ defined in (4b)) at each point y depends, of course, on the section σ and can be found in /18/. We choose $\sigma(0) = e \in S$ in order to have $K_m(0) = \tilde{T}_m(0)$ for all m=1,...,d and thus $\tilde{\gamma}_o(K_m, K_n) = \tilde{\gamma}_o(\tilde{T}_m, \tilde{T}_n) := \tilde{\gamma}_{o\,mn}$.

The decomposition of AdR($\mathcal{M}$) into irreducible representations of R can be written in the form

$$AdR\mathcal{M} = \sum_{r=1}^{N_1} \sum_{p=1}^{N_2(r)} \psi^{rp} , \qquad (11a)$$

$$\mathcal{M} = \bigoplus_{r=1}^{N_1} \bigoplus_{p=1}^{N_2(r)} \mathcal{M}^{rp} , \qquad (11b)$$

where the representations $\{\psi^{rp}\}_{p=1}^{N_2(r)}$ are all equivalent while ψ^{rp} and $\psi^{r'p'}$ are nonequivalent for $r \neq r'$.

The set of S-invariant symmetric tensors of the second rank forms a linear space of dimension $N = \sum_{r=1}^{N_1} N_2(r)(N_2(r)+1)/2$ /19/. Although up to now we used the rule of summation over repeated **indices** in the following we will not be able to keep to it. According to (11) we introduce composite **indices** m=(i,rp) in $\mathcal{M}$ and choose an orthonormal basis $T_{(i,rp)}$ for $\mathcal{M}$ so as $T_{(i,rp)} \approx T_{(i,rq)}$ (with respect to $AdR(\mathcal{M})$). Then the S-invariance yields that $\gamma_{o(i,rp)(j,sq)}$, $R_{o(i,rp)(j,sq)}$ and $Q_{o(i,rp)(j,sq)}$ are proportional to $\delta_{ij}\delta_{rs}$. If the corresponding coefficients of proportionality for $\gamma_{o\,mn}$ are given

$$\gamma_{o(i,rp)(j,sq)} = (1/M^2_{r,pq})\, \delta_{rs}\, \delta_{ij} , \qquad (12a)$$

those for $R_{o\,mn}$ and $Q_{o\,mn}$ can be evaluated

$$R_{o(i,rp)(j,sq)} = R_{r,pq}(\{M\})\, \delta_{rs}\, \delta_{ij} , \qquad (12b)$$

$$Q_{o(i,rp)(j,sq)} = Q_{r,pq}(\{M\}, \{f\})\, \delta_{rs}\delta_{ij} , \qquad (12c)$$

and the Einstein equation (2a) reduces to

$$R_{r,pq} = (4\pi\kappa/e^2)\, Q_{r,pq} . \qquad (13)$$

Let us now find the explicit dependence of $R_{r,pq}$, $Q_{r,pq}$ and V on $\{M\}$. The presence of equivalent representations in (11) leads in (13) to a set of complicated nonlinear matrix equations for the matrices $\| 1/M^2_r \|_{pq} := 1/M^2_{r,pq}$. For the sake of simplicity we restrict ourselves to the case $N_2(r) = 1$ for all r.

The formula for the Ricci tensor at the point $y = 0$ in the basis of Killing vectors $\{K_m\}$ reads /24/

$$R_{omn}=\sum_{m'm''n'n''}(1/2)\gamma^{m'm''}\Big[\sum_{l'l''}(1/2)\gamma^{n'n''}\gamma_{ml'}\gamma_{nl''}C^{l'}_{m'n'}C^{l''}_{m''n''}-$$
$$-\gamma_{n'n''}C^{n'}_{mm'}C^{n''}_{nm''}\Big]-\sum_{n'm'}(1/2)C^{n'}_{mm'}C^{m'}_{nn'}-\sum_{m'a}C^{a}_{mm'}C^{m'}_{na}. \tag{14}$$

In accordance with (5) the two last terms in the r.h.s. of (14) are equal to $(1/2)\,C_2^S(Ad)\,\delta_{mn}$

Using (12a) we transform (14) to

$$R_{o(ir)(js)}=\sum_{r's'=1}^{N}(1/2)\,D^{s'}_{r'(ir)(js)}\left\{\frac{M_{s'}^2M_{r'}^2}{2M_s^2M_r^2}-\frac{M_{r'}^2}{M_{s'}^2}\right\}+(1/2)C_2^S(Ad)\,\delta_{ij}\delta_{rs}, \tag{15}$$

where $D^{s'}_{r'(ir)(js)}=\sum_{i'j'}C^{(j's')}_{(ir)(i'r')}\,C^{(j's')}_{(js)(i'r')}$.

In the particular case of a normal metric, i.e. $M_r^2=M^2$ for all $r=1,\dots,N$, (15) transforms to the corresponding formulae given in ref /13,15/.

Utilizing (6), (12a) and the definition of Q we find

$$Q_{o(ir)(js)}=\sum_{t=1}^{N}M_t^2\,q_{t(ir)(js)}$$

$$q_{t(ir)(js)}=-\sum_{K}\Big\langle[\phi_{(ir)},\phi_{(Kt)}]-\sum_{a}C^{a}_{(ir)(Kt)}\tau_a-\sum_{i'r'}C^{(i'r')}_{(ir)(Kt)}\phi_{(i'r')},$$
$$[\phi_{(js)},\phi_{(Kt)}]-\sum_{a'}C^{a'}_{(js)(Kt)}\tau_{a'}-\sum_{j's'}C^{(j's')}_{(js)(Kt)}\phi_{(j's')}\Big\rangle. \tag{16}$$

Introducing the totally symmetric quantities

$$D^{s'}_{r'r}=\sum_{i=1}^{n(r)}D^{s'}_{r'(ir)(ir)},$$

$$q_{tr}=\sum_{i=1}^{n(r)}q_{t(ir)(ir)},$$

where $n(r)=\dim\mathcal{M}^r$, and (8), (12) and (15) we obtain for R_r, Q_r and V the expressions

$$R_r=(1/2n(r))\sum_{r's'=1}^{N}D^{s'}_{r'r}\left\{\frac{M_{r'}^2M_{s'}^2}{2M_r^4}-\frac{M_{r'}^2}{M_{s'}^2}\right\}+(1/2)C_2^S(Ad), \tag{17a}$$

$$Q_r=(1/n(r))\sum_{s=1}^{N}M_s^2\,q_{sr}(\{f\}), \tag{17 b}$$
$$V(\{f\})=\sum_{r,s=1}^{N}M_s^2M_r^2\,q_{sr}(\{f\}). \tag{17 c}$$

From (17a) and (17b) it can be shown that if we search in (13) only for solutions corresponding to the normal metric and G=R then this equation reduces to equation (4) of ref. /15/. Obviously if the normal metric is not a solution of (13) then equation (4) of ref /15/ has no solution, while (13) in principle always has a solution.

If M_r^2 are solutions of (13) then the dimensionless quantities $\mu_r^2=M_r^2(4\pi K/e^2)$ are solutions of (13) with $4\pi K/e^2=1$.

Let us introduce a simplifying vector notation

$\vec{\nu}:=\{\mu_1^2,\dots,\mu_N^2\},\vec{R}:=\{R_1,\dots,R_N\}$ and $\vec{Q}:=\{Q_1,\dots,Q_N\}\in\mathbb{R}^N$.

The dimensionless equation (13) takes the form

$$\vec{R}(\vec{\nu})=\vec{Q}(\vec{\nu}) \quad . \tag{18}$$

The number of equations in (18) can be reduced from N to N- 1 . Actually let $\vec{\beta}\in\mathbb{R}^{N-1}$ be a solution of

$$L_t=Q_1(\{1,\vec{\beta}\})R_{t+1}(\{1,\vec{\beta}\})-R_1(\{1,\vec{\beta}\})Q_{t+1}(\{1,\vec{\beta}\})=0, \tag{19}$$

where $t=1,\dots,N-1$. Then $\vec{\nu}\in\mathbb{R}^N$ defined by $\vec{\nu}:=\nu_1\{1,\vec{\beta}\}$, where $\nu_1=R_1(\{1,\vec{\beta}\})/Q_1(\{1,\vec{\beta}\})$ is a solution of (18) and every physically meaningful solution $(\nu_i\neq 0)$ can be obtained in this way.

In conclusion we analyse different variants of spontaneous compactification of extra dimensions to a six-dimensional non--symmetric space frequently used in literature

4. Non-Symmetric Space $S/R=SO(5)/SU(2)\times U(1)$

The embedding $R=SU(2)\times U(1)\subset S=SO(5)$ is defined by the branching rule

$$Ad\,SO(5)\downarrow_{SU(2)\times U(1)} = \underline{3}(0) + \underline{1}(0) + \underline{2}(1) + \underline{2}(-1) + \tag{20}$$
$$+ \underline{1}(2) + \underline{1}(-2) .$$

Using (12a) we define the general form of a SO(5) invariant metric

$$\gamma_{o(i\,r)(j\,s)} = (1/M_r^2)\,\delta_{ij}\,\delta_{rs} \quad , \quad r,s=1,2 \tag{21}$$

and choose in (11) $\Psi^1 = 2(1) + 2(-1)$ and $\Psi^2 = 1(2) + 1(-2)$. Thus in (21) $i,j = 1,2,3,4$ when $r=s=1$ ($n(1)=4$) and $i,j = 1,2$ when $r=s=2$ ($n(2)=2$).

From (17a) we find the quantities R_r corresponding to the metric (21)

$$R_1 = 3 - M_1^2/2M_2^2 \;,$$
$$R_2 = 2 + M_1^4/2M_2^4 \;. \tag{22}$$

Now we are ready to study various possibilities of spontaneous compactification into $SO(5)/SU(2)\times U(1)$ which are induced by different background gauge fields (ϕ^{ext}) in different models $[\tau, G]$.

First we briefly recall the model discussed in /15/ with G=SU(2)XU(1). We find that

$$Q_{o(i\,r)(j\,s)} = 2M_r^2\,\delta_{ij}\,\delta_{rs} \;, \tag{23}$$

i.e. $Q_r = 2M_r^2$ and thus the equation (19) takes the form

$$(6\beta^2 + \beta + 1)(\beta - 1) = 0 \quad . \tag{24}$$

The only solution of (24) is $\beta = 1$, which corresponds to the solution in the form of normal metric found in /15/.

If we enlarge G this solution will not in general be stable and therefore it is worthwile to find another models with SCS.

We shall consider now the interesting model proposed within the framework of CSDR scheme in /16/.

Let G=SU(5) and $\tau(SU(2)\times U(1))$ be defined by the branching rule

$$Ad\,SU(5)\downarrow_{\tau(SU(2)\times U(1))} = \underline{3}(0) + \underline{1}(0) + 3\,\underline{2}(1) +$$
$$+ 3\,\underline{2}(-1) + 8\,\underline{1}(0) .$$

To find new CS we shall follow the algorithm proposed in sect.2 using at the same time some results of /16/.

Step.1. The solutions of constraints (3) form a scalar triplet f of the gauge group of the reduced theory H = $= C_{SU(5)}(\tau(SU(2)XU(1))) = SU(3)XU(1)$.

Step 2. For the calculation of both potential and Q_r we have to find the quantities q_{rs}. Notice that q_{rs} can be easily calculated (see /17/) with the help of mappings similar to those introduced in /6/. The result is

$$q_{11} = 3|f|^4 - 8|f|^2 + 36/5 \;,$$
$$q_{12} = q_{21} = 2|f|^2 \;, \tag{25}$$
$$q_{22} = 12/5 .$$

Using (17b), (17c) and (25) we find

$$Q_1 = M_1^2\left[(3/4)|f|^4 - (2-\beta/2)|f|^2 + 8/5\right] , \tag{26a}$$
$$Q_2 = M_1^2\left[|f|^2 + 6\beta/5\right] , \tag{26b}$$
$$V = M_1^4\left[3|f|^4 - 4(2-\beta)|f|^2 + (12/5)(3+\beta^2)\right] , \tag{26c}$$

where $\beta = M_2^2/M_1^2$. Our result (26c) differs slightly from the corresponding formula (15) of ref /16/. In /16/ the term of the zeroth order in $|f|$ is the following: $M_1^4\,(6/5)\left(3+2\beta^2\right)$.

We see that if $\beta \geqslant 2$, the potential (26c) has the absolute minimum at $f^{vac} = 0$ and no other extrema exist. If $\beta < 2$, the absolute minimum is attained at the points satisfying $|f^{vac}|^2 = (2/3)(2-\beta)$ with positive v.e.v. of the potential whereas $f^{ext} = 0$ is a local maximum of V.

It is interesting to find out how the dynamics (i.e. equation (19)) answers the question whether the spontaneous symmetry breaking occurs ($\beta < 2$) or not ($\beta \geqslant 2$). Notice that in principle both cases can be realized if for $|f^{ext}|^2 = (2/3)(2-\beta)$ we get $\beta < 2$ and for $f^{ext} = 0$ $\beta \geqslant 2$ (in such a case both solutions would be SCS, the first being SBSCS and the second NSBSCS).

Equation (19) for $f^{ext} = 0$ reads

$$12\beta^3 - 14\beta^2 - 3 = 0 \,. \tag{27}$$

The only real solution of (27) is approximately equal to

$$\beta \simeq 1.312 \quad , \tag{28}$$

and thus a solution of (2) is given by (6) and (21) with

$$\begin{aligned} \tilde{A} &= \phi = 0 \,, \\ M_1^2 &\simeq 1.455\left(e^2/4\pi K\right), \\ M_2^2 &\simeq 1.909\left(e^2/4\pi K\right). \end{aligned} \tag{29}$$

For $|f^{ext}|^2 = (2/3)(2-\beta)$ we have

$$8\beta^3 + 84\beta^2 - 30\beta - 7 = 0 \,. \tag{30}$$

The only positive solution of (30) is

$$\beta = 1/2 \,, \tag{31}$$

and the solution of (2) corresponding to it is given by (6) and (21) with

$$\begin{aligned} \tilde{A} &= 0 \\ \phi &= \sum_L f_L \phi^L \,, \quad |f|^2 = 1 \,, \\ M_1^2 &= (5/2)\left(e^2/4\pi K\right) \\ M_2^2 &= (5/4)\left(e^2/4\pi K\right). \end{aligned} \tag{32}$$

From (28) and (31) we see that both solutions (29) and (32) are of symmetry breaking type ($\beta < 2$); (29) is a NSCS and (32) is a SBSCS.

Acknowledgements.

I am grateful to M. Fabbrichesi and D.V. Shirkov for helpful discussions and to D.I. Kazakov for his support at the Laboratory of Theoretical Physics, JINR. I would like to thank particularly Yu.A. Kubyshin and I.P. Volobujev for their permanent interest in the present work and for many interesting and useful discussions.

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Received by Publishing Department on March 3, 1988.