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S.G.Donev*

GEODESIC VECTOR FIELDS ON MINKOWSKI SPACE-TIME AND (3+1)-SOLITARY WAVES

*Institute for Nuclear Research and Nuclear Energy, Bulg.Acad.Sci., boul.Lenin 72, 1784, Sofia, Bulgaria

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Let us consider the Minkowski space-time $(R^4, \eta) \equiv M$, where η is the Lorentz-Minkowski metric, $\operatorname{diag} \eta = (-1,-1,-1,1)$.
We shall denote by $(x^1, x^2, x^3, x^4) = (x, y, z, \xi = ct)$ the standard coordinates on M, so that

$$ds^2 = -dx^2 - dy^2 - dz^2 + d\xi^2 = \eta_{\mu\nu} dx^\mu dx^\nu .$$

The metric η induces corresponding covariant derivative ∇ with components $\Gamma^\sigma_{\mu\nu} = 0$ in these coordinates, but in general curvelinear coordinates (y^1, y^2, y^3, y^4), $\Gamma^\sigma_{\mu\nu}(y^\alpha) \neq 0$. In particular, in spherical coordinates $(r, \theta, \varphi, \xi)$

$$\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r \cos\theta \\ \xi &= \xi \end{aligned}$$

we have

$$\Gamma^r_{\theta\theta} = -r, \quad \Gamma^r_{\varphi\varphi} = -r^2 \sin^2\theta, \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$$

/1/
$$\Gamma^\theta_{r\theta} = \frac{1}{r}, \quad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \quad \Gamma^\varphi_{\theta\varphi} = \cot\theta .$$

All other $\Gamma^\sigma_{\mu\nu}$ are identically zero.
<u>Definition</u>. A vector field $u = u^\sigma \frac{\partial}{\partial y^\sigma}$ on M is called geodesic (or autoparallel) if $\nabla_u u = 0$, i.e. if

/2/
$$u^\sigma \frac{\partial u^\mu}{\partial y^\sigma} + \Gamma^\mu_{\sigma\nu} u^\sigma u^\nu = 0 .$$

Note that /2/ is a system of 4 nonlinear partial differential equations for the functions $u^\sigma(y^1, y^2, y^3, y^4)$. We are going to consider the equations /2/ in the following two cases: first in standard coordinates (x, y, z, ξ), where $\Gamma^\sigma_{\mu\nu} = 0$ and /2/ reduces to

/3/ $$u^\sigma \frac{\partial u^\mu}{\partial x^\sigma} = 0 \ ,$$

and, second, in spherical coordinates, where $\Gamma^\sigma_{\mu\nu}$ are given by /1/. The equations /2/ appear in physics if we consider the well known energy-momentum tensor

/4/ $$T^{\mu\nu} = u^\mu u^\nu \ ,$$

satisfying the local conservation law

/5/ $$\nabla_\nu T^{\mu\nu} = 0 \ .$$

Then

$$\nabla_\nu T^{\mu\nu} = u^\nu \nabla_\nu u^\mu + u^\mu \nabla_\nu u^\nu$$

and the most natural "field equations" are

/6/ $$u^\nu \nabla_\nu u^\mu = 0$$
$$u^\mu \nabla_\nu u^\nu = 0 \ .$$

We are going to show that this system of nonlinear equations admits solutions of (3 + 1) - solitary wave type, i.e. the components may be chosen to be concentrated in finite regions $\Omega \subset R^3$ with respect to the three space-coordinates (x, y, z) or (r, θ, φ) and to depend on $\xi = ct$ like a "running wave".

Consider first the case of standard coordinates (x, y, z, ξ) with $\Gamma^\sigma_{\mu\nu}(x,y,z,\xi) = 0$. Let the solution run along the z-coordinate. Hence, $u^1 = u^2 = 0$ and the system /6/ reduces to

a/ $$u^3 \frac{\partial u^3}{\partial z} + u^4 \frac{\partial u^3}{\partial \xi} = 0$$ c/ $$u^3 \left(\frac{\partial u^3}{\partial z} + \frac{\partial u^4}{\partial \xi} \right) = 0$$

b/ $$u^3 \frac{\partial u^4}{\partial z} + u^4 \frac{\partial u^4}{\partial \xi} = 0$$ d/ $$u^4 \left(\frac{\partial u^3}{\partial z} + \frac{\partial u^4}{\partial \xi} \right) = 0 \ .$$

From a/ and c/ it follows

$$u^3 \frac{\partial u^4}{\partial \xi} = u^4 \frac{\partial u^3}{\partial \xi} \quad \Longrightarrow \quad \ln u^3 = \ln u^4 + f(x,y,z).$$

From b/ and d/ it follows

$$u^4 \frac{\partial u^3}{\partial z} = u^3 \frac{\partial u^4}{\partial z} \quad \Longrightarrow \quad \ln u^3 = \ln u^4 + g(x,y,\xi) \ .$$

So $f(x,y,z) = g(x,y,\xi) = \varphi(x,y)$, and

$$u^3 = \beta(x,y)\, u^4, \qquad \text{where} \qquad \beta(x,y) = \exp[\varphi(x,y)] \ .$$

Now the equations a/ - d/ reduce to just one equation, namely

$$\beta \frac{\partial u^4}{\partial z} + \frac{\partial u^4}{\partial \xi} = 0 \ .$$

The general solution of this equation is

$$u^4 = u^4(x, y, z - \beta\xi) \ , \quad \beta = \beta(x,y) \ ,$$

where the dependance on (x, y, z) is arbitrary and, therefore, we may require u^4 to be concentrated with respect to (x, y, z) in a small region $\Omega \subset R^3$. Assuming the simple case $\beta(x,y) =$ con we have the solution

/7/ $$u^1 = 0, \ u^2 = 0, \ u^3 = \beta u^4 \ , \ u^4 = u^4(x, y, z - \beta\xi) \ .$$

If $u^\sigma u_\sigma > 0$ then $u^3 u_3 + u^4 u_4 = -(u^3)^2 + (u^4)^2 = (u^4)^2(1 - \beta^2) > 0$, i.e. $\beta^2 < 1$. Putting $\beta = \frac{v}{c}$, where v is some 3-velocity and c is the velocity of light, we obtain a solution, describing a time-stable localized object, moving as a whole along the z-coordinate with velocity $v < c$. If $u^\sigma u_\sigma = 0$ the object moves with the speed of light.

The energy-momentum tensor /4/ looks like

$$T^{\mu\nu} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (u^3)^2 & u^3 u^4 \\ 0 & 0 & u^4 u^3 & (u^4)^2 \end{vmatrix}$$

Assume u^4 is a bounded function, i.e. $|u^4| \leq const < \infty$, and concentrated in the small region $\Omega \subset \mathbb{R}^3$. Then the corresponding total energy E and momentum $\vec{P}$ are finite and given by

$$E = \int_{R^3} (u^4)^2 dx\,dy\,dz \;, \quad \vec{P} = \left(0,\, 0,\, \frac{v}{c^2} \int_{R^3} (u^4)^2 dx\,dy\,dz\right) .$$

If $v < c$ we can define the mass m of the object by

$$m = \frac{1}{c^2}\sqrt{E^2 - c^2(\vec{P})^2} \;.$$

It follows

$$m = \frac{1}{c^2}\sqrt{E^2 - c^2\frac{v^2}{c^4}E^2} = \frac{E}{c^2}\sqrt{1 - \frac{v^2}{c^2}} \;,$$

i.e.

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \;.$$

If $v = c$ the corresponding mass m is zero. Thus the solution ~~is~~ /7/ describes localized free objects with finite constant total energy and momentum coming from $(-\infty)$ and going to $(+\infty)$. Assuming $\beta = -\frac{v}{c}$ we can reverse the direction of motion.

Now consider the case of spherical coordinates. Physically this means the following. We isolate a small region D, e.g. a ball, around the origin of the coordinate system (x, y, z) or (r, θ, φ). Let some physical processes ocurr in D, so that some particles leave D, and other particles enter D. Let these particles move radially, i.e. along the r-coordinate. The question is, do equations /2/ admit corresponding $(3+1)$-solitary wave solutions? We shall show that the answer is positive.

Since the motion is along the r-coordinate we assume $u^\theta = u^\varphi = 0$. In view of /1/ we get from /2/ just two equations, namely for $\mu = 1, 4$ (The other two equations for $\mu = 2, 3$ are identically satisfied.):

$$u^1 \frac{\partial u^1}{\partial r} + u^4 \frac{\partial u^1}{\partial \xi} = 0$$

$$u^1 \frac{\partial u^4}{\partial r} + u^4 \frac{\partial u^4}{\partial \xi} = 0 \;.$$

We have the solution

$$u^1 = \pm\beta(\theta,\varphi)\, u^4, \qquad \text{where} \qquad u^4 = u^4(r + \beta\xi, \theta, \varphi) \;,$$

and the dependance on (r, θ, φ) is arbitrary. Hence, choosing $\beta = const.$ and u^4 to be concentrated in a small region $\Omega \subset \mathbb{R}^3$ out of D, we obtain the desired radially moving $(3+1)$- solitary wave solution. If $u^\sigma u_\sigma > 1$ then $\beta^2 < 1$ and again we may put $\beta = \frac{v}{c}$, $v < c$. If $u^\sigma u_\sigma = 0$ then $\beta^2 = 1$, i.e. $v = c$. In this case the tensor $T^{\mu\nu} = u^\mu u^\nu$ is not the energy-momentum tensor, since

$\nabla_\sigma u^\sigma = u^1 \frac{2}{r} \neq 0$ for these solutions. But if $|u^1| \ll r$, approximately $\nabla_\sigma u^\sigma \cong 0$, i.e. far enough from D we obtain again energy-momentum tensor, and the total energy and momentum of the solution can be introduced in the same way.

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