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It is well-known that the causality condition plays the most important role in Bogoliubov's axiomatic approach to the quantum field theory [1,2] . This condition can be formulated, for instance, in terms of variational derivatives with respect to asymptotic fields of $S$-matrix extended off the mass shell. Furthermore, it is assumed that the momentum space, in which the coefficient functions of the extended $S$-matrix, the off the mass shell field operators, etc., are defined, is ordinary Minkowsky space.

In the paper [3] a hypothesis has been put forward that the momentum space corresponding to the off the mass shell $S$-matrix be a space of constant curvature. And the Bogolubov causality condition has been generalized accordingly.

Proceeding from the results of the papers [3,4] (and in a certain sense implementing the programm outlined in [3] ) we shall consider a theory in which the motion group of the momentum space is de-Sitter group $SO(4,1)$ (unlike $SO(3,2)$ in [3] ). The momentum space with such a motion group has been considered in the paper [5].

Further, making use of causality condition we shall construct an integral representation for a matrix element of scalar field commutator - an analogue of Iost-Lehmann-Dyson representation [6] . In this paper spectrality condition is not taken into account.

The momentum space of constant curvature is embodied as a hyperboloid in pseudoeuclidean 5-space of variables

$(p_0, p_1, p_2, p_3, p_4)$:

$$p_0^2 - p_1^2 - p_2^2 - p_3^2 - p_4^2 = -\frac{\hbar^2}{\ell_0^2} ,$$

$\ell_0$ being the fundamental length. We shall use the system of units $\hbar = c = \ell_0 = 1$ .

Casimir operator of the group $SO(4,1)$ is interpreted as the operator of squared interval. In the unitary representations it possesses the following spectrum:

$$C = \begin{cases} \left(\frac{3}{2}\right)^2 + \Lambda^2 & 0 \leq \Lambda < \infty \\ -L(L+3) & L = -1, 0, 1, 2. \end{cases}$$

The continuous series corresponds to the timelike region, and the discrete - to the spacelike one.

The transition to configuration space is carried out with the help of "plane waves":

$$\langle \Lambda, N | P \rangle = (P_4 - PN)_+^{-\frac{3}{2} - i\Lambda}, \quad N^2 = 1 \tag{1 a}$$

$$\langle L, N | P \rangle = (P_4 - ipN)^{-(L+3)}, \quad N^2 = -1 . \tag{1 b}$$

These objects are generalized functions with power singularities. Therefore, a regularization is needed to calculate integrals with them. We use the regularization corresponding to the generalized function $x_+^\lambda$ which is defined as

$$x_+^\lambda = \begin{cases} x^\lambda, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

when $\lambda > -1$ and as an analytic continuation from this domain for other values of $\lambda$ [7].

Further we shall denote the sets $(\Lambda, N)$, $(L, N)$ that correspond to the "points" of quantized space by greek letters $\xi$, $\eta$, etc.

Let us consider a commutator of scalar operators

$$[\varphi(\xi), \varphi(0)],$$

where $\varphi(0)$ is defined by

$$\varphi(0) = \frac{1}{(2\pi)^{3/2}} \int \varphi(P, P_4)\, d\Omega_p .$$

It is not difficult to proove that this commutator possesses the property of locality. This emerges from the new causality condition and solvability condition (independence of the second variational derivative on the order of variation). Therefore

$$[\varphi(\xi), \varphi(0)] = 0 \quad , \quad \xi = (L, N).$$

We take a matrix element of this commutator between certain states $\alpha$ and $\beta$ :

$$\langle \alpha | [\varphi(\xi), \varphi(0)] | \beta \rangle = f_{\alpha\beta}(\xi),$$

and suppose that it is not equal zero identically. Then we obtain a function $f_{\alpha\beta}(\xi)$ that vanishes beyond the time-like region. Thus, the problem reduces to the construction of the most general representation for functions of this type (we omit indices $\alpha$ and $\beta$ ).

Let us introduce the characteristic function of the time-like region:

$$\theta(\xi^2) = \begin{cases} 1, & \xi = (\Lambda, N) \\ 0, & \xi = (L, N). \end{cases} \tag{2}$$

This function enables us to write down the main equation for the functions that vanish beyond the continuous series:

$$f(\xi) = \theta(\xi^2)\, f(\xi). \tag{3}$$

Transforming (3) to the momentum representation we shall derive an integral equation "in convolutions". The general solution of this equation can be constructed by virtue of specific form of $\tilde{\Theta}(\rho^2)$ .

It turns out that[x]

$$\tilde{\Theta}(\rho^2) = \frac{1}{\pi^3} \mathcal{P} \frac{1}{[2-2\rho_4]^2} \qquad (4)$$

Finally we find the main equation (3) in the momentum representation:

$$f(p) = \frac{1}{\pi^3} \int \frac{f(q)\, d\Omega_q}{[(p_L - q_L)^2]^2} \qquad (5)$$

where $q_L = (q, q_4)$, $p_L = (p, p_4)$ and $p_L^2 = q_L^2 = -1$ .

From here on our reasoning will remind the procedure of Dyson [6] . First of all let us note the fact that

$$\mathcal{D}_1(p) = \mathcal{P} \frac{1}{[p^2]^2}$$

is the even invariant solution of the equation

$$\Box_6 \mathcal{D}_1(p) = 0,$$

$\Box_6$ being D'Alembert's operator in the space of 6 dimensions. Next we shall introduce into consideration 6-vectors $Q$, $\mathcal{P} = (p_0, p_1, p_2, p_3, p_4, p_5)$, $\hat{Q}$ and $\hat{\mathcal{P}}$ being the vectors of special form $\hat{Q} = (q_L, 0)$, $\hat{\mathcal{P}} = (p_L, 0)$ , and a new function

$$F(\mathcal{P}) = \frac{1}{\pi^3} \int \frac{f(q)\, d\Omega_q}{[(\mathcal{P} - \hat{Q})^2]^2} \ . \qquad (6)$$

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[x] The proof of this fact see in Appendix.

This equation represents $F(\mathcal{P})$ in the form of convolution of two generalized functions:

$$F(\mathcal{P}) = \mathcal{D}_1 * f(\mathcal{P}).$$

The rule of convolution differentiation and the definition of $\mathcal{D}_1(\rho)$ give

$$\Box_6 F(\mathcal{P}) = 0. \tag{7}$$

Moreover, the definition of $F(\mathcal{P})$ (6) shows that $F(\mathcal{P})$ is invariant under the reflection of the fifth axis $\rho_5 \to -\rho_5$ and $F(\hat{\mathcal{P}}) = f(\rho)$. These properties are sufficient to construct the general representation of $f(\rho)$.

Let us write down the generalized Kirchhof's formulae for the function $F(\mathcal{P})$ (it is possible due to (7)):

$$F(\mathcal{P}) = \int d\Sigma \left[\mathcal{D}(\mathcal{P}-u)\frac{\partial}{\partial\Gamma}\tilde{F}(u) - \tilde{F}(u)\frac{\partial}{\partial\Gamma}\mathcal{D}(\mathcal{P}-u)\right] \tag{8}$$

where $\Sigma$ is a spacelike surface.

To satisfy the parity condition it is sufficient to choose the surface $\Sigma$ and the initial conditions $\tilde{F}(u)$ to be even functions of $u_5$. In particular, when $\Sigma$ does not depend on $u_5$, the formulae (8) can be rewritten in the following way:

$$F(\mathcal{P}) = \int_{-\infty}^{\infty} du_5 \int d\sigma \left[\mathcal{D}(\mathcal{P}-u)\frac{\partial}{\partial\gamma}\tilde{F}(u) - \tilde{F}(u)\frac{\partial}{\partial\gamma}\mathcal{D}(\mathcal{P}-u)\right], \tag{9}$$

$\sigma$ being a spacelike surface of by unity less dimension than that of $\Sigma$.

Introducing generalized functions $\delta(\sigma)$ and $\frac{\partial}{\partial\gamma}(\tilde{F}(u)\delta(\sigma))$ [7], we can transform (9) to be

$$F(\mathcal{P}) = \pi^2 \int_{-\infty}^{\infty} du_5 \int du' \, \mathcal{D}(\mathcal{P}-u)\,\Phi(u', u_5), \tag{10}$$

where $U' = (u_0, u_1, u_2, u_3, u_4)$ and

$$\Phi(u', u_5) = \frac{1}{\pi^3}\left[\delta(\sigma)\frac{\partial}{\partial\gamma}\tilde{F}(u) + \frac{\partial}{\partial\gamma}\left(\tilde{F}(u)\delta(\sigma)\right)\right].$$

Here from using the explicit form of $\mathcal{D}(\mathcal{P})$ [6]

$$\mathcal{D}(\mathcal{P}) = \frac{1}{2\pi^2}\,\varepsilon(p_0)\,\delta'(\mathcal{P}^2)$$

and $\Phi(u', u_5)$ being an even function of $u_5$ we get a representation for $f(p)$ :

$$f(p) = \int_0 du_5 \int du'\,\varepsilon(p_0 - u_0)\,\delta'(-1 - 2p_L u' + u'^2 - u_5^2)\times \tag{11}$$
$$\times\,\Phi(u', u_5).$$

Further, the surface $\sigma$ can be cho...n so that it is sufficient to integrate in the formulae (11) only over the region $u'^2 < 0$ . In this case we have:

$$f(p) = \int_0^\infty du_5 \int_0^\infty ds^2 \int du'\,\delta(u'^2 + s^2)\,\varepsilon(p_0 - u_0)\times$$
$$\times\,\delta'(-1 - 2p_L u' + u'^2 - u_5^2)\,\Phi(u', u_5).$$

We make a substitution in this integral according to $u' = s u_L$ :

$$f(p) = \int_0^\infty du_5 \int_0^\infty ds^2 \int s^5 du_L\,\delta(s^2(u_L^2 + 1))\,\varepsilon(p_0 - u_0)\times$$
$$\times\,\delta'(-1 - 2p_L u_L s - s^2 - u_5^2)\,\Phi(s u_L, u_5) =$$
$$= \int_0^\infty du_5 \int_0^\infty s\,ds^2 \int d\Omega_u\,\varepsilon(p_0 - u_0)\,\delta'\left(2(p(-)u)_4 - \frac{1 + s^2 + u_5^2}{2s}\right)\times \tag{12}$$
$$\times\,\Phi(s u_L, u_5).$$

It is convenient at this stage to introduce in (12) a new variable

$$\chi = \frac{1}{2}\,\frac{1+S^2+u_s^2}{S}\qquad 1\le \chi < \infty .$$

The explicit expression of Jacobian

$$\frac{\mathcal{D}(S^2, u_s)}{\mathcal{D}(\chi, u_s)} = \frac{4S^2}{S^2 - 1 - u_s^2}$$

shows that the transformation under consideration is not unique. Dividing the region of integration in (12) into subregion where it is unique and making in each one transition to the new variable we shall obtain the following representation:

$$f(p) = \int_1^\infty d\chi \int d\Omega_u\, \varepsilon(p_0 \cdot u_0)\, \delta\big(\lambda(p(-)u)_4 - \lambda\chi\big)\ \times \tag{13}$$
$$\times\ \Psi(u, \chi).$$

In this formulae

$$\Psi(u,\chi) = \lambda\frac{\partial}{\partial\chi}\int_0^{\sqrt{\chi^2-1}} \frac{du_s}{\sqrt{\chi^2-1-u_s^2}}\Big[\big(\chi-\sqrt{\chi^2-1-u_s^2}\big)^3\times$$
$$\times\,\Phi\big((\chi-\sqrt{\chi^2-1-u_s^2})u_L, u_s\big) - \big(\chi+\sqrt{\chi^2-1-u_s^2}\big)^3\Phi\big((\chi+\sqrt{\chi^2-1-u_s^2})u_L, u_s\big)\Big].$$

The representation (13) for $f(p)$ corresponds exactly to the Iost-Lehmann-Dyson representation in ordinary theory (and coincides with it as $\ell_0 \to 0$ ). The spectrality condition would cause supplementary restrictions on the function $\Psi(u,\chi)$. The corresponding problems will be considered in a separate paper.

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## Appendix

Let us prove that the choice

$$\hat{\Theta}(p^2) = \frac{1}{\pi^3}\mathcal{P}\frac{1}{[2-2p_4]^2} \tag{A 1}$$

is consistent with the formulae (2).
That is

$$\int \langle \xi | p \rangle \tilde{\Theta}(p^2)\, d\Omega_p = \begin{cases} 1, & \xi = (\Lambda, N) \quad \text{(A 2a)} \\ 0, & \xi = (L, N). \quad \text{(A 2b)} \end{cases}$$

We shall turn to the proof of (A 2a) first.

It is clear that due to the relativistic invariance of $\tilde{\Theta}(p^2)$ we can prove (A 2a) under a special choice of $\xi = (\Lambda, (1, \vec{0}))$. Let us introduce on the hyperboloid

$$p_0^2 - \vec{p}^2 - p_4^2 = -1$$

modified orispherical coordinates:

$$\begin{aligned} p_4 - p_0 &= e^{-\omega} & -\infty < \omega < \infty \\ \vec{p} &= e^{-\frac{\omega}{2}}\vec{q} & \vec{q} \in R^3 \\ p_4 + p_0 &= e^{\omega} - \vec{q}^2 . \end{aligned}$$

It can be easily found that

$$\frac{\mathcal{D}(p_0, \vec{p})}{\mathcal{D}(\omega, \vec{q})} = e^{-\frac{3\omega}{2}}$$

In the coordinates (A 3) the "plane wave" (1a) and $\tilde{\Theta}(p^2)$ (A 3) respectively take the form

$$\langle \Lambda,(1,\vec{0})|\omega,\vec{q}\rangle = e^{(\frac{3}{2}+i\Lambda)\omega} \qquad \text{(A 4)}$$

$$\tilde{\theta}(\omega,\vec{q}) = \frac{1}{\pi^3}\,\mathcal{P}\frac{1}{[-4sh^2\frac{\omega}{2}+\vec{q}^{\,2}]^2} \qquad \text{(A 5)}$$

exactly analogous to that of the "flat theory". Substituting (A 4) and (A 5) into (A 2a) we shall obtain:

$$\tilde{\theta}(\xi^2) = \frac{1}{\pi^3}\int e^{i\Lambda\omega}\frac{d\omega\, d\vec{q}}{[-4sh^2\frac{\omega}{2}+\vec{q}^{\,2}]^2}, \quad \xi = (\Lambda,(1,\vec{0})). \qquad \text{(A 6)}$$

We can easily prove that

$$I = \int\frac{d\vec{q}}{[-a^2+\vec{q}^{\,2}]^2} = \pi^3\,\delta(a). \qquad \text{(A 7)}$$

Really, using the formulae

$$\mathcal{P}\frac{1}{[-a^2+\vec{q}^{\,2}]^2} = \frac{1}{2}\lim_{\varepsilon\to 0}\left\{\frac{1}{[-a^2+\vec{q}^{\,2}+i\varepsilon]^2}+\frac{1}{[-a^2+\vec{q}^{\,2}-i\varepsilon]^2}\right\} \qquad \text{(A 8)}$$

we can represent the integral (A 7) in the form

$$I = \lim_{\varepsilon\to 0}\frac{1}{2}\left[I_1+I_2\right], \qquad \text{(A 9a)}$$

where
$$I_1 = \int\frac{d\vec{q}}{[-a^2+\vec{q}^{\,2}-i\varepsilon]^2} \qquad \text{(A 9b)}$$

$$I_2 = I_1^*. \qquad \text{(A 9c)}$$

Introducing in (A 9b) spherical coordinates we get

$$I_1 = 2\pi\int_0^\infty\frac{\sqrt{q}\,dq}{[-a^2+q-i\varepsilon]^2}. \qquad \text{(A 10)}$$

This integral can be easily calculated by contour integration (see fig.1).
We find ( $a > 0$ ):

$$2\int_0^\infty \frac{\sqrt{q}\,dq}{[-a^2+q-i\varepsilon]^2} = \frac{\pi i}{a+i\varepsilon} . \quad \text{(A 11)}$$

By combination of (A 9a), (A 9b) and (A 11) we derive:

$$I = \lim_{\varepsilon \to 0} \frac{1}{2} \frac{2\pi^2 \varepsilon}{a^2+\varepsilon^2} = \pi^3 \delta(a).$$

Thus we have proved the validity of (A 7). Substituting (A 7) into (A 6) we obtain

$$\tilde{\theta}(\xi^2) = \int e^{i\Lambda\omega} \delta(2 sh \tfrac{\omega}{2})\, d\omega = 1 \quad \text{(A 12)}$$

$$\xi = (\Lambda, (1, \vec{0})).$$

which probves (A 2a).

Now let us consider the formulae (A 2b). The explicit form of the "plane wave" and the relativistic invariance of $\tilde{\theta}(p^2)$ lead to the result:

$$I = \int (p_4 - i p_3)^{-(L+3)} \frac{2\delta(p_0^2 - \vec{p}^2 - p_4^2 + 1)}{[2 - 2p_4]^2} dp_0\, d\vec{p}\, dp_4 . \quad \text{(A 13)}$$

Then we introduce on the plane $(p_3 p_4)$ polar coordinates

$$p_4 = z\cos\varphi \qquad 0 \le \varphi \le 2\pi$$

$$p_3 = z\sin\varphi \qquad 0 \le z < \infty$$

and integrate over $dp_1 dp_2$ with the help of $\delta$-function. It gives:

$$I = \pi \int_{z \ge 0} z \, e^{g(z)}{}^{-(L+2)} e^{i(L+3)\varphi} \frac{\theta(p_0^2 - z^2 + 1)}{[2 - 2z\cos\varphi]^2} dp_0\, dz\, d\varphi . \quad \text{(A 14)}$$

Substituting in (A 14) $z \to -z$ , $\varphi' \to \pi + \varphi'$ and adding the result back to (A 14) we find:

$$I = \frac{\pi}{2}\int w_g(z)^{-(L+2)}\,\mathcal{E}(z)\,e^{i(L+3)\varphi'}\frac{\theta(R^2-z^2+1)}{[2-2z\cos\varphi']^2}\,dp_i\,dz\,d\varphi', \qquad (A\ 15)$$

where the region of integration over $dz$ is $(-\infty, \infty)$. Now we shall regularize $\theta$-function in (A 15) by integration over $dp_i$ in finite region $(-A, A)$.
A simple transformation of integral (A 15) yields

$$I = I_1 + I_2 \qquad (A\ 16)$$

where

$$I_1 = A\pi\int_0^{2\pi} e^{i(L+3)\varphi}d\varphi\int_{-\infty}^{\infty}\mathcal{E}(z)\,w_g(z)^{-(L+2)}\frac{dz}{[2-2z\cos\varphi]^2}, \qquad (A\ 17)$$

$$I_2 = -\pi\int_0^{2\pi} e^{i(L+3)\varphi}d\varphi\left\{\int_1^{\infty} z^{-(L+2)}\frac{\sqrt{z^2-1}\,dz}{[2-2z\cos\varphi]^2} - \int_{-\infty}^{-1} z^{-(L+2)}\frac{\sqrt{z^2-1}\,dz}{[2-2z\cos\varphi]^2}\right\}. \qquad (A\ 18)$$

It is evident that

$$I_1 = C\int_0^{2\pi} e^{i(L+3)\varphi}(\cos\varphi)^{L+1}d\varphi = 0. \qquad (A\ 19)$$

The integral (A 18) can be treated as a contour integral (see fig.2). Or in the explicite form:

$$I_2 = -\pi\int_0^{2\pi} e^{i(L+3)\varphi}d\varphi\int_C z^{-(L+2)}\frac{\sqrt{z^2-1}\,dz}{[2-2z\cos\varphi]^2}. \qquad (A\ 20)$$

Calculating the residues we find that the pole at zero gives a polynomial in $\cos\varphi$ of the power $L+1$ , and therefore the integration over $d\varphi$ gives zero (cp.A (19) ). It is easy to find that owing to the formulae (A 8) the poles of $\frac{1}{[\lambda - \lambda_2(\omega)f]^2}$ give no contribution at all. Really, in accordance with (A 11) and (A 12) the residues are taken alternately above and below the cut and therefore differ in sign. This proves (A 2b).

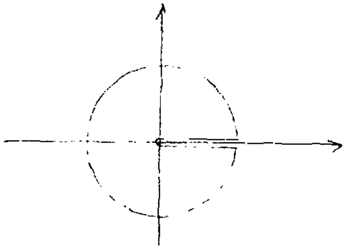

fig.1.

The contour of integration in ( A 11).

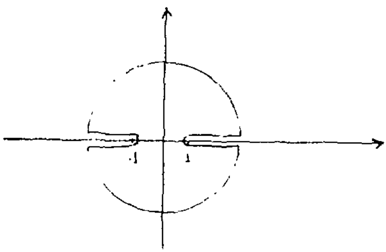


fig. 2.

Contour of integration in (A 20).

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