объединенный
институт
ядерных
исследований
дубна

E2-88-136

A.B.Govorkov

ON THE IMPOSSIBILITY OF A SMALL VIOLATION OF THE PAULI PRINCIPLE WITHIN THE LOCAL QUANTUM FIELD THEORY

Submitted to "Physics Letters A"

1988

Recently, Ignatiev and Kuzmin /1/ have again raised the problem of verification of the accuracy and possible small violation of the Pauli principle, for instance, when electrons occupy the atomic levels. Greenberg and Mohapatra /2/ have formulated trilinear relations for field operators that, as they proposed, should be a generalization of the one-level Ignatiev-Kuzmin model for the local quantum field theory of violation of the Pauli principle. However, earlier, in /3/ I proved, on the basis of general assumptions (also valid for the Greenberg-Mohapatra theory), the uniqueness theorem of the Green paraquantization /4,5/ as an extension of the conventional quantization, and consequently, the uniqueness of the para-Fermi and para--Bose-statistics as an extension of the usual statistics (with allowance for the possibility of infinite parastatistics when the number of particles in symmetric and antisymmetric states is not limited). When proving that theorem I made use of the condition of the positiveness of the state-vector norms that was earlier formulated by Greenberg and Messiah /5/ in demonstrating sufficiency of the Green quantization for the description of parastatistics.

In this note it is shown that under that condition the Greenberg-Mohapatra theory reduces either to the usual Fermi statistics or to the second-order para-Fermi statistics, and thus, cannot be the local field theory of small violation of the Pauli principle. At the same time we shall convince ourselves in the possibility of generalization of conventional statistics to para-Fermi and para-Bose statistics which, in turn, happen to be equivalent to usual ones with including some extra internal quantum numbers (see /6/ and /7/).

Greenberg and Mohapatra /2/ based their theory on the following commutation relations (with the Hermitian conjugate ones):

$$(2-\beta^2)[a_k, a_\ell]_- a_m^+ + (1-2\beta^2) a_m^+ [a_k, a_\ell]_- =$$

$$= (1-\beta^2+\beta^4)(\delta_{\ell m} a_k - \delta_{km} a_\ell) + 3(1-\beta^2)(a_\ell a_m^+ a_k - a_k a_m^+ a_\ell), \quad (1a)$$

$$(2-\beta^2)[a_k, a_\ell]_+ a_m^+ - (1-2\beta^2) a_m^+ [a_k, a_\ell]_+ =$$
$$= -(1-\beta^2+\beta^4)(\delta_{\ell m} a_k + \delta_{km} a_\ell) + (1+\beta^2)(a_\ell a_m^+ a_k + a_k a_m^+ a_\ell), \tag{1b}$$

where a_k and a_m^+ are the operators of annihilation and creation of a particle resp. in states k and m (k and m runs over a certain discrete set of one particle states); the bracket with "$-$" and "$+$" mean a commutator and an anticommutator, resp. Summing (1a) and (1b) we get

$$(2-\beta^2) a_k a_\ell a_m^+ - (1-2\beta^2) a_m^+ a_\ell a_k =$$
$$= -(1-\beta^2+\beta^4)\delta_{km} a_\ell + (2-\beta^2) a_\ell a_m^+ a_k - (1-2\beta^2) a_k a_m^+ a_\ell . \tag{2}$$

Subtracting (1a) from (1b) we arrive at the same expression (2) with the change $k \rightleftarrows \ell$. Thus, instead of (1a) and (1b) one may employ a relation (2). It can be rewritten in the form

$$[[a_m^+, a_\ell]_\varepsilon, a_k]_- = -\alpha \delta_{km} a_\ell ,$$
$$[a_m^+, a_\ell]_\varepsilon \equiv a_m^+ a_\ell + \varepsilon a_\ell a_m^+ . \tag{3}$$

where

$$\varepsilon = (2-\beta^2)/(1-2\beta^2), \quad \alpha = (1-\beta^2+\beta^4)/(-1+2\beta^2). \tag{4}$$

Relation (3) is a general expression for the basic commutation relation in the local quantum theory of a free field with arbitrary real parameters ε and α /3/. By free fields we mean the fields for which observables (the Hamiltonian, currents, etc.) are bilinear in form. Thus, the Greenberg-Mohapatra scheme is a particular case of the general scheme with parametrization (4) leaving only one parameter independent. It can be obtained from the general scheme (3) by setting

$$\beta^2 = (2-\varepsilon)/(1-2\varepsilon), \quad \alpha = (1-\varepsilon+\varepsilon^2)/(1-2\varepsilon). \tag{5}$$

For the general scheme the following theorem has been proved /3/:

If $\varepsilon \neq 0$ or $\varepsilon \not\to \infty$ and 1) there exists a unique vector $|0\rangle$ such that $a_m |0\rangle = 0$ for all m , (6)

2) the norm of vectors is positive definite, and
3) the number of particles in a symmetric (antisymmetric) state does not exceed a given integer $M \geq 2$, then $\varepsilon = -1$ ($\varepsilon = +1$) and $\alpha > 0$. In this case,

$$a_m a_n^+ |0\rangle = p\,\delta_{mn} |0\rangle , \tag{7}$$

where $p \geq 0$ is a real number connected with number M as follows:

$$M = 2p/\alpha . \tag{8}$$

The exception is the Fermi (Bose) statistics: $M = 1$. In this case, when $p=1$, any ε are admissible under the condition

$$\alpha = 1-\varepsilon \quad (\alpha = 1+\varepsilon). \tag{9}$$

In other words, under the conditions of the theorem the only feasible generalization of the conventional quantization of free fields is the Green paraquantization (with the account taken of the above exceptions $\varepsilon = 0$ and $\varepsilon \to \infty$, α/ε = const) /4/.

In the parametrization (5) the admissible solutions are

$$\varepsilon = -1, \quad \beta^2 = 1, \quad \alpha = 1 , \tag{10}$$

$$\varepsilon = 0, \quad \beta^2 = 2, \quad \alpha = 1 , \tag{11}$$

$$\varepsilon \to \infty, \quad \beta^2 = 1/2 , \quad \alpha/\varepsilon \to -1/2 \tag{12}$$

(the value of $\varepsilon = 1$ corresponds to the para-Bose statistics and is not allowed in that parametrization since in this case $\beta^2 = -1$ and $\alpha = -1 < 0$).

The solution (10) represents the para-Fermi statistics. If, following /2/, we set p=1, then from (8) we find that M=2, i.e. we have the second-order para-Fermi statistics. In general case we might set $p = 1/2, 1, 3/2, 2, \ldots$ to which there correspond the conventional Fermi statistics ($M = 1$) and para-Fermi statistics of orders $M = 2, 3, 4, \ldots$.

For the solution (11) relations (2) or (3) are not sufficient for the complete determination of the commutation relations, whereas for (12) a scheme arises, analogous to the para-Fermi statistics but with trilinear relations different from the Green relations. In /3/ the latter case was not considered, however its consideration does not influence the conclusions to be given below.

For the Fermi statistics $\beta^2 = 0$ /2/ and in accordance with (4) and (9) we have $\varepsilon = 2$ and $\alpha = -1$.

In any of the above-listed cases admitted by the theorem the parameter β^2 has a fixed finite value and cannot be made arbitrarily small.

To demonstrate the procedure of proving the theorem, we briefly describe it for the Greenberg-Mohapatra scheme (at $p = 1$ in (7)). The relation, Hermitian conjugate to (2):

$$(2-\beta^2) a_m a_\ell^+ a_k^+ = (1-2\beta^2) a_k^+ a_\ell^+ a_m + (2-\beta^2) a_k^+ a_m a_\ell^+ -$$
$$- (1-2\beta^2) a_\ell^+ a_m a_k^+ - (1-\beta^2+\beta^4)\delta_{km} a_\ell^+ \tag{12}$$

allows us to shift the annihilation operators to the right towards vacuum and with the conditions (6) and (7), to calculate the norms of state vectors. We shall be interested in the computation of the norms of symmetric state vectors.

The norm of a symmetric two-particle vector is given by

$$\| \sum_{\mathcal{P}} a^+_{\mathcal{P}\ell} a^+_{\mathcal{P}m} |0\rangle \| \equiv 2! \sum_{\mathcal{P}} \langle 0 | a_m a_\ell a^+_{\mathcal{P}\ell} a^+_{\mathcal{P}m} |0\rangle =$$
$$= 2!\, \beta^2 (1+\delta_{\ell m}), \tag{13}$$

where summation runs over arbitrary permutations of indices $\ell, m \to \mathcal{P}\ell, \mathcal{P}m$ (in this case, over ℓ, m or m, ℓ). The condition for the norm (13) being positive definite implies

$$\beta^2 \geq 0 .$$

The value $\beta^2 = 0$ stands for the Fermi statistics /2/.

Further, by induction it may be shown that any projections of a vector containing three symmetrized creation operators which stand one after the other equal zero:

$$\langle 0 | a_\ell a_m \dots a_n a_k^+ \dots (\sum_{\mathcal{P}} a^+_{\mathcal{P}q} a^+_{\mathcal{P}r} a^+_{\mathcal{P}s}) \dots a_t^+ |0\rangle = 0 .$$

Thus, in the Fock representation there will always vanish a symmetric combination

$$\sum_{\mathcal{P}} a^+_{\mathcal{P}q} a^+_{\mathcal{P}r} a^+_{\mathcal{P}s} = 0$$

and any symmetric vector with a number of particles greater than two.

However, if symmetrization is performed over three but not successive creation operators, projections of a vector of that type

cannot automatically vanish. Consider two orthogonal vectors with four particles, three of which are in the same state ℓ , and the fourth is in another state k ($k \neq \ell$)*):

$$| x_\pm \rangle \equiv (1/2) a_\ell^+ (a_k^+ a_\ell^+ \pm a_\ell^+ a_k^+) a_\ell^+ |0\rangle . \tag{14}$$

Calculating the norms of these vector we find that the norm of the vector $|x_+\rangle$ disappears automatically, whereas the norm of the vector $|x_-\rangle$ is

$$\langle x_- | x_- \rangle = \beta^2 (-1+\beta^2)(1+\beta^2)(1-2\beta^2)^2 / (2-\beta^2)^2 . \tag{15}$$

For small $\beta^2 \ll 1$ this norm becomes negative! Thus, the parameter β^2 cannot be small, and the Greenberg-Mohapatra scheme cannot be the theory of small violation of the Pauli principle. If we then require the number of particles in one state to be not larger than two, the norm (15) will vanish in three cases:

$$\beta^2 = 0, \quad \beta^2 = 1, \quad \beta^2 = 1/2,$$

which coincide with the above-mentioned results obtained on the basis of the general theorem /3/. (Note that at $\beta^2 = 2$ denominator in (15) becomes zero and theory gets inconsistent. In accordance with (11), $\varepsilon = 0$ corresponds to this case when the commutation relations become incomplete).

Vanishing of the norm of a vector in the Fock space means vanishing of the vector itself. For vectors (14) this can be realized by operator algebras themselves.

For $\beta^2 = 0$, $a_k^+ a_\ell^+ + a_\ell^+ a_k^+ = 0$ (Fermi statistics), (16)

For $\beta^2 = 1$, $a_k^+ a_\ell^+ a_k^+ = 0$ (para-Fermi-statistics of second order), (17)

In the case $\beta^2 = 1/2$ we have two relations

$$a_\ell^+ a_k^+ a_\ell^+ + (a_\ell^+)^2 a_k^+ + a_k^+ (a_\ell^+)^2 = 0, \tag{18a}$$

$$a_\ell\, a_k^+ (a_\ell^+)^2 = 0 . \tag{18b}$$

*)The following arguments are applicable to systems having at least two different states (levels). For this reason they cannot be applied to the initial one-level Ignatiev-Kuzmin model /1/.

As is easily convinced, algebras (16)-(18) provide vanishing of vectors $|X_\pm\rangle$. The vector $|X_-\rangle$ could be put equal to zero also with

$$a_k^+ a_\ell^+ - a_\ell^+ a_k^+ = 0 \qquad \text{(Bose-statistics).} \quad (19)$$

But this solution is unsatisfactory. Indeed, making use of (12) and the conditions (6) and (7) (at $p=1$) we get ($k \neq \ell$)

$$a_\ell (a_\ell^+ a_k^+ - a_k^+ a_\ell^+)|0\rangle = (2-\beta^2)\, a_k^+ |0\rangle .$$

But according to (19) it should vanish. Since we a priori reject the case $\beta^2 = 2$, we have $a_k^+ |0\rangle = 0$, which signifies the representation being trivial:

$$a_k^+ = a_k \equiv 0 .$$

Thus, we directly verified the impossibility of small violation of the Pauli principle in the local quantum field theory, in agreement with the results of the earlier theorem /3/. We also verified the possibility of its "big" violation in the form of the para-Fermi statistics. There remains still an open problem of formulating, within the parafield scheme, the physical symmetries observed in Nature and the corresponding gauge theories /8,9/. Nevertheless, the uniqueness proved above for this possible generalization of the conventional statistics seems rather attractive.

Finally note that Okun /10/ proposed another generalization of the Ignatiev-Kuzmin model of small violation of the Pauli principle within multi(infinite)-level scheme. It was suggested to assume relations of the type (2) to be valid only for equal ($k = \ell = m$) states, whereas for different states all the operators were set anticommuting

$$a_k^+ a_\ell^+ = - a_\ell^+ a_k^+ .$$

Besides this theory is nonlocal and CPT-noninvariant, it has, as noted by Okun /10/, a more serious shortcoming: there is no continuous transition between the possibility for two particles being in the same state (for instance, with the same momentum) and the prohibition for these particles being in a symmetric state with values k and ℓ (momenta) infinitely close to each other. A scheme like that is a priori tied to a definite representation and does not permit superpositions of states.

The author is sincerely grateful to A.Yu.Ignatiev, V.A.Kuzmin, V.A.Matveev, L.B.Okun, and V.A.Rubakov for preliminary discussions of the results presented here.

References

1. A.Yu.Ignatiev, V.A.Kuzmin, Yad.Fys. 1987, 46, p.786.
2. O.W.Greenberg, R.N.Mohapatra. Phys.Rev.Lett., 1987, v.59. p.2507.
3. A.B.Govorkov. Teor.Mat.Fys., 1983, v.54, p.361.
4. H.S.Green. Phys.Rev., 1953, v.90, p.170.
5. O.W.Greenberg, A.M.L.Messiah. Phys.Rev., 1965, v.B138, p.1155.
6. Y.Ohnuki, S.Kamefuchi, Quantum field theory and parastatistics (University of Tokyo Press, Springer-Verlag, Berlin-Heidelberg-New York, 1982).
7. A.B.Govorkov. Elem. Chast. Atom.Yad., 1983, v.14, p. 1229.
8. O.W.Greenberg, K.I.Macrae. Nucl.Phys., 1983, v.B129, p.358.
9. A.B.Govorkov. Teor.Mat.Fys., 1982, v.53, p.283; ibid, 1983. v.55, p.3.
10. L.B.Okun, ITEP preprints No.195, No.203, No.209, Moscow-Atominform, 1987; Pis'ma Zh.Ehksp.Teor.Fiz., 1987, v.46, p.420.

Received by Publishing Department
on February 25, 1988.