ОБЪЕДИНЕННЫЙ
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ДУБНА

E2 - 8700

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MATRIX CANONICAL REALIZATIONS OF THE LIE ALGEBRA o(m,n)

II. CASIMIR OPERATORS

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Submitted to *Annales de l'Institut Henri Poincaré*,

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1. INTRODUCTION

In the first part of this paper [1] we expressed the generators of the Lie algebra of the pseudoorthogonal group $O(m,n)$ by means of matrices, the elements of which were the polynomials in the quantum canonical variables p^i and q_i . This is what we call the matrix canonical realization of the algebra $o(m,n)$ *. We proved a.o. that these realizations are Schur-realizations, i.e., that all Casimir operators are realized by multiples of the identity element. Now we are interested in their "eigenvalues".

In ref. [1] we described two sets of matrix canonical realizations of $o(m,n)$. Every realization from the first set was determined by a sequence of n real numbers and if $m-n \geq 2$ by some finite-dimensional skew-hermitean irreducible representation of the compact Lie algebra $o(m-n)$. As any such representation is uniquely (up to equivalence) determined by its signature $(a_1, \ldots, a_{[\frac{m-n}{2}]})$, i.e., by a certain sequence of integrals of half-integers ** [2], we can say that every realization of $o(m,n)$ from the first set is determined by the sequence $a_{m,n} = (n; a_1, \ldots, a_{[\frac{m+n}{2}]})$, where the first $[\frac{m-n}{2}]$ numbers correspond to the signature of the representation of $o(m-n)$ and the remaining n numbers are the mentioned real parameters; we call this sequence the signature of realization.

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* For the exact definitions of all the concepts used here and details we refer to ref. [1].

** The only exception concerns the algebra $o(2)$ when the number $a_{[\frac{2}{2}]}$ assumes any real value.

The realizations of the second set are the usual canonical realizations, i.e., generators of $o(m,n)$ in them are realized as polynomials in canonical variables only. They are similarly determined by the signature $(d;\alpha_1,\dots\alpha_{[\frac{m+n}{2}]})$, $d=1,2,..,n-1$, where now $\alpha_1=\dots=\alpha_{[\frac{m+n}{2}]-d}=0$ and the rest are real numbers.

In this paper we shall give simple formula for calculation of generating Casimir operators. They are expressed in it as the sum of matrix elements of powers of a certain matrix. The "exceptional" generating Casimir operator $\tilde{I}^{(m,n)}$ in the case of $o(m,n)$ with $m+n$ even is given explicitly (theorem I).

It will be shown that, with the exception of $\tilde{I}^{(m,n)}$, all generating Casimir operators are certain symmetric polynomials in variables $(\beta_1)^2,\dots,(\beta_{[\frac{m-n-2d}{2}]})^2$, $(i\alpha_{[\frac{m+n-2d}{2}]+1})^2,\dots,(i\alpha_{[\frac{m+n}{2}]})^2$, where β_s, $s=1,\dots,[\frac{m-n-2d}{2}]$, is a certain linear function of α_s. Casimir operator $\tilde{I}^{(m,n)}$ is also a symmetric polynomial, however, only in the first degrees of constants $\beta_1,\dots,\alpha_{[\frac{m+n}{2}]}$. Due to this symmetry property there is a finite number of realizations in both the sets with the same "eigenvalues" of Casimir operators only. As the order of numbers in the "subsignature" $(\alpha_1,..,\alpha_{[\frac{m-n}{2}]})$ is fixed, the signatures of all these realizations differ, with the exception of some cases if $m+n$ is even, either in the permutation of the last d components or in the signms $d'\le d$ of them.

In the last part of the paper the connection with our earlier results /3/ is briefly discussed.

2. PRELIMINARIES

A. For $o(m,n)$, $m\ge n\ge 1$, we conventionally use the metric tensor in the form $g_{\mu\nu} = \mathrm{diag}(g_{11},\ldots,g_{m+n-2\ m+n-2},-1,+1)$. Together with the tensor basis $L_{\mu\nu}=-L_{\nu\mu}(\mu,\nu=1,2,\ldots,m+n)$, the elements of which obey the commutation relations

$$[L_{\mu\nu}, L_{\rho\tau}] = g_{\nu\rho}L_{\mu\tau} - g_{\mu\rho}L_{\nu\tau} + g_{\nu\tau}L_{\rho\mu} - g_{\mu\tau}L_{\rho\nu} \tag{1}$$

we use also the following one:

$$L_{ij},\ P_i = L_{i,m+n} + L_{i,m+n-1},\ Q_i = L_{i,m+n} - L_{i,m+n-1},$$

$$R = L_{m+n-1,\ m+n}$$

$i,j = 1,2,\ldots,m+n-2$ *. As we said in the introduction, to every signature $\alpha_{m,n} = (d;\alpha_1,\ldots,\alpha_{[\frac{m+n}{2}]})$ there corresponds the Schur-realization $r \equiv r(\alpha_{m,n})$ of $o(m,n)$ in $W_{2(m+n-2+N),M}$**. We obtain this realization using the recurrent formulae (see theorems 1,3 of /1/):

$$
\begin{aligned}
&r(P_i) = p_i, \quad r(L_{ij}) = q_i p_j - q_j p_i + M_{ij}, \\
&r(R) = -(qp) - [\tfrac{1}{2}(m+n-2) - i\alpha]I, \quad \alpha \in \mathbf{R}, \\
&r(Q_i) = -q^2 p_i - 2q_i r(R) - 2q^k M_{ki}, \\
&(qp) = q^i p_i, \quad q^2 = q_i q^i,
\end{aligned} \tag{2}
$$

where $p_i = g_{ij}p^j$, $q^i = g^{ij}q_j$ and $M_{ij} = -M_{ji}$ is the realization of generators of $o(m-1,n-1)$ in $W_{2N,M}$***. The difference between both the sets of realizations is that in the first

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* Indices i,j,k,ℓ will run always from 1 to $m+n-2$.

** Remember that $W_{2N',M}$ (matrix Weyl-algebra) denotes the associative algebra generated by N' canonical pairs p^i, q_i, $[p^i, q_j] = \delta^i_j I$, with complex $M\times M$-matrix coefficients; r is a homomorphism of $o(m,n)$ into $W_{2(m+n-2+N),M}$.

*** For $m+n=2,3$ we define $M_{ij}=0$.

case we continue the "reduction" to realization of the compact algebra $o(m-n)$, while in the second one $(d<n)$ we use the trivial realization of $o(m-d, m-d)$.

B. The number of generating Casimir operators of the algebra $o(m,n)$ equals to $[\frac{m+n}{2}]$. For $m+n$ odd they can be all found among Casimir operators

$$I_r^{(m,n)} = L_{\mu_1}^{\ \mu_2} L_{\mu_2}^{\ \mu_3} \dots L_{\mu_r}^{\ \mu_1} \quad , \quad r = 1,2,3,\dots$$

(we understand $I_1^{(m,n)} = L_{\mu_1}^{\ \mu_1} = 0$ and define also $I_0^{(m,n)} \equiv m+n$). For $m+n$ even we must add to them the Casimir operator

$$\tilde{I}^{(m,n)} = \epsilon^{\mu_1 \nu_1 \dots \mu_{\frac{m+n}{2}} \nu_{\frac{m+n}{2}}} L_{\mu_1 \nu_1} \dots L_{\mu_{\frac{m+n}{2}} \nu_{\frac{m+n}{2}}} ,$$

where $\epsilon^{\mu_1 \nu_1 \dots}$ is the completely antisymmetric Levi-Civita tensor in $m+n$ indices with normalization: $\epsilon^{12\dots m+n} = 1$.

C. The statements of the part B are, of course, valid also for compact algebra $o(m,0) \equiv o(m)$. As we have reminded, every irreducible skew-hermitean representation of this algebra is uniquely (up to equivalence) determined by the signature $(a_1, \dots, a_{[m/2]})$. Values of the generating Casimir operators in this representation can be expressed explicitly by means of its signature [4,5].

To this purpose we shall define special sort of symmetric polynomials in $[m/2]$ variables $x_1, \dots, x_{[m/2]}$. Let us firstly define recurrently the $m \times m$ – matrices $S_m(x_1, \dots, x_{[m/2]})$:

$$S_1 \equiv 0, \quad S_2(x_1) = \begin{pmatrix} x_1, 0 \\ 0, -x_1 \end{pmatrix},$$

$$S_m(x_1, \dots, x_{[m/2]}) = \begin{pmatrix} x_{[m/2]} + \frac{m-2}{2}, & -e^+_{m-2}, & 0 \\ 0, & S_{m-1}(x_1, \dots, x_{[\frac{m-2}{2}]}) + E_{m-2}, & -e_{m-2} \\ 0, & 0, & -x_{[\frac{m}{2}]} + \frac{m-2}{2} \end{pmatrix}. \tag{3}$$

Here $e^+_{m-2} = (1, 1, \dots, 1)$ (e_{m-2}) is the $(m-2)$-dimensional row (column) consisting of unities and E_{m-2} is the identity $(m-2) \times (m-2)$-matrix. This recurrent relation is solved explicitly in /4/ (see eq. (16) and Table 1). The polynomials $\sigma_r^{(m)} \equiv \sigma_r^{(m)}(x_1, \dots, x_{[m/2]})$ are defined as follows

$$\sigma_r^{(m)} = \begin{cases} e^+_m \cdot S^r_m(x_1, \dots, x_{[m/2]}) e_m, & r = 1, 2, \dots \\ m, & r = 0. \end{cases} \tag{4}$$

Note 1. The main important property of $\sigma_r^{(m)}$ is that any $\sigma_r^{(m)}$ is a polynomial function of Newton sums of even degree $s_2, s_4, \dots, s_{2[r/2]}$ where

$$s_r = \sum_{i=1}^{[m/2]} x_i^r$$

and on the contrary any Newton sum s_{2r} is a polynomial function of $\sigma_1^{(m)}, \dots, \sigma_{2r}^{(m)}$ (see eq. (90) of /5/).

The value of the Casimir operator $I_r^{(m,0)} \equiv I_r^{(m)}$, $m \geq 2$, in the representation characterized by signature $(\alpha_1, \ldots, \alpha_{[m/2]})$ is [4] *

$$I_r^{(m)} = \sigma_r^{(m)}(\beta_1, \ldots, \beta_{[m/2]}), \quad r = 0, 1, \ldots$$

$$\beta_s = \alpha_s + \gamma_s, \quad \gamma_s = \frac{m}{2} - s, \quad s = 1, 2, \ldots, [m/2].$$

The value of Casimir operator $\tilde{I}^{(m,0)} = \tilde{I}^{(m)}$ (for m even) in this representation is the following:

$$\tilde{I}^{(m)} = (2i)^{\frac{m}{2}} \left(\frac{m}{2}\right)! \cdot \beta_1 \cdots \beta_{\frac{m}{2}} .$$

3. *REALIZATIONS OF CASIMIR OPERATORS OF* $o(m,n)$.

Lemma 1. Let $J_r^{(m,n)} = \sum_{s=0}^{r} \binom{r}{s} I_s^{(m,n)}$ and $I_r^{(1,0)} = \delta_{r0}$. Then in the realization of $o(m,n)$, $m+n \geq 3$, given recurrently by the formulae (2) the following formulae are valid:

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* In paper [4] the Casimir operators C_r, $r=1,2,\ldots$ and $C'_{m/2}$, m– even, of the Lie algebra $o(m)$ are defined. The definitions of operators C_r and $C'_{m/2}$ are formally the same as the definitions of our $I_r^{(m)}$ and $\tilde{I}^{(m)}$, however, another (twoindexed) basis is used. The connection between these two bases has the usual tensorial character so that, as C_r behave as scalars, $I_r^{(m)} = C_r$, $r=1,2,\ldots$. On the other hand, Casimir operator $C'_{m/2}$ is a pseudoscalar and therefore the connection with $\tilde{I}^{(m)}$ has the form

$$\tilde{I}^{(m)} = (-1)^{\frac{m(m-2)}{8}} (i)^{\frac{m}{2}} C'_{m/2} ,$$

where $(i)^{m/2}$ is the determinant of the linear transformation (eq. (3) in [4]) inducing the mentioned tensorial transformation of bases. The signm factor $(-1)^{\frac{m(m-2)}{8}}$ arises due to distinct normalization of the Levi-Civita tensor.

$$I_r^{(m,n)} = \beta^r + \bar\beta^r - \sum_{s=0}^{r-2}\left(\beta^{r-s-1} + \bar\beta^{r-s-1} + \frac{\bar\beta^{r-s-1} - \beta^{r-s-1}}{\beta - \bar\beta}\right)\times$$

$$\times \tilde J_s^{(m-1,n-1)} - 2 J_{r-1}^{(m-1,n-1)} + J_r^{(m-1,n-1)}, \quad r = 0,1,\ldots \tag{5}$$

where $\beta = i\alpha + \frac{1}{2}(m+n-2)$ and in the case when $m+n$ is even

$$\tilde I^{(m,n)} = i\alpha\,(m+n)\,\tilde I^{(m-1,n-1)} . \tag{6}$$

Proof: Any element $\mathcal{P} \in W_{2(m+n-2+N),M}$ can be written in the form

$$\mathcal{P} = \sum_{r,s} a_{rs} \cdot q^r \cdot p^s$$

$$(a_{rs} \cdot q^r \cdot p^s \equiv a_{r_1 \ldots r_{m+n-2} s_1 \ldots s_{m+n-2}} \times$$

$$\times\ q_1^{r_1} \ldots q_{m+n-2}^{r_{m+n-2}} \cdot p_1^{s_1} \ldots p_{m+n-2}^{s_{m+n-2}} ,$$

where $a_{rs} \in W_{2N,M} \subset W_{2(m+n-2+N),M}$).

Let us introduce the "projection" operator "abs" in $W_{2(m+n-2+N),M}$ by the relation

$$\text{abs}\ \mathcal{P} = a_{00} .$$

Directly from the definition we see that

$$\text{abs}\; q_i \mathcal{P} = \text{abs}\, \mathcal{P}\, p_i = 0$$

$$\text{abs}\, M_{ij} \mathcal{P} = M_{ij}\; \text{abs}\, \mathcal{P} \;, \tag{7}$$

$$\text{abs}(\mathcal{P} + \mathcal{P}') = \text{abs}\, \mathcal{P} + \text{abs}\, \mathcal{P}',$$

$$\text{abs}(p_1 q_1) = \text{abs}(q_1 p_1 + 1) = 1, \text{ etc}.$$

As we proved in ref. /1/ (see proof of theorem 1), the realization $r(z)$ of any Casimir operator z of $o(m,n)$ in the realization (2) does not depend on canonical variables q_i, p_i. We can write therefore for $r(I_r^{(m,n)})$ the relation

$$r(I_r^{(m,n)}) = \text{abs}\, r(I_r^{(m,n)}) = g^{\mu\nu}\; \text{abs}\, r(T_{\mu\nu}^{(r)}).$$

Here $T_{\mu\nu}^{(r)}$ is defined recurrently:

$$T_{\mu\nu}^{(r)} = L_\mu^{\ \rho}\, T_{\rho\nu}^{(r-1)}, \quad T_{\mu\nu}^{(0)} = g_{\mu\nu}.$$

As the proof of formula (5) for $r = 0$ is trivial we can assume $r \geq 1$ and further write:

$$r(I_r^{(m,n)}) = \text{abs}\, r\, \{ R\, (T_{m+n-1,\, m+n}^{(r-1)} - T_{m+n,\, m+n-1}^{(r-1)}) +$$

$$+ \frac{1}{2}(P^i + Q^i)\, (T_{m+n,i}^{(r-1)} - T_{i,m+n}^{(r-1)}) + \frac{1}{2}(P^i - Q^i) \times$$

$$\times (T_{i,m+n-1}^{(r-1)} - T_{m+n-1,\, i}^{(r-1)}) + (M^{ij} + q^i p^j - q^j p^i)\, T_{ji}^{(r-1)} \}.$$

This expression can be, due to the special form of realization of the basis elements (2), simplified by means of the relations (7) to

$$r(I_r^{(m,n)}) = -\bar{\beta}\; \text{abs}\, r\, (T_{m+n-1,\, m+n}^{(r-1)} - T_{m+n,\, m+n-1}^{(r-1)}) +$$

$$+ \frac{1}{2}\; \text{abs}\, r\, ([\, P^i, T_{m+n,\, i}^{(r-1)} - T_{i,m+n}^{(r-1)} + T_{i,m+n-1}^{(r-1)} - T_{m+n-1,\, i}^{(r-1)}\,]) +$$

$$+ M^{ij} \operatorname{abs} \tau (T_{ji}^{(r-1)}),$$

where $\operatorname{abs} R = -\beta = i\alpha - \frac{1}{2}(m+n-2)$. Using the commutation relations

$$[L_{\rho\tau}, T_{\mu\nu}^{(r)}] = g_{\tau\mu} T_{\rho\nu}^{(r)} - g_{\rho\mu} T_{\tau\nu}^{(r)} + g_{\tau\nu} T_{\mu\rho}^{(r)} - g_{\rho\nu} T_{\mu\tau}^{(r)}$$

we further obtain

$$\tau (I_r^{(m,n)}) = \beta \operatorname{abs} \tau \; (T_{m+n-1,m+n}^{(r-1)} - T_{m+n,m+n-1}^{(r-1)}) +$$

$$+ M^{ij} \operatorname{abs} \tau (T_{ji}^{(r-1)}).$$

In order to prove the formula (5) we need to express the right-hand side of the last equation in terms of Casimir operators of $o(m-1), n-1)$. Let us define

$$A_r = \operatorname{abs} \tau \,(T_{m+n-1,m+n}^{(r)} - T_{m+n,m+n-1}^{(r)}),$$

$$B_r = \operatorname{abs} \tau \,(T_{m+n,m+n}^{(r)} - T_{m+n-1,m+n-1}^{(r)}).$$

Using the same calculation as above we derive easily the recurrent relations for these quantities:

$$A_r = i\alpha B_{r-1} + \frac{1}{2}(m+n-2)A_{r-1} - g^{ij} \operatorname{abs} \tau (T_{ij}^{(r-1)}),$$

$$B_r = i\alpha A_{r-1} + \frac{1}{2}(m+n-2)B_{r-1} - g^{ij} \operatorname{abs} \tau (T_{ij}^{(r-1)}).$$

It further gives

$$A_r - B_r = \bar{\beta}\,(A_{r-1} - B_{r-1}) \qquad (8)$$

from which

$$B_r = A_r + 2\bar{\beta}^r.$$

Substituting it into the above relations for A_r, B_r, we obtain the single relation

$$A_r = \beta A_{r-1} + 2ia \cdot \bar{\beta}^{r-1} - g^{ij} \operatorname{abs} r (T_{ij}^{(r-1)}) =$$

$$= \beta A_{r-1} + (\beta - \bar{\beta}) \bar{\beta}^{r-1} - g^{ij} \operatorname{abs} r (T_{ij}^{(r-1)}).$$

Using once more the above calculation and eq. (8) we derive easily the recurrent relation for $\operatorname{abs} r (T_{ij}^{(r)})$:

$$\operatorname{abs} r (T_{ij}^{(r)}) = \tilde{M}_i^k \operatorname{abs} r (T_{kj}^{(r-1)}) - g_{ij} \bar{\beta}^{r-1},$$

where

$$\tilde{M}_{ij} = M_{ij} + g_{ij} .$$

One can solve this relation as follows

$$\operatorname{abs} r (T_{ij}^{(r)}) = \tilde{M}_{ij}^{(r)} - \sum_{s=0}^{r-1} \bar{\beta}^{r-s-1} \tilde{M}_{ij}^{(s)},$$

where

$$\tilde{M}_{ij}^{(r)} = \begin{cases} g_{ij} + rM_{ij} & r = 0, 1 \\ \tilde{M}_i^{s_1} \tilde{M}_{s_1}^{s_2} \ldots \tilde{M}_{s_{r-1}, j} & r > 1 . \end{cases}$$

Using it we obtain

$$C_r \equiv g^{ij} \operatorname{abs} r (T_{ij}^{(r)}) = \tilde{M}^{(r)} - \sum_{s=0}^{r-1} \bar{\beta}^{r-s-1} \tilde{M}^{(s)},$$

where $\tilde{M}^{(r)} = \tilde{M}_{ij}^{(r)} \cdot g^{ij}$ and

$$M^{ij} \operatorname{abs} r (T_{ji}^{(r)}) = \tilde{M}^{(r+1)} - 2\tilde{M}^{(r)} + (1 - \bar{\beta}) \sum_{s=1}^{r-1} \bar{\beta}^{r-s-1} \tilde{M}^{(s)} +$$

$$+ (m + n - 2) \bar{\beta}^{r-1} = \tilde{M}^{(r+1)} - 2\tilde{M}^{(r)} +$$

$$+ (1 - \bar{\beta}) \sum_{s=0}^{r-1} \bar{\beta}^{r-s-1} \tilde{M}^{(s)} + (m + n - 2) \bar{\beta}^{r} .$$

The relation for A_r we shall now write in the form

$$A_r = \beta A_{r-1} + (\beta - \bar\beta)\bar\beta^{r-1} - C_{r-1}$$

which is solved by

$$A_r = \beta^r - \bar\beta^r - \sum_{s=0}^{r-1} \beta^{r-s-1} C_s$$

Substituting now for M^{ij} abs $\tau(T_{ji}^{(r-1)})$ and A_r into the equation

$$\tau(I_r^{(m,n)}) = \beta A_{r-1} + M^{ij} \text{ abs}\,\tau(T_{ji}^{(r-1)})$$

we finally obtain

$$\tau(I_r^{(m,n)}) = \beta^r + \bar\beta^r - \sum_{s=0}^{r-2} [\beta^{r-s-1} + \bar\beta^{r-s-1} +$$

$$+ \frac{\bar\beta^{r-s-1} - \beta^{r-s-1}}{\beta - \bar\beta}]\tilde M^{(s)} + \tilde M^{(r)} - 2\tilde M^{(r-1)}.$$

From the definitions of $\tilde M_{ij}, \tilde M^{(r)}$ we obtain directly

$$\tilde M^{(r)} = \sum_{s=0}^{r} \binom{r}{s} M^{(s)},$$

where

$$M^{(r)} = \begin{cases} (m+n-2)(1-r) & r = 0,1 \\ M_{s_r}^{s_1} M_{s_1}^{s_2} \dots M_{s_{r-1}}^{s_r} & r > 1 . \end{cases}$$

As the elements M_{ij}, $i, j = 1,2,\dots,m+n-2$, generate a given realization of $o(m-1, n-1)$, the quantities $M^{(r)}$ are just the Casimir operators (more exactly: their realizations) of $o(m-1, n-1)$, i.e.,

$$M^{(r)} = I_r^{(m-1,n-1)} \Rightarrow \tilde M^{(r)} = J_r^{(m-1,n-1)}$$

and formula (5) is proved.

As to the formula (6), the realization of generating

Casimir operator $\tilde{I}^{(m,n)}$ does not depend also on canonical variables and we can write

$$\tau(\tilde{I}^{(m,n)}) = \text{abs}\,\tau\left(\epsilon^{\mu_1\nu_1\cdots\mu_{\frac{m+n}{2}}\nu_{\frac{m+n}{2}}} L_{\mu_1\nu_1}\cdots L_{\mu_{\frac{m+n}{2}}\nu_{\frac{m+n}{2}}}\right).$$

Let us denote $h=\frac{1}{2}(m+n)$ and notice that since the only non-zero terms are those having all the indices μ_1, $\nu_1,\dots,\mu_h,\nu_h$ mutually different, we are absolutely free in interchanging $L'_{\mu\nu}$s (see the commutation relations (1)) so that we can write

$$\tau(\tilde{I}^{(m,n)}) = 2h\,\text{abs}\,\tau\left(\epsilon^{m+n-1,\,m+n,\,i_2,j_2,\dots,i_h,j_h} L_{m+n-1,m+n} L_{i_2j_2}\cdots\right.$$

$$\left.\cdots L_{i_hj_h}\right) + \left[2\binom{2h}{2} - 2h\right]\text{abs}\,\tau\left(\epsilon^{i,\,m+n-1,\,j,\,m+n,\,i_3,j_3,\dots,i_h,j_h}\times\right.$$

$$\left.\times L_{i,m+n-1} L_{j,m+n} L_{i_3j_3}\cdots L_{i_hj_h}\right)$$

where the latin indices run from 1 to $2h-2$. Further with the help of eqs. (1), (2), (7) we have:

$$\tau(\tilde{I}^{(m,n)}) = \epsilon^{i_2j_2\cdots i_hj_h}\left[-2h\bar{\beta}\,\text{abs}\,\tau(L_{i_2j_2}\cdots L_{i_hj_h}) - \right.$$

$$\left. - h(h-1)\text{abs}\,\tau([P_{i_2}, Q_{j_2}] L_{i_3j_3}\cdots L_{i_hj_h}\right) =$$

$$= 2h(-\bar{\beta}+h-1)\epsilon^{i_2j_2\cdots i_hj_h}\,\text{abs}\,\tau\left[(M_{i_1j_2} + q_{i_2}p_{j_2} - \right.$$

$$\left. - q_{j_2}p_{i_2}) L_{i_3j_3}\cdots L_{i_hj_h}\right] =$$

$$=2h(-\bar\beta+h-1)\,\epsilon^{i_2 j_2 \dots i_h j_h} M_{i_2 j_2}\dots M_{i_h j_h} .$$

But since M_{ij}, $i,j=1,2,\dots,2h-1$, generates the realization of $o(m-1,n-1)$, the last equation one can write in the form

$$r(\bar I^{(m,n)})=[-2h\bar\beta+2h(h-1)]\tilde I^{(m-1,n-1)} .$$

According to the definition

$$-\bar\beta+h-1=i\alpha-\frac{1}{2}(m+n-2)+\frac{1}{2}(m+n)-1=i\alpha\ ,\ 2h=m+n$$

and the validity of the formula (6) is proved.

Lemma 2: Let a realization of $o(m,n), m+n\geq 3$, of the type (2) be given. If the corresponding Schur-realization of $o(m-1,n-1)$ is such that the values of the Casimir operators can be expressed as

$$I_r^{(m-1,n-1)}=\sigma_r^{(N)}(\delta_1,\dots,\delta_{[N/2]}),\ N=m+n-2,\ r=0,1,\dots$$

for some complex numbers $(\delta_1,\dots,\delta_{[N/2]})$ then the values of Casimir operators in the realization of $o(m,n)$ are

$$I_r^{(m,n)}=\sigma_r^{(N+2)}(\delta_1,\dots,\delta_{[N/2]},i\alpha).$$

Proof. From the definition (3) one can prove easily by induction the relation between r-th powers of the matrices $S_N(x_1,\dots,x_{[N/2]})=S_N$ and $S_{N+2}(x_1,\dots,x_{[\frac{N+2}{2}]})=S_{N+2}$:

$$S_{N+2}^r=\begin{pmatrix} y^r, & -e_N^+\sum_{s=0}^{r-1}y^{r-s-1}(S_N+E_N)^s, & e_N^+\sum_{s=0}^{r-2}\frac{y^{r-s-1}-z^{r-s-1}}{y-z}(S_N+E_N)^s e_N \\ 0, & (S_N+E_N)^r, & -\sum_{s=0}^{r-1}z^{r-s-1}(S_N+E_N)^s e_N \\ 0, & 0, & z^r \end{pmatrix}\tag{9}$$

where $y=x_{[\frac{N+2}{2}]}+\frac{N}{2}$, $z=-x_{[\frac{N+2}{2}]}+\frac{N}{2}$.

Using the definition (4) of the polynomials $\sigma_r^{(N)}$ we obtain the relation between $\sigma_r^{(N)}(x_1, \dots, x_{[N/2]})$ and $\sigma_r^{(N+2)}(x_1, \dots, x_{[\frac{N+2}{2}]})$:

$$\sigma_r^{(N+2)}(x_1, \dots, x_{[\frac{N+2}{2}]}) =$$

$$= y^r + z^r - \sum_{s=0}^{r-2} \left(y^{r-s-1} + z^{r-s-1} + \frac{y^{r-s-1} - z^{r-s-1}}{z-y} \right) \times$$

$$\times \omega_s^{(N)} - 2\omega_{r-1}^{(N)} + \omega_r^{(N)}, \tag{10}$$

where

$$\omega_r^{(N)} \equiv \omega_r^{(N)}(x_1, \dots, x_{[N/2]}) = e_N^+ (S_N + E_N)^r e_N =$$

$$= \sum_{s=0}^{r} \binom{r}{s} \sigma_s^{(N)}(x_1, \dots, x_{[N/2]}), \quad r = 0, 1 \dots$$

Substituting into the relation (10)

$$x_1 = \delta_1, \dots, x_{[N/2]} = \delta_{[N/2]}, \quad N = n + m - 2,$$

$$x_{[\frac{N+2}{2}]} = i\alpha \quad \Rightarrow y = \beta, \quad z = \bar{\beta},$$

$$I_r^{(m-1,n-1)} = \sigma_r^{(N)}(\delta_1, \dots, \delta_{[N/2]}), \quad J_r^{(m-1,n-1)} = \omega_r^{(N)}(\delta_1, \dots, \delta_{[N/2]})$$

we obtain with the help of formula (5)

$$I_r^{(m,n)} = \sigma_r^{(m+n)}(\delta_1, \dots, \delta_{[\frac{m+n-2}{2}]}, i\alpha)$$

which just proves the lemma.

Now we are in the position to prove our main theorem.

Theorem 1. Let $a_{m,n}=(d; a_1, \dots, a_{[\frac{m+n}{2}]})$ be signature of the realization (2) of Lie algebra $o(m,n), m \geq n \geq 1$. Then the values of Casimir operators are

i. $$I_r^{(m,n)} = \sigma_r^{(m+n)}(\beta_1, \dots, \beta_{[\frac{m+n-2d}{2}]}, i a_{[\frac{m+n-2d}{2}]+1}, \dots, i a_{[\frac{m+n}{2}]}),$$

$r = 0, 1, \dots,$ (11)

where

$$\beta_s = a_s + \gamma_s, \quad \gamma_s = \frac{m+n-2d}{2} - s, \; s = 1, 2, \dots, [\frac{m+n-2d}{2}], \tag{12}$$

ii. for $m+n$ even

$$\tilde{I}^{(m,n)} = \delta_{dn} \cdot (2i)^{\frac{m+n}{2}} (\frac{m+n}{2})! \, \beta_1 \dots \beta_{\frac{m-n}{2}} a_{\frac{m-n+2}{2}} \dots a_{\frac{m+n}{2}}.$$

Proof. By induction: i. a) Let us firstly consider the realization of the type (2) of the algebra $o(m,1)$ with signature $a_{m,1} = (1, a_1, a_2, \dots, a_{[\frac{m+1}{2}]})$, $m>2$. As it was pointed out in the part C of Preliminaries the Casimir operators $I_r^{(m-1,0)}$ in the realization of $o(m-1,0) \equiv o(m-1)$ characterized by signature $(a_1, \dots, a_{[\frac{m-1}{2}]})$ have just the form (11) in variables $\beta_1, \dots, \beta_{[\frac{m-1}{2}]}$ so that lemma 2 can be applied. In the case of $o(2,1)$ the assertion follows also from lemma 2 if we put $I_r^{(1,0)} = \sigma_r^{(1)} = \delta_{r0}$ (see remark p.5 and eq. (4)) and for $o(1,1)$ it can be verified directly. b) Suppose now that the assertion i. is valid for $o(m-1, n-1)$, $m \geq n \geq 2$, and let us take realization of $o(m,n)$ corresponding to signature $a_{m,n} = (d; a_1, \dots, a_{[\frac{m+n}{2}]})$. For $d>1$ the realization of $o(m-1,n-1)$ from the formulae (2) corresponds to the signature

$$(d-1; a_1, \dots, a_{[\frac{m+n}{2}]-1})$$

and because, by induction assumption, Casimir operators have the desired form, the lemma 2 can be applied.

If signature $a_{m,n} = (1; 0, \ldots, 0, a_{[\frac{m+n}{2}]})$, the realization of $o(m-1,n-1)$ used in eqs. (2) it trivial and we have to prove that Casimir operators $I^{(m-1,n-1)}_{r} \geq 0$ can be expressed as the values of polynomials $\sigma_k^{(m+n-2)}$ at the point $(\gamma_1, \ldots, \gamma_{[\frac{m+n-2}{2}]})$. This fact is however proved in ref. /5/ (see, e.g., relations (55)-(57)) so that lemma 2 again can be applied and the proof of assertion i is completed.

ii. The proof is a simple consequence of eq. (6) and of the form of the Casimir operator $I^{(m-n)}$ given in Preliminaries, part C.

Now we shall deal with the question how the values of Casimir operators differ for different signatures of realizations. We denote by $\Omega_{m,n}$ the following subset of the set of all signatures with fixed m and n:

$$\Omega_{m,n} = \{ (d; a_1, \ldots, a_{[\frac{m+n}{2}]}) \mid 0 \leq z_K + \delta_{mn} (|a_K| - a_K) \leq a_{K+1} \leq \ldots$$

$$\ldots \leq a_{[\frac{m+n}{2}]}, \quad K = [\frac{m-n}{2}] + 1;$$

if $m-n$ is even then $d \neq n-1$ and $a_{[\frac{m-n}{2}]+1} = 0 \Rightarrow a_{[\frac{m-n}{2}]} \geq 0$*}.

Theorem 2: i. For every signature $a_{m,n}$ there exists $a'_{m,n} \in \Omega_{m,n}$ such that the values of any Casimir operator in the corresponding realizations are the same.

ii. The signature $a'_{m,n} \in \Omega_{m,n}$ is determined uniquely, i.e., for two different signatures from $\Omega_{m,n}$ the corresponding realizations differ by the value of at least one Casimir operator.

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* This condition is automatically satisfied if either $d<n$ or $m=n$.

P r o o f: i. The assertion is a simple consequence of the symmetry of polynomials in the last d squared components of the signature $a_{m,n}$. If $m-n$ is even, the signatures $(n-1;0,\dots,0,a_{[\frac{m-n}{2}]+2},\dots,a_{[\frac{m+n}{2}]})$ may be excluded from

$\Omega_{m,n}$ because they give the same values of Casimir operators as the signature $(n;1,\dots 1,0,a_{[\frac{m-n}{2}]+2},\dots a_{[\frac{m+n}{2}]})$ (see eqs. (11)-(12)). As to signature $a_{m,n}=(n;a_1,\dots,a_{[\frac{m+n}{2}]})$, $m-n$ even, $a_{[\frac{m-n}{2}]+1+\delta_{mn}}\dots a_{[\frac{m+n}{2}]}\neq 0$, when also exceptional invariant $\tilde{I}^{(m,n)}$ has to be considered, the signature $a'_{m,n}\in\Omega_{m,n}$ has the form

$a'_{m,n}=(n;a_1,\dots,\epsilon a_{[\frac{m-n}{2}]+\delta_{mn}},|a_{s_1}|,\dots,|a_{s_{n'}}|)$,

where $\epsilon=\operatorname{sgn} a_{[\frac{m-n}{2}]+1+\delta_{mn}}\dots a_{[\frac{m+n}{2}]}$ and $s_1,\dots,s_{n'}$, $n'=n-\delta_{mn}$, is such permutation of indices $[\frac{m-n}{2}]+1+\delta_{mn},\dots,[\frac{m+n}{2}]$ that $|a_{s_1}|\leq|a_{s_2}|\dots\leq|a_{s_{n'}}|$.

ii. As we pointed out in Preliminaries, any Newton's sum of even degree $s_{2r}=\sum_{s=1}^{N}(x_s)^{2r}$ can be written as the polynomial in variables $\sigma_s^{(N)}=\sigma_s^{(N)}(x_1,\dots,x_N)$, $s=1,2,\dots,2r$. Even Newton's sum s_{2r} can be considered as the Newton's sum s_r in variables $x'_s=x_s^2$, $s=1,\dots,N$.

Consider now the so-called elementary symmetric polynomials $\xi_r^{(N)}$, $r=1,2,\dots,N$, in variables x'_i defined as follows;

$$\xi_r^{(N)}=\xi_r^{(N)}(x'_1,\dots,x'_N)=\sum_{(s_1,\dots,s_r)}x'_{s_1}\dots x'_{s_r},$$

where summation runs over all sequences $(s_1,\dots,s_r)$ with $1\leq s_1<s_2<\dots<s_r\leq N$. It is known [6] that every symmetric polynomial $\xi_r^{(N)}$ can be expressed by means of Newton's sums $s_t=\sum_{s=1}^{N}(x'_s)^t$ and therefore any symmetric poly-

nomial $\xi_r^{(N)}$ can be expressed also by means of polynomials $\sigma_s^{(N)}$.

So, two signatures $\alpha'_{m,n}$, $\alpha''_{m,n}$ giving the same values of any Casimir operator give also the same values of $\xi^{[\frac{m+n}{2}]}$ – polynomials:

$$\xi_r \equiv \xi_r^{[\frac{m+n}{2}]}(\beta_1'^2, \dots, (i\alpha'_{[\frac{m+n-2d'}{2}]+1})^2, \dots, (i\alpha'_{[\frac{m+n}{2}]})^2) \equiv$$

$$\equiv \xi_r^{[\frac{m+n}{2}]}(\beta_1''^2, \dots, (i\alpha''_{[\frac{m+n-2d''}{2}]+1})^2, \dots, (i\alpha''_{[\frac{m+n}{2}]})^2).$$

It is however, further known /6/ that the set of all solutions of the $[\frac{m+n}{2}]$ – th order equation

$$y^{[\frac{m+n}{2}]} + \xi_1 y^{[\frac{m+n}{2}]-1} + \dots + \xi_{[\frac{m+n}{2}]-1} y + \xi_{[\frac{m+n}{2}]} = 0$$

equals just to

$$\{\beta_1'^2, \dots, (i\alpha_{[\frac{m+n}{2}]})^2\} \equiv \{\beta_1''^2, \dots, (i\alpha''_{[\frac{m+n}{2}]})^2\}.$$

As $\alpha'_{m,n}, \alpha''_{m,n} \in \Omega_{m,n}$, the elements of these sets are ordered*:

$$\beta_1'^2 > \beta_2'^2 > \dots > \beta'^2_{[\frac{m+n-2d'}{2}]} \geq 0 \geq (i\alpha'_{[\frac{m+n-2d'}{2}]+1})^2 \geq \dots \geq (i\alpha'_{[\frac{m+n}{2}]})^2,$$

---

*See also eq. (12) and remember that for $d = n$ and $m-n \geq 2$ the components $\alpha_1, \dots, \alpha_{[\frac{m-n}{2}]}$ form the signature of an irreducible skew-hermitean representation of $o(m-n)$ and they are ordered: $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{[\frac{m-n}{2}]} \geq 0$ if $m-n$ is odd and $\alpha_1 \geq \dots \geq |\alpha_{[\frac{m-n}{2}]}|$ if $m-n$ is even.

$$\beta_1''^2 > \beta_2''^2 > \ldots > \beta''^2_{[\frac{m+n-2d''}{2}]} \geq 0 \geq (i\alpha''_{[\frac{m+n-2d''}{2}]+1})^2 \geq \ldots \geq (i\alpha''_{[\frac{m+n}{2}]})^2.$$

For $m+n$ odd $\beta'_{[\frac{m+n-2d'}{2}]} > 0$, $\beta''_{[\frac{m+n-2d''}{2}]} > 0$ even (see eq. (12)) and therefore $d'' = d'$ and, consequently, $\alpha'_{m,n} = \alpha''_{m,n}$, i.e., assertion ii. is proved.

If however $m-n$ is even, then, beside possibility $d' = d''$ which implies again $\alpha'_{m,n} = \alpha''_{m,n}$ *, also $\beta'_{[\frac{m+n-2d'}{2}]} =$ $= 0 = \alpha''_{[\frac{m+n-2d''}{2}]+1}$ (or $\beta_{[\frac{m+n-2d''}{2}]} = 0 = \alpha'_{[\frac{m+n-2d'}{2}]+1}$) could be allowed which implies $d'' = d' - 1$ ($d' = d'' - 1$). For $d' < n$ it contradicts the equation $\beta_1' \equiv \gamma_1' = \beta_1'' \equiv \gamma_1''$ so that $d' = n$, $d'' = n-1$. The signatures with $d'' = n-1$ are not however included in the set $\Omega_{m,n}$ and uniqueness of $\alpha'_{m,n}$ is proved in this last case too.

4. CONCLUSION

In the first part of this paper we proved that two described realizations τ and τ' of the Lie algebra $o(m,n)$ characterized by different signatures are non-related, i.e., no endomorphism θ of $W_{2N,M}$, $\theta(1) = 1$, exists such that either $\theta \circ \tau = \tau'$ or $\theta \circ \tau' = \tau$. It may happen, of course, that by a proper embedding of $W_{2N,M}$ in a larger structure (e.g., in the case of W_{2N} embedding

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* The uncertainty $\alpha'_{[\frac{m-n}{2}]+\delta_{mn}} = \pm \alpha''_{[\frac{m-n}{2}]+\delta_{mn}}$ which may arise for $d' = d'' = n$ is excluded either by definition of $\Omega_{m,n}$ ($\alpha_{[\frac{m-n}{2}]+1} = 0 \Rightarrow \alpha_{[\frac{m-n}{2}]} = \beta_{[\frac{m-n}{2}]} = 0$) or by means of Casimir operator $\tilde{I}^{(m,n)}$.

in its quotient division ring) when more general endomorphisms are allowed, the non-related realizations appear as related in the generalized sense, (e.g., non-related realizations (2) of $o(2,1)$ in W_2 with opposite a's are related in quotient division ring $D_2 \supset W_2$; the endomorphism θ has the form: $\theta(p_1) = p_1$, $\theta(q_1) = q_1 - i\frac{2a}{p_1}$). This possibility is, however, excluded in the case of our realizations, the signatures of which lie in $\Omega_{m,n}$. The reason is that the element z from the centre of the enveloping algebra of $o(m,n)$ exists such that $r(z) = a_z 1$, $r'_z(z) = a'_z 1$, $a_z, a'_z \in C$ with $a_z \neq a'_z$ and therefore for no endomorphism θ, $\theta(1) = 1$ of any structure containing $W_{2N,M}$ equation $\theta \circ r(z) = r'(z)$ is valid because it implies immediately $a_z = a'_z$.

It means that as related realizations in the generalized sense the realizations with signatures differing only in permutation of the last n components and their signs (with the exception of some cases if $m+n$ is even) can appear.

In our earlier paper /3/, dealing with the minimal canonical realizations of the complexification $o_C(m,n)$ of the Lie algebra $o(m,n)$*, we studied a.o. also the question of the mutual dependence of Casimir operators in canonical realization in $W_{2(n+m-2)}$ (i.e., when generators of $o(m,n)$ are expressed as polynomials in $m+n-2$ pairs of canonical variables). We showed that if $m+n \geq 7$ in any such realization r, realization of any generating Casimir operator $r(I_{2r}^{(m,n)})$ (and square $r(\tilde{I}^{(m,n)})^2$ if $m+n$ is even) depends polynomially on $r(I_2^{(m,n)})$; there are at most two types of these polynomials and they do not depend on realization r. The one-parametrical set of realizations with signatures $(1;0,...,0,a_{[\frac{m+n}{2}]})$ lies in $W_{2(m+n-2)}$ and we can

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* Note that in Cartan classification of simple Lie algebra $o_C(m,n) \simeq D_{\frac{m+n}{2}}$ if $m+n$ is even and $o_C(m,n) \simeq B_{\frac{m+n-1}{2}}$ if $m+n$ is odd.

easily see that the above assertion is valid in this case. The realizations $r(I_{2r}^{(m,n)})$ are now symmetric polynomials in one variable a^2 only and a^2 is a linear function of $r(I_2^{(m,n)})$; the fact that this polynomial dependence is really one of the two above-mentioned dependences needs, of course, a special proof. The realization of Casimir operator $\tilde{I}^{(m,n)}$ equals zero.

Increasing d, the number of independent Casimir operators in realization also increases.

If $d < n$ then $r(I_{2r}^{(m,n)})$, $r > d$, is the polynomial function in the variables $r(I_2^{(m,n)}), \dots, r(I_{2d}^{(m,n)})$, which considered as the functions of the parameters $a_{[\frac{m+n}{2}]-d+1}, \dots, a_{[\frac{m+n}{2}]}$, are mutually independent and $r(\tilde{I}^{(m,n)}) = 0$ if $m+n$ is even.

In accordance with note 1 and theorem 1 Newton's sums $s_2, \dots, s_{2d}$ polynomially depend on $r(I_{2s}^{(m,n)}) = \sigma_{2s}^{(m+n)}$, $s \leq d$. The remaining Newton's sums $s_{2(d+1)}, \dots$ depend on the first d even ones, as they are, following our assumption, functions of d variables only. Therefore all $r(I_{2r}^{(m,n)})$ depend in this case on Newton's sums $s_2, \dots, s_{2d}$ only, i.e., on $r(I_2^{(m,n)}), \dots, r(I_{2d}^{(m,n)})$.

If $d = n$ the realizations of all $[\frac{m+n}{2}]$ generating Casimir operators $r(I_2^{(m,n)}), \dots, r(I_{2[\frac{m+n}{2}]-2}^{(m,n)})$ and $r(I_{[\frac{m+n}{2}]}^{(m,n)})$ (or $r(\tilde{I}^{(m,n)})$ if $m+n$ is even) are independent *. The

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* In the case $d = n$ when part of the parameters can allow only discrete values we generalize the concept of independent polynomials in the following way:

a) Let subset $\Omega \subset R^N$ have the property: if a polynomial $P(x) = 0$ for all $x \in \Omega$ then $P(x) \equiv 0$ for all $x \in R^N$

b) the set $\{P_1^\Omega, \dots, P_M^\Omega\}$ of functions on Ω which are restrictions of some polynomials $P_1, \dots, P_M$ to Ω are called independent if $P_1, \dots, P_M$ are independent.

The condition (a) guarantes uniqueness of extension P_i to any P_i^Ω. It is clear that the condition (a) is respected by the set of all signatures $(n; a_1, \dots, a_{[\frac{m+n}{2}]})$ considered as the subset of $R^{[\frac{m+n}{2}]}$.

proof is the same as in the preceeding case; only if $m+n$ is even the $[\frac{m+n}{2}]$ -th Casimir operator $I^{(m,n)}_{2[\frac{m+n}{2}]}$ can be substituted by $\tilde{I}^{(m,n)}$.

If $m-n=0,1,2$ then no "right" matrix canonical realizations of $o(m,n)$ exist in our set, i.e., the realization with any signature is an usual canonical one. In this case the maximal number $[\frac{m+n}{2}]$ of independent Casimir operators is achieved taking maximal $d=n$, i.e., considering the set of realizations with maximal number of canonical pairs $N(n)=n(m-1)$.

On the contrary if $m-n>2$ the canonical realizations form the proper subset in the described set which is characterized by the signatures with $d<n$ or $d=n$ and $a_1 = \dots = a_{[\frac{m-n}{2}]} = 0$.

In this case at most $n<[\frac{m+n}{2}]$ independent Casimir operators can be obtained in the set of canonical realizations with $N(n)=n(m-1)$ canonical pairs.

So to reach the full number $[\frac{m+n}{2}]$ of independent Casimir operators the use of right matrix canonical realizations is necessary.

Formulae for the eigenvalues of Casimir operators in matrix canonical realizations of noncompact Lie algebra $o(m,n), n\geq 1$ derived in this paper are closely related to formulae for the eigenvalues of Casimir operators in irreducible representations of compact Lie algebra $o(m+n)$ derived by Perelomov and Popov [4,5]. Our formulae (11) and (12) arise, essentially from the formulae of Perelomov and Popov (see Preliminaries part C) simply by substitution of $\beta_{[\frac{m+n-2d}{2}]+1}, \dots \beta_{[\frac{m+n}{2}]}$ by $i a_{[\frac{m+n-2d}{2}]+1}, \dots, i a_{[\frac{m+n}{2}]}$. This interesting circumstance should indicate some sort of exceptionality of the matrix canonical realizations of $o(m,n)$ described and investigated in our paper.

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Received by Publishing Department
on March 18, 1975.